13
Currents and Varifolds

The classical reference for the overall theory of the currents in the Euclidean spaces is [203]. Currents and varifolds are very well presented also in [282], [297] and [397], and more informally in [351]. De Lellis's survey [161] enlightens the background and recent developments for this and the next two chapters.

13.1 Currents in Euclidean Spaces

Federer and Fleming begin their ground-breaking paper [205] with the following quotation:

> 'Long has been the search for a satisfactory analytic and topological formulation of the concept "*k* dimensional domain of integration in euclidean *n*-space." Such a notion must partake of the smoothness of differentiable manifolds and of the combinatorial structure of polyhedral chains with integer coefficients. In order to be useful for the calculus of variations, the class of all domains must have certain compactness properties. All these requirements are met by the integral currents studied in this paper.'

So the currents they introduced are generalized surfaces which, as they expected, have turned out to be extremely useful for the calculus of variations, and in many other topics too. De Giorgi's theory of sets of finite perimeter already gave such a setting for codimension one surfaces. Currents can be of any dimension and they have many other advantages over sets of finite perimeter, but also some disadvantages.

Analytic theory of currents was developed by De Rham in the 1950s. They are just distributions over differential forms. Federer and Fleming introduced geometric aspects. The idea of how they are related to smooth surfaces is simple. If M is a smooth m-dimensional submanifold of $\mathbb{R}^n$, then one can integrate

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differential m-forms ω over it and define the linear functional $[M]$:

$$[M](\omega) = \int_M \omega.$$

By the Stokes theorem, $\int_{\partial M} \omega = \int_M d\omega$, where $d\omega$ is the exterior derivative of ω. Thus if we define the boundary ∂T of a general current T by

$$\partial T(\omega) = T(d\omega),$$

then $\partial[M] = [\partial M]$.

The integral $\int_M \omega$ can be written as $\int_M \langle \omega(x), \xi(x) \rangle \, d\mathcal{H}^m x$, where $\xi(x)$ is an m-vector associated with the tangent plane of M at x; differential m-forms can be defined as functions with values in the dual of the m-vectors. This means that $\xi(x)$ is of the form $v_1 \wedge \cdots \wedge v_m$, where $\{v_1, \ldots, v_m\}$ is an orthonormal basis of the tangent m-plane of M at x.

Let $\mathcal{D}^m(\mathbb{R}^n)$ be the space of differential m-forms on $\mathbb{R}^n$ with compact support. They can be written as $\omega = \sum_\alpha \omega_\alpha dx^\alpha$, where α runs through the sequences $a(1) < \cdots < \alpha(m), \alpha(i) \in \{1, \ldots, n\}$, the ω_α are smooth functions and $dx^\alpha = dx^{\alpha(1)} \wedge \cdots \wedge dx^{\alpha(m)}$.

By definition, an *m-current* $T, T \in \mathcal{D}_m(\mathbb{R}^n)$ is a continuous linear functional on $\mathcal{D}^m(\mathbb{R}^n)$. The *support* $\operatorname{spt} T$ of a current T is the smallest closed set such that $T(\omega) = 0$ for every $\omega \in \mathcal{D}^m(\mathbb{R}^n)$ for which $\operatorname{spt} \omega \subset \mathbb{R}^n \setminus \operatorname{spt} T$. The *mass*, generalizing area, is

$$M(T) = \sup \Big\{ |T(\omega)| \colon \|\omega\|_\infty \le 1, \omega \in \mathcal{D}^m(\mathbb{R}^n) \Big\}.$$

Differential forms can be pulled back, so currents can be pushed forward by maps (with proper conditions) f; $f_\# T(\omega) = T(f^\# \omega)$. We have the easy weak compactness theorem: if $\sup_j M(T_j) < \infty$, $j = 1, 2, \ldots$, then there is a subsequence (T_{j_i}) and a current T such that $T_{j_i}(\omega) \to T(\omega)$ for all $\omega \in \mathcal{D}^m(\mathbb{R}^n)$.

