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Asymptotic Properties of Solutions to Semilinear Equations Involving Multiple Critical Exponents

Dongsheng Kang

Abstract. In this paper, we investigate a semilinear elliptic equation that involves multiple Hardy-type terms and critical Hardy–Sobolev exponents. By the Moser iteration method and analytic techniques, the asymptotic properties of its nontrivial solutions at the singular points are investigated.

1 Introduction

In this paper, we study the following elliptic problem:

$$(1.1)\qquad \begin{cases} -\Delta u - \displaystyle\sum_{i=1}^{k} \frac{\mu_i\, u}{|x-\xi_i|^2} = \sum_{i=1}^{k} \frac{|u|^{p_i-2}u}{|x-\xi_i|^{t_i}} & x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with the smooth boundary $\partial\Omega$, $\xi_i \in \Omega$, $0 < t_i < 2$, $p_i = 2^*(t_i) := \frac{2(N-t_i)}{N-2}$, $\mu_i < \bar{\mu} := (\frac{N-2}{2})^2$, $i = 1, 2 \dots, k, k \geq 2$. Note that $\bar{\mu}$ is the best Hardy constant, $2^*(t_i)$ are the critical Hardy–Sobolev exponents, and $2^*(0) = 2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent. We work in the space $H_0^1(\Omega)$, the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\| := (\int_\Omega |\nabla u|^2\, dx)^{1/2}$.

The energy functional corresponding to (1.1) is defined as follows:

$$J(u) := \frac{1}{2} \int_\Omega \Big(|\nabla u|^2 - \sum_{i=1}^{k} \frac{\mu_i\, u^2}{|x-\xi_i|^2} \Big)\, dx - \sum_{i=1}^{k} \frac{1}{p_i} \int_\Omega \frac{|u|^{p_i}}{|x-\xi_i|^{t_i}}\, dx.$$

Then $J \in C^1(H_0^1(\Omega), \mathbb{R})$. The function $u \in H_0^1(\Omega)$ is said to be a solution of (1.1) if

$$\langle J'(u), v\rangle := \int_\Omega \Big(\nabla u \nabla v - \sum_{i=1}^{k} \Big(\frac{\mu_i\, uv}{|x-\xi_i|^2} + \frac{|u|^{p_i-2}uv}{|x-\xi_i|^{t_i}} \Big) \Big)\, dx = 0, \quad \forall v \in H_0^1(\Omega).$$

It is clear that singularity occurs in the problem (1.1). Furthermore, problem (1.1) is related to the Caffarelli–Kohn–Nirenberg inequality [4]:

$$(1.2)\qquad \Big(\int_{\mathbb{R}^N} \frac{|u|^{\tilde{p}}}{|x-\xi|^{b\tilde{p}}}\, dx \Big)^{\frac{2}{\tilde{p}}} \leq C_{a,b} \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x-\xi|^{2a}}\, dx, \quad \forall\, u \in C_0^\infty(\mathbb{R}^N),$$

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which is also referred to as the Hardy–Sobolev inequality, where $\xi \in \mathbb{R}^N$, $a < \sqrt{\bar{\mu}}$, $a \le b < a+1$, $\tilde{p} = \tilde{p}(a,b) := \frac{2N}{N-2(1+a-b)}$ is the critical Hardy–Sobolev exponent and $C_{a,b} > 0$ is a constant depending on a and b. Note that (1.2) has another form:

$$(1.3)\qquad \Big(\int_{\mathbb{R}^N} \frac{|u|^{p(a,t)}}{|x-\xi|^t}\,dx\Big)^{2/p(a,t)} \le C\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x-\xi|^{2a}}\,dx, \quad \forall\, u \in C_0^\infty(\mathbb{R}^N),$$

where $a < \sqrt{\bar{\mu}}$, $2^* a \le t \le 2(a+1)$, and $p(a,t) := \frac{2(N-t)}{N-2-2a}$. Furthermore, the following Hardy inequality holds [6, 11]:

$$(1.4)\qquad \int_{\mathbb{R}^N} \frac{|u|^2}{|x-\xi|^{2(1+a)}}\,dx \le \frac{1}{(\sqrt{\bar{\mu}}-a)^2}\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x-\xi|^{2a}}\,dx, \quad \forall\, u \in C_0^\infty(\mathbb{R}^N).$$

According to (1.2)–(1.4) for all $\xi \in \mathbb{R}^N$, $0 \le t < 2$ and $2^*(t) := p(0,t) = \frac{2(N-t)}{N-2}$, the following Hardy and Hardy–Sobolev inequalities hold:

$$(1.5)\qquad \int_{\mathbb{R}^N} \frac{|u|^2}{|x-\xi|^2}\,dx \le \frac{1}{\bar{\mu}}\int_{\mathbb{R}^N} |\nabla u|^2\,dx, \quad \forall\, u \in C_0^\infty(\mathbb{R}^N),$$

$$(1.6)\qquad \Big(\int_{\mathbb{R}^N} \frac{|u|^{2^*(t)}}{|x-\xi|^t}\,dx\Big)^{2/2^*(t)} \le C\int_{\mathbb{R}^N} |\nabla u|^2\,dx, \quad \forall\, u \in C_0^\infty(\mathbb{R}^N).$$

For all $\mu < \bar{\mu}, 0 \le t < 2$ and $\xi \in \mathbb{R}^N$, by (1.5) and (1.6), the following best Hardy–Sobolev constant is well defined:

$$S_{\mu,t} := \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}\big(\,|\nabla u|^2 - \frac{\mu u^2}{|x-\xi|^2}\,\big)\,dx}{\big(\int_{\mathbb{R}^N} \frac{|u|^{2^*(t)}}{|x-\xi|^t}\,dx\big)^{2/2^*(t)}},$$

where $D^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to $(\int_{\mathbb{R}^N} |\nabla u|^2\,dx)^{1/2}$. The constant $S_{\mu,t}$ is crucial for the study of the problems as (1.1) and its minimizers were investigated in [6, 15].

In this paper we always set

$$(1.7)\qquad \mu^* := \sum_{\mu_i > 0,\ 1 \le i \le k} \mu_i, \quad \nu_i := \sqrt{\bar{\mu}} - \sqrt{\bar{\mu}-\mu_i}, \quad 1 \le i \le k.$$

The elliptic operator L is defined on $H_0^1(\Omega)$ as:

$$L := -\triangle\cdot - \sum_{i=1}^k \mu_i \frac{\cdot}{|x-\xi_i|^2}.$$

According to the Hardy inequality, L is a positive operator if $\mu^* < \bar{\mu}$.

The following assumptions are needed in this paper:

$$(\mathcal{H}_1)\qquad k \ge 2, \quad \mu^* < \bar{\mu}, \quad 0 < t_i < 2, \quad i = 1, 2, \dots, k.$$

$$(\mathcal{H}_2)\qquad \xi_i \in \Omega, \quad \xi_i \ne \xi_j, \quad i, j = 1, 2, \dots, k,\ i \ne j.$$

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It should be mentioned that the singular elliptic problems involving the Hardy–Sobolev inequality were studied extensively, and many important results were obtained providing good insight into these problems; see for example [1,4–10,13–16,19, 20] and the references therein. In particular, the problem (1.1) was studied by Kang and Li [14], and the existence of the positive solutions to (1.1) was established by the concentration compactness principle [17,18] and Mountain-Pass theorem [2,3]. In a recent paper [5], a singular problem involving a critical Sobolev exponent and multiple Hardy-type terms was studied by Cao and Han, the existence of nontrivial solutions was established by the variational methods, and the asymptotic properties of solutions were proved by applying a result of Smets [19, Theorem 2.3] and the Moser iteration method.

The study of the problem (1.1) is motivated by its various applications. See for example [8] for the physical applications of this kind of problems. The mathematical interest lies in the fact that there are multiple nonlinear terms with critical exponents in (1.1) (The exponent is critical in the sense of the weighted Sobolev embeddings). The multiple critical phenomena and the singularities in (1.1) cause more difficulties for its investigation and thus make (1.1) attractive and challenging.

In this paper, we prove the asymptotic properties of nontrivial solutions to (1.1) by an argument different than [5]. We first establish a crucial L^p estimating result (see (2.11) below). Then we improve the integrability of solutions to (1.1) by the Moser iteration methods and finally get the desired results. These asymptotic properties are crucial for the further study of (1.1), and to the best of our knowledge, the conclusions are new.

The main results of this paper are summarized in the following theorems.

Theorem 1.1 *Suppose that* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$ *hold and* $\mu_{i_0} \in (0, \bar{\mu})$ *for some* $i_0 \in \{1, 2, \dots, k\}$. *Assume that* $u(x) \in H_0^1(\Omega)$ *is a solution of the problem* (1.1). *Then there exist positive constants* C *and* ρ *small enough such that* $B_\rho(\xi_{i_0}) \subset \Omega$ *and*

$$|u(x)| \le C \left| x - \xi_{i_0} \right|^{-(\sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu_{i_0}})}, \quad \forall\, x \in B_\rho(\xi_{i_0}) \setminus \{\xi_{i_0}\}.$$

Theorem 1.2 *Suppose that* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$ *hold,* $\mu_{i_0} \in (0, \bar{\mu})$ *for some* $i_0 \in \{1, 2, \dots, k\}$, *and* $u(x) \in H_0^1(\Omega)$ *is a positive solution of the problem* (1.1). *Assume that either* $\mu_i = 0$ *for* $1 \le i \le k, i \ne i_0$, *or*

$$(\mathcal{H}_3) \qquad \sum_{1 \le i \le k,\ i \ne i_0} \frac{\mu_i}{|\xi_i - \xi_{i_0}|^2} > 0.$$