A consequence of the Riesz representation theorem is that if $M(T) < \infty$, then there is a Radon measure μ_T and an m-vector-valued Borel function $\overrightarrow{T}$, the tangent vector field of T, such that

$$T(\omega) = \int \Big\langle \omega(x), \overrightarrow{T}(x) \Big\rangle \, d\mu_T x \text{ for } \omega \in \mathcal{D}^m(\mathbb{R}^n). \tag{13.1}$$

If $B \subset \mathbb{R}^n$ is a Borel set, then for T as above we define the restriction of T to B, $T \llcorner B$, by integrating only over B, that is, $\mu_{T \llcorner B} = \mu_T \llcorner B$. Similarly one defines $T \llcorner f$ for functions f.

The currents with $N(T) := M(T) + M(\partial T) < \infty$ are called *normal*. Although our main interest is in m-currents with $m < n$, the n-currents in $\mathbb{R}^n$ too are interesting. In particular, the normal n-currents in $\mathbb{R}^n$ can be identified with BV-functions:

Theorem 13.1 *Let $T \in \mathcal{D}_n(\mathbb{R}^n)$ be normal. Then there is $g \in BV(\mathbb{R}^n)$ such that*

$$T(f\,dx^1 \wedge \cdots \wedge dx^n) = \int f g \, d\mathcal{L}^n \text{ for } f \in C_c^\infty(\mathbb{R}^n).$$

The proof can be done approximating T by usual convolutions, cf. [297, 7.1.9]. The converse also is true.

Based on this connection, Federer presented almost the whole theory of BV-functions in [203, Theorem 4.5.9] and its 31 statements.

Let P_α be the projection $P_\alpha(x_1, \ldots, x_n) = (x_{\alpha(1)}, \ldots, x_{\alpha(m)})$. Then one checks for $T \in \mathcal{D}_m(\mathbb{R}^n)$ and $\omega = \sum_\alpha \omega_\alpha dx^\alpha \in \mathcal{D}^m(\mathbb{R}^n)$ that

$$T(\omega) = \sum_\alpha P_{\alpha\#}\left(T \llcorner \omega_\alpha\right)(dy^1 \wedge \cdots \wedge dy^m),$$

where $dy^1, \ldots, dy^m$ are the coordinate 1-forms on $\mathbb{R}^m$. With some extra work this yields

Lemma 13.2 *If $T \in \mathcal{D}_m(\mathbb{R}^n), N(T) < \infty$ and B is a Borel set satisfying $\mathcal{L}^m(P_\alpha(B)) = 0$ for all α, then $T \llcorner B = 0$. In particular this holds if $\mathcal{H}^m(B) < \infty$ and B is purely m-unrectifiable.*

The second statement follows from the Besicovitch–Federer projection Theorem 4.17, since that allows us to choose the appropriate coordinate axis.

Of the rich theory of currents I now only discuss rectifiable currents.

Definition 13.3 An m-current T in $\mathbb{R}^n$ with finite mass is called *m-rectifiable* if there are an m-rectifiable $\mathcal{H}^m$ measurable set $E \subset \mathbb{R}^n$ and an $\mathcal{H}^m$ measurable positive function θ on E with $\int_E \theta \, d\mathcal{H}^m < \infty$ such that the values of $\overrightarrow{T}$ are simple m-vectors associated with the approximate tangent planes of E and we have

$$T(\omega) = \int_E \left\langle \omega(x), \overrightarrow{T}(x) \right\rangle \theta(x) \, d\mathcal{H}^m x \text{ for } \omega \in \mathcal{D}^m(\mathbb{R}^n). \tag{13.2}$$

If in addition the values of θ are integers, T is called *integer multiplicity m-rectifiable current*. We say that T is an *integral current* if both T and ∂T are integer multiplicity rectifiable currents.

We denote the set of m-rectifiable currents in $\mathbb{R}^n$ by $R_m(\mathbb{R}^n)$, the set of integer multiplicity m-rectifiable currents in $\mathbb{R}^n$ by $\mathcal{R}_m(\mathbb{R}^n)$, and the set of integral m-currents in $\mathbb{R}^n$ by $\mathcal{I}_m(\mathbb{R}^n)$.