Then there exist positive constants ρ *small enough and* C, *such that* $B_\rho(\xi_{i_0}) \subset \Omega$ *and*

$$u(x) \ge C \left| x - \xi_{i_0} \right|^{-(\sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu_{i_0}})}, \ \forall\, x \in B_\rho(\xi_{i_0}) \setminus \{\xi_{i_0}\}.$$

Theorem 1.3 *Suppose that* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$ *hold, and* $-\infty < \mu_{i_0} \le 0$ *for some* $i_0 \in \{1, 2, \dots, k\}$. *Assume that* $u(x) \in H_0^1(\Omega)$ *is a solution of the problem* (1.1). *Then there exists a constant* $\rho > 0$ *small enough such that* $B_\rho(\xi_{i_0}) \subset \Omega$ *and* $u(x) \in L^\infty(B_\rho(\xi_{i_0}))$, *that is,* $u(x)$ *has no singularity at the point* ξ_{i_0}.

Remark 1.4 Note that $(\mathcal{H}_3)$ is weaker than the condition $\mu_i > 0, 1 \le i \le k$, $i \ne i_0$, and depends on the location of the singular points ξ_i and the values of μ_i, $1 \le i \le k, i \ne i_0$. According to Theorems 1.1–1.3, the singularities of solutions to the problem (1.1) are caused by the Hardy-type terms $\mu_i\, u/|x-\xi_i|^2,\ \mu_i > 0$. The Hardy-type terms $\mu_i\, u/|x-\xi_i|^2,\ \mu_i \le 0$, and the critical Hardy–Sobolev terms $|u|^{p_i-2}u/|x-\xi_i|^{t_i}$, do not cause the singularity of solutions.

This paper is organized as follows. In Section 2, Theorem 1.1 is proved. In Section 3, Theorems 1.2 and 1.3 are verified. In the following argument, $L^q(\Omega, |x-\xi|^\alpha)$ is the usual weighted $L^q(\Omega)$ space with the weight $|x-\xi|^\alpha$ and $H_0^1(\Omega, |x-\xi|^\alpha)$ is the weighted $H_0^1(\Omega)$ space with the weight $|x-\xi|^\alpha$. Then $O(\varepsilon^t)$ denotes the quantity satisfying $|O(\varepsilon^t)|/\varepsilon^t \le C$, $o(\varepsilon^t)$ means $|o(\varepsilon^t)|/\varepsilon^t \to 0$ as $\varepsilon \to 0$, $o(1)$ is a generic infinitesimal value and the quantity $O_1(\varepsilon^t)$ means that there exist constants $C_1, C_2 > 0$ such that $C_1\varepsilon^t \le O_1(\varepsilon^t) \le C_2\varepsilon^t$ as ε small. In the following argument, we always denote the positive constants as C and omit dx in integrals for convenience.

2 Proof of Theorem 1.1

We follow the Moser iteration argument.

Suppose that $\mu_{i_0} \in (0, \bar{\mu})$. Without loss of generality, we may assume $\xi_{i_0} = 0 \in \mathbb{R}^N$. Let $u \in H_0^1(\Omega)$ be a solution of (1.1) and define the function

$$v(x) := |x|^{\nu_{i_0}} u(x) = |x|^{\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu_{i_0}}}\, u(x), \quad x \in \Omega \setminus \{\xi_i,\ i = 1, 2, \dots, k\}.$$

Then direct calculation shows that $v(x)$ solves the equation

$$-\operatorname{div}(|x|^{-2\nu_{i_0}}\nabla v) = \frac{|v|^{p_{i_0}-2}v}{|x|^{p_{i_0}\nu_{i_0}+t_{i_0}}} + \sum_{1\le i\le k,\ i\ne i_0} \frac{\mu_i\, v}{|x|^{2\nu_{i_0}}|x-\xi_i|^2} + \sum_{1\le i\le k,\ i\ne i_0} \frac{|v|^{p_i-2}v}{|x|^{p_i\nu_{i_0}}|x-\xi_i|^{t_i}}. \tag{2.1}$$

Since $\xi_i \ne \xi_{i_0} = 0$ for all $i = 1, 2, \dots, k, i \ne i_0$, we can choose $R > 0$ small enough such that

$$R < \frac{1}{2}\min\{|\xi_i|,\ 1 \le i \le k, i \ne i_0\}.$$

Note that $|x-\xi_i| = O_1(1) > R > |x|$, $\forall\, x \in B_R(0), i \ne i_0$. Let $\eta \in C_0^\infty(B_R(0))$ be a cut-off function and set $\phi = \eta^2 v v_n^{2(s-1)}$, where $s,\ n > 1$, and $v_n = \min\{|v|,\ n\}$. Note that

$$\int_\Omega |x|^{-2\nu_{i_0}}\nabla v\nabla\phi = 2\int_\Omega |x|^{-2\nu_{i_0}}\eta v v_n^{2(s-1)}\nabla\eta\nabla v + \int_\Omega |x|^{-2\nu_{i_0}}\eta^2 v_n^{2(s-1)}|\nabla v|^2 + 2(s-1)\int_\Omega |x|^{-2\nu_{i_0}}\eta^2 v_n^{2(s-1)}|\nabla v_n|^2. \tag{2.2}$$

By the Cauchy inequality we get

$$\Big|2\int_\Omega |x|^{-2\nu_{i_0}}\eta v v_n^{2(s-1)}\nabla\eta\nabla v\Big| \le \frac{1}{2}\int_\Omega |x|^{-2\nu_{i_0}}\eta^2 v_n^{2(s-1)}|\nabla v|^2 + 2\int_\Omega |x|^{-2\nu_{i_0}}|\nabla\eta|^2 v^2 v_n^{2(s-1)}. \tag{2.3}$$

Multiplying (2.1) by ϕ and from (2.1)–(2.3), we have

$$\begin{aligned}\int_\Omega |x|^{-2\nu_{i_0}}\eta^2|\nabla v|^2 v_n^{2(s-1)} &+ 4(s-1)\int_\Omega |x|^{-2\nu_{i_0}}\eta^2|\nabla v_n|^2 v_n^{2(s-1)}\\ &\le C\int_\Omega |x|^{-2\nu_{i_0}}|\nabla\eta|^2 v^2 v_n^{2(s-1)} + C\int_\Omega \frac{\eta^2|v|^{p_{i_0}}v_n^{2(s-1)}}{|x|^{p_{i_0}\nu_{i_0}+t_{i_0}}}\\ &\quad + C\sum_{1\le i\le k,\ i\ne i_0}\Big(\int_\Omega \frac{\eta^2 v^2 v_n^{2(s-1)}}{|x|^{2\nu_{i_0}}|x-\xi_i|^2} + \int_\Omega \frac{\eta^2|v|^{p_i}v_n^{2(s-1)}}{|x|^{p_i\nu_{i_0}}|x-\xi_i|^{t_i}}\Big)\\ &\le C\int_\Omega |x|^{-2\nu_{i_0}}|\nabla\eta|^2 v^2 v_n^{2(s-1)} + C\int_\Omega \frac{\eta^2|v|^{p_{i_0}}v_n^{2(s-1)}}{|x|^{p_{i_0}\nu_{i_0}+t_{i_0}}}\\ &\quad + C\sum_{1\le i\le k,\ i\ne i_0}\Big(\int_\Omega \frac{\eta^2 v^2 v_n^{2(s-1)}}{|x|^{2\nu_{i_0}}} + \int_\Omega \frac{\eta^2|v|^{p_i}v_n^{2(s-1)}}{|x|^{p_i\nu_{i_0}+t_i}}\Big).\end{aligned} \tag{2.4}$$

From (1.2) it follows that

$$\Big(\int_\Omega \frac{|\eta v v_n^{s-1}|^{\tilde p}}{|x|^{(b+\nu_{i_0})\tilde p}}\Big)^{\frac{2}{\tilde p}} \le C\int_\Omega |x|^{-2(a+\nu_{i_0})}|\nabla(\eta v v_n^{s-1})|^2. \tag{2.5}$$

Take $a=0$, $b=\frac{t_i}{p_i}$ and $\tilde p = p_i$ in (2.5). Then we have

$$\Big(\int_\Omega \frac{|\eta v v_n^{s-1}|^{p_i}}{|x|^{p_i\nu_{i_0}+t_i}}\Big)^{\frac{2}{p_i}} \le C\int_\Omega |x|^{-2\nu_{i_0}}|\nabla(\eta v v_n^{s-1})|^2, \quad 1\le i\le k. \tag{2.6}$$

From (2.4)–(2.6) it follows that

$$\begin{aligned}\sum_{i=1}^k\Big(\int_\Omega \frac{|\eta v v_n^{s-1}|^{p_i}}{|x|^{p_i\nu_{i_0}+t_i}}\Big)^{\frac{2}{p_i}} &\le C\int_\Omega |x|^{-2\nu_{i_0}}|\nabla(\eta v v_n^{s-1})|^2\\ &\le Cs\int_\Omega |x|^{-2\nu_{i_0}}|\nabla\eta|^2 v^2 v_n^{2(s-1)} + Cs\int_\Omega \frac{|\eta v v_n^{s-1}|^2}{|x|^{2\nu_{i_0}}}\\ &\quad + Cs\sum_{i=1}^k\int_\Omega \frac{\eta^2|v|^{p_i}v_n^{2(s-1)}}{|x|^{p_i\nu_{i_0}+t_i}}.\end{aligned} \tag{2.7}$$

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For $i = 1, 2, \dots, k$, by the Hölder inequality we have

$$\int_\Omega \frac{\eta^2 |v|^{p_i} v_n^{2(s-1)}}{|x|^{p_i \nu_{i_0} + t_i}} \le \Big(\int_\Omega \frac{|\eta v v_n^{s-1}|^{p_i}}{|x|^{p_i \nu_{i_0} + t_i}} \Big)^{\frac{2}{p_i}} \Big(\int_{B_R(0)} \frac{|v|^{p_i}}{|x|^{p_i \nu_{i_0} + t_i}} \Big)^{\frac{p_i - 2}{p_i}}, \tag{2.8}$$

$$\int_\Omega \frac{|\eta v v_n^{s-1}|^2}{|x|^{2\nu_{i_0}}} \le \Big(\int_\Omega \frac{|\eta v v_n^{s-1}|^{p_i}}{|x|^{p_i \nu_{i_0} + t_i}} \Big)^{\frac{2}{p_i}} \Big(\int_{B_R(0)} |x|^{\frac{2t_i}{p_i - 2}} \Big)^{\frac{p_i - 2}{p_i}}. \tag{2.9}$$

Taking R small and $1 < s \le \max\{\frac{p_i}{2},\ i = 1, 2, \dots, k\}$, by (2.4)–(2.9) we have

$$\sum_{i=1}^k \Big(\int_\Omega \frac{|\eta v v_n^{s-1}|^{p_i}}{|x|^{p_i \nu_{i_0} + t_i}} \Big)^{\frac{2}{p_i}} \le C \int_\Omega |x|^{-2\nu_{i_0}} |\nabla \eta|^2 v^2 v_n^{2(s-1)},$$

which implies that

$$\Big(\int_\Omega \frac{|\eta v v_n^{s-1}|^{p_i}}{|x|^{p_i \nu_{i_0} + t_i}} \Big)^{\frac{2}{p_i}} \le C \int_\Omega |x|^{-2\nu_{i_0}} |\nabla \eta|^2 v^2 v_n^{2(s-1)}, \quad 1 \le i \le k. \tag{2.10}$$

Choose $s = \frac{p_i}{2}$ in (2.10). Then

$$\Big(\int_\Omega \frac{|\eta v v_n^{\frac{p_i}{2} - 1}|^{p_i}}{|x|^{p_i \nu_{i_0} + t_i}} \Big)^{\frac{2}{p_i}} \le C \int_\Omega \frac{|v|^{p_i}}{|x|^{p_i \nu_{i_0} + t_i}} |x|^{(p_i - 2)\nu_{i_0} + t_i} |\nabla \eta|^2, \quad 1 \le i \le k.$$

Note that $(p_i - 2)\nu_{i_0} + t_i > 0$. We can choose $\eta \in C_0^\infty(B_R(0))$ such that

$$|x|^{(p_i - 2)\nu_{i_0} + t_i} |\nabla \eta|^2 \le C, \quad \forall x \in B_R(0).$$

Consequently,

$$\Big(\int_\Omega \frac{|\eta v v_n^{\frac{p_i}{2} - 1}|^{p_i}}{|x|^{p_i \nu_{i_0} + t_i}} \Big)^{\frac{2}{p_i}} \le C \int_\Omega \frac{|v|^{p_i}}{|x|^{p_i \nu_{i_0} + t_i}} \le C \int_\Omega |x|^{-2\nu_{i_0}} |\nabla v|^2 \le C.$$

Taking $n \to \infty$, we deduce that

$$v \in L^{\frac{p_i^2}{2}}(B_R(0), |x|^{-(p_i \nu_{i_0} + t_i)}), \quad i = 1, 2, \dots, k. \tag{2.11}$$

Now let η be a cut-off function in $B_{R+r}(0) \subset \Omega$, such that $0 \le \eta \le 1$, $|\nabla \eta| \le 4/r$ in $B_{R+r}(0)$ and $\eta = 1$ in $B_R(0)$. For $i = 1, 2, \dots, k$, set

$$\delta_i := \frac{p_i^2}{2(p_i - 2)}, \quad \bar{p}_i := \frac{2\delta_i}{\delta_i - 1}, \quad \tau_i := 2\nu_{i_0}\delta_i - (p_i \nu_{i_0} + t_i)(\delta_i - 1).$$

Direct calculation shows $\bar{p}_i < p_i$, $\tau_i < N$. From the Hölder inequality it follows that

$$\begin{aligned} \int_\Omega |x|^{-2\nu_{i_0}} |\nabla \eta|^2 v^2 v_n^{2(s-1)} & \\ &\le C r^{-2} \Big(\int_{B_{R+r}(0)} \frac{1}{|x|^{\tau_i}} \Big)^{\frac{1}{\delta_i}} \Big(\int_{B_{R+r}(0)} \frac{|v v_n^{s-1}|^{\frac{2\delta_i}{\delta_i - 1}}}{|x|^{p_i \nu_{i_0} + t_i}} \Big)^{1 - \frac{1}{\delta_i}} \\ &\le C r^{-2} \Big(\int_{B_{R+r}(0)} \frac{|v v_n^{s-1}|^{\frac{2\delta_i}{\delta_i - 1}}}{|x|^{p_i \nu_{i_0} + t_i}} \Big)^{\frac{\delta_i - 1}{\delta_i}}, \end{aligned} \tag{2.12}$$

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$$\int_\Omega \frac{\eta^2|v|^{p_i}v_n^{2(s-1)}}{|x|^{p_i\nu_{i_0}+t_i}} \le \Big(\int_{B_{R+r}(0)} \frac{|v|^{(p_i-2)\delta_i}}{|x|^{p_i\nu_{i_0}+t_i}}\Big)^{\frac{1}{\delta_i}} \Big(\int_{B_{R+r}(0)} \frac{|vv_n^{s-1}|^{\frac{2\delta_i}{\delta_i-1}}}{|x|^{p_i\nu_{i_0}+t_i}}\Big)^{\frac{\delta_i-1}{\delta_i}}, \tag{2.13}$$

$$\int_\Omega \frac{|\eta v v_n^{s-1}|^2}{|x|^{2\nu_{i_0}}} \le \Big(\int_{B_{R+r}(0)} \frac{1}{|x|^{\tau_i}}\Big)^{\frac{1}{\delta_i}} \Big(\int_{B_{R+r}(0)} \frac{|vv_n^{s-1}|^{\frac{2\delta_i}{\delta_i-1}}}{|x|^{p_i\nu_{i_0}+t_i}}\Big)^{\frac{\delta_i-1}{\delta_i}}. \tag{2.14}$$

Take $R+r$ small and $n\to\infty$. From (2.7) and (2.11)–(2.14) it follows that

$$\Big(\sum_{i=1}^k\Big(\int_{B_R(0)} \frac{|v|^{p_is}}{|x|^{p_i\nu_{i_0}+t_i}}\Big)^{\frac{2}{p_i}}\Big)^{\frac{1}{2s}} \le C^{\frac{1}{2s}}s^{\frac{1}{2s}}r^{-\frac{1}{s}}\Big(\sum_{i=1}^k\Big(\int_{B_{R+r}(0)} \frac{|v|^{\bar p_is}}{|x|^{p_i\nu_{i_0}+t_i}}\Big)^{\frac{2}{p_i}}\Big)^{\frac{1}{2s}}. \tag{2.15}$$

Without loss of generality we may assume $p_1 = \min\{p_i | i = 1, 2, \dots, k\}$. Choose $s=\chi^j$, $\chi = p_1/\bar p_1$, and $r=\rho^j$, $j\ge 1$. Note that for all $i=1,2,\dots,k$,

$$\chi>1,\quad \bar p_1\chi^j = p_1\chi^{j-1},\quad \bar p_i\chi\le p_i,\quad \bar p_i\chi^j\le p_i\chi^{j-1},\quad p_i\nu_{i_0}+t_2<N.$$

Take R,ρ small enough such that $R+\rho<1$ and the measure $|B_{R+\rho}(0)|<1$. For any $r=\rho^j$, $j\ge1$, by the Hölder inequality we deduce that

$$\begin{aligned}\Big(\int_{B_{R+r}(0)} \frac{|v|^{\bar p_i\chi^j}}{|x|^{p_i\nu_{i_0}+t_i}}\Big)^{\frac{1}{\bar p_i\chi^j}} &\le \Big(\int_{B_{R+r}(0)} \frac{|v|^{p_i\chi^{j-1}}}{|x|^{p_i\nu_{i_0}+t_i}}\Big)^{\frac{1}{p_i\chi^{j-1}}}\Big(\int_{B_{R+r}(0)} \frac{1}{|x|^{p_i\nu_{i_0}+t_i}}\Big)^{\frac{1-\frac{\bar p_i\chi}{p_i}}{\bar p_i\chi^j}}\\ &\le \Big(\int_{B_{R+r}(0)} \frac{|v|^{p_i\chi^{j-1}}}{|x|^{p_i\nu_{i_0}+t_i}}\Big)^{\frac{1}{p_i\chi^{j-1}}}.\end{aligned} \tag{2.16}$$

Note that the following elementary inequality holds:

$$a^\tau+b^\tau\le(a+b)^\tau,\quad \forall\, a,b>0,\tau\ge1. \tag{2.17}$$

From (2.16) and (2.17) it follows that

$$\begin{aligned}\Big(\sum_{i=1}^k\Big(\int_{B_{R+r}(0)} \frac{|v|^{\bar p_i\chi^j}}{|x|^{p_i\nu_{i_0}+t_i}}\Big)^{\frac{2}{p_i}}\Big)^{\frac{1}{2\chi^j}} &\le \Big(\sum_{i=1}^k\Big(\int_{B_{R+r}(0)} \frac{|v|^{p_i\chi^{j-1}}}{|x|^{p_i\nu_{i_0}+t_i}}\Big)^{\frac{2\chi}{p_i}}\Big)^{\frac{1}{2\chi^j}}\\ &\le \Big(\sum_{i=1}^k\Big(\int_{B_{R+r}(0)} \frac{|v|^{p_i\chi^{j-1}}}{|x|^{p_i\nu_{i_0}+t_i}}\Big)^{\frac{2}{p_i}}\Big)^{\frac{1}{2\chi^{j-1}}}.\end{aligned} \tag{2.18}$$

Employing (2.15) and (2.18) recursively, we have

$$\begin{aligned}\Big(\int_{B_R(0)} |v|^{p_1\chi^j}\Big)^{\frac{1}{p_1\chi^j}} &\le R^{\frac{p_1\nu_{i_0}+t_1}{p_1\chi^j}}\Big(\int_{B_R(0)} \frac{|v|^{p_1\chi^j}}{|x|^{p_1\nu_{i_0}+t_1}}\Big)^{\frac{1}{p_1\chi^j}}\\ &\le R^{\frac{p_1\nu_{i_0}+t_1}{p_1\chi^j}}\Big(\sum_{i=1}^{k}\Big(\int_{B_{R+\rho^j}(0)} \frac{|v|^{p_i\chi^j}}{|x|^{p_i\nu_{i_0}+t_i}}\Big)^{\frac{2}{p_i}}\Big)^{\frac{1}{2\chi^j}}\\ &\le R^{\frac{p_1\nu_{i_0}+t_1}{p_1\chi^j}} C^{\sum_{i=1}^{j}\frac{1}{2\chi^i}}\chi^{\sum_{i=1}^{j}\frac{i}{2\chi^i}}\rho^{-\sum_{i=1}^{j}\frac{i}{\chi^i}}\\ &\quad\times\Big(\sum_{i=1}^{k}\Big(\int_{B_{R+\rho}(0)} \frac{|v|^{p_i}}{|x|^{p_i\nu_{i_0}+t_i}}\Big)^{\frac{2}{p_i}}\Big)^{\frac{1}{2}}.\end{aligned}$$

Since $\chi > 1$, we have $\chi^j \to \infty$ as $j \to \infty$. Furthermore, the infinite sums in the right-hand side of the last inequality all converge. Taking $j \to \infty$, we conclude that $v \in L^\infty(B_R(0))$. ∎

3 Proofs of Theorems 1.2 and 1.3

We first establish the following strong maximum principle.

Lemma 3.1 ([12]) *Suppose that $\tau > 2 - N$, $u \in C^2(\Omega \setminus \{0\})$, $u > 0$ in $\Omega \setminus \{0\}$, and*

$$-\operatorname{div}(|x|^\tau \nabla u) \ge 0.$$

Then for any $\rho > 0$ such that $B_\rho(0) \subset \Omega$ we have

$$u(x) \ge \min_{|x|=\rho} u(x), \quad \forall\, x \in B_\rho(0) \setminus \{0\}.$$

Proof of Theorem 1.2 If $u \in H_0^1(\Omega)$ is a solution of the problem (1.1), a standard elliptic regularity argument shows that

$$u \in C^2(\Omega \setminus \{\xi_1, \xi_2, \dots, \xi_k\}) \cap C^1(\bar\Omega \setminus \{\xi_1, \xi_2, \dots, \xi_k\}). \tag{3.1}$$

Suppose that $0 < \mu_{i_0} < \bar\mu$ for some $i_0 \in \{1, 2, \dots, k\}$ and $u \in H_0^1(\Omega)$ is a positive solution of (1.1). Define

$$v(x) = |x - \xi_{i_0}|^{\nu_{i_0}} u(x), \quad \nu_{i_0} = \sqrt{\bar\mu} - \sqrt{\bar\mu - \mu_{i_0}}.$$

Then $v \in H_0^1(\Omega, |x - \xi_{i_0}|^{-2\nu_{i_0}})$. For all $x \in \Omega \setminus \{\xi_i,\ i = 1, 2, \dots, k\}$, direct calculation shows that v satisfies

$$\begin{aligned}-\operatorname{div}(|x - \xi_{i_0}|^{-2\nu_{i_0}} \nabla v) = &\sum_{i=1}^{k} \frac{v^{p_i - 1}}{|x - \xi_{i_0}|^{p_i\nu_{i_0}} |x - \xi_i|^{t_i}}\\ &+ \Big(\sum_{1\le i\le k,\ i\ne i_0} \frac{\mu_i}{|x - \xi_i|^2}\Big) \frac{v}{|x - \xi_{i_0}|^{2\nu_{i_0}}}.\end{aligned} \tag{3.2}$$

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Choose $\rho > 0$ small enough such that

$$B_{2\rho}(\xi_{i_0}) \subset \Omega \setminus \{\xi_i \mid 1 \le i \le k, i \ne i_0\}.$$

(i) Assume $\mu_i = 0,\ 1 \le i \le k,\ i \ne i_0$. By (3.2) we have that

$$-\operatorname{div}(|x-\xi_{i_0}|^{-2\nu_{i_0}}\nabla v) > 0, \quad \forall x \in B_{2\rho}(\xi_{i_0}) \setminus \{\xi_{i_0}\}. \tag{3.3}$$

(ii) Assume that

$$\sum_{1\le i\le k,\ i\ne i_0} \frac{\mu_i}{|\xi_i - \xi_{i_0}|^2} > 0.$$

By the continuity argument, we can choose $\rho > 0$ small enough such that

$$\sum_{1\le i\le k,\ i\ne i_0} \frac{\mu_i}{|x - \xi_i|^2} > 0, \quad \forall x \in B_{2\rho}(\xi_{i_0}).$$

By (3.2) we also have

$$-\operatorname{div}(|x-\xi_{i_0}|^{-2\nu_{i_0}}\nabla v) > 0, \quad \forall x \in B_{2\rho}(\xi_{i_0}) \setminus \{\xi_{i_0}\}. \tag{3.4}$$

Note that $-2\nu_{i_0} > 2 - N$. From (3.1)–(3.4) and Lemma 3.1 it follows that

$$v(x) \ge C_2 := \min_{|x-\xi_{i_0}|=\rho} v(x), \quad \forall x \in B_\rho(\xi_{i_0}) \setminus \{\xi_{i_0}\} > 0.$$

Consequently, $u(x) \ge C_2\,|x-\xi_{i_0}|^{-\nu_{i_0}}$, $\forall x \in B_\rho(\xi_{i_0}) \setminus \{\xi_{i_0}\}$. ■

Proof of Theorem 1.3 Suppose that $\mu_{i_0} \le 0$ and $u \in H_0^1(\Omega)$ is a solution of (1.1). For any $\rho > 0$ small enough that $B_\rho(\xi_1) \in \Omega$, let $\eta \in C_0^\infty(B_\rho(\xi_1))$ be a cut-off function and set $\phi = \eta^2 u u_n^{2(s-1)}$, where $s,\ n > 1$, and $u_n = \min\{|u|,\ n\}$. Since $\xi_i \ne \xi_{i_0}$, for $i \ne i_0$, we can choose $R > 0$ small enough such that

$$R < \frac{1}{2}\min\{|\xi_i - \xi_{i_0}|, 1 \le i \le k, i \ne i_0\}.$$

Note that $|x-\xi_i| = O_1(1) > R > |x - \xi_{i_0}|$, $\forall x \in B_R(\xi_{i_0}), i \ne i_0$. Multiplying (1.1) by ϕ and arguing as in (2.2) and (2.3), we have

$$\begin{aligned}
&\int_\Omega \eta^2|\nabla u|^2u_n^{2(s-1)} + 4(s-1)\int_\Omega \eta^2|\nabla u_n|^2u_n^{2(s-1)} \\
&\qquad \le \int_\Omega \eta^2|\nabla u|^2u_n^{2(s-1)} - \mu_{i_0}\int_\Omega \frac{\eta^2u^2u_n^{2(s-1)}}{|x-\xi_{i_0}|^2} \\
&\qquad\qquad + 4(s-1)\int_\Omega \eta^2|\nabla u_n|^2u_n^{2(s-1)} \\
&\qquad \le C\int_\Omega |\nabla\eta|^2u^2u_n^{2(s-1)} + C\sum_{i=1}^k\int_\Omega \frac{\eta^2|u|^{p_i}u_n^{2(s-1)}}{|x-\xi_i|^{t_i}} \\
&\qquad\qquad + C\sum_{1\le i\le k,\ i\ne i_0}\mu_i\int_\Omega \frac{\eta^2u^2u_n^{2(s-1)}}{|x-\xi_i|^2} \\
&\qquad \le C\int_\Omega (|\nabla\eta|^2+\eta^2)u^2u_n^{2(s-1)} + C\sum_{i=1}^k\int_\Omega \frac{\eta^2|u|^{p_i}u_n^{2(s-1)}}{|x-\xi_{i_0}|^{t_i}}.
\end{aligned} \tag{3.5}$$

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By (1.6) and (3.5) we get

$$(3.6)\quad \sum_{i=1}^{k}\Big(\int_{\Omega}\frac{|\eta u u_n^{s-1}|^{p_i}}{|x-\xi_{i_0}|^{t_i}}\Big)^{\frac{2}{p_i}} \leq Cs\int_{\Omega}(|\nabla\eta|^2+\eta^2)u^2u_n^{2(s-1)}+Cs\sum_{i=1}^{k}\int_{\Omega}\frac{\eta^2|u|^{p_i}u_n^{2(s-1)}}{|x-\xi_{i_0}|^{t_i}}.$$

From (3.6) and arguing as in the proof of Theorem 1.1 we finally have that

$$u\in L^{\frac{p_i^2}{2}}\left(B_R(\xi_{i_0}),|x-\xi_{i_0}|^{-t_i}\right),\quad i=1,2,\dots,k,$$

and furthermore, $u\in L^{\infty}(B_\rho(\xi_{i_0}))$. The details are omitted for simplicity. ■

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School of Mathematics and Statistics, South-Central University for Nationalities, Wuhan 430074, P.R. China
e-mail: dongshengkang@yahoo.com.cn