Again the condition on $\overrightarrow{T}$ means that for $\mathcal{H}^m$ almost all $x \in E$, the m-vector $\overrightarrow{T}(x)$ is of the form $v_1 \wedge \cdots \wedge v_m$, where $\{v_1, \ldots, v_m\}$ is an orthonormal basis of the approximate tangent m-plane of E at x. Then $\overrightarrow{T}(x)$ is uniquely determined up to sign. Choosing the sign means orienting E and T.

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The terminology and notation differ across books and papers.

Here are some of the main tools to study rectifiable currents:

The *deformation theorem*: if T is an integral current, then $T = P + Q + \partial S$, where P is a polyhedral chain, and Q and S are integral currents with small masses. The proof consists of carefully projecting T into the skeletons of a cubical decomposition of $\mathbb{R}^n$. It gives a useful approximation of integral currents by polyhedral chains as well as the *isoperimetric theorem*: if $T \in \mathcal{I}_{m-1}(\mathbb{R}^n)$ and $\partial T = 0$, then there is $S \in \mathcal{I}_m(\mathbb{R}^n)$ such that $\partial S = T$ and $M(S) \lesssim M(T)^{m/(m-1)}$.

Slicing is a very useful operation on currents. Let $T \in \mathcal{D}_m(\mathbb{R}^n)$ with $N(T) < \infty$ and $f\colon \mathbb{R}^n \to \mathbb{R}^k, k \leq n$ be a Lipschitz map. Then the slice of T at $t \in \mathbb{R}^k$, $\langle T, f, t\rangle$, is an $(n-k)$-current with support in $\operatorname{spt} T \cap f^{-1}\{t\}$ such that T is obtained as an integral of the $\langle T, f, t\rangle$. For simplicity we only consider slicing with real-valued Lipschitz maps $f\colon \mathbb{R}^n \to \mathbb{R}$. Then we can define

$$\langle T, f, t\rangle = (\partial T) \llcorner \{x\colon f(x) > t\} - \partial(T \llcorner \{x\colon f(x) > t\}).$$

For almost all $t \in \mathbb{R}$,

$$\operatorname{spt}\langle T, f, t\rangle \subset \operatorname{spt} T \cap f^{-1}\{t\},\ \partial\langle T, f, t\rangle = -\langle \partial T, f, t\rangle,\ N(\langle T, f, t\rangle) < \infty \text{ and}$$

$$\langle T, f, t\rangle \in \mathcal{R}_{m-1}(\mathbb{R}^n) \text{ if } T \in \mathcal{R}_m(\mathbb{R}^n).$$

The first line is rather easy to prove, and the second can be proven with the help of Theorem 4.3, see [397, Section 28].

Here is a general, not very hard, rectifiability theorem, see [397, Theorem 32.1]. Recall similar results in 12.14 and 12.17.

Theorem 13.4 *If $T \in \mathcal{D}_m(\mathbb{R}^n)$ is normal and $\Theta^{*m}(\mu_T, x) > 0$ for μ_T almost all $x \in \mathbb{R}^n$, then $T \in R_m(\mathbb{R}^n)$.*

We will say a few words about the proof. We should establish (13.2) starting with (13.1). By Theorem 1.3 $\Theta^{*m}(\mu_T, x) < \infty$ for $\mathcal{H}^m$ almost all $x \in \mathbb{R}^n$, and so also for μ_T almost all $x \in \mathbb{R}^n$ by Lemma 13.2. The set

$$E = \left\{x\colon \Theta^{*m}(\mu_T, x) > 0\right\}$$

has σ-finite $\mathcal{H}^m$ measure, and μ_T and $\mathcal{H}^m \llcorner E$ are mutually absolutely continuous. Hence $\mu_T = \theta\mathcal{H}^m \llcorner E$ for some θ. By Lemma 13.2, $\mu_T(B) = 0$ for every purely m-rectifiable set $B \subset E$, whence E is m-rectifiable. That $\overrightarrow{T}$ is associated with the approximate tangent planes of E requires more work. It can be established by a blow-up method, see [397, Theorem 32.1].

We shall discuss mass minimizing rectifiable currents in Chapter 15. For their existence we need the compactness theorem:

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Theorem 13.5 *If $T_j \in \mathcal{I}_m(\mathbb{R}^n)$, $j = 1, 2, \ldots$, with $\sup_j N(T_j) < \infty$, then there is a subsequence (T_{j_i}) and a current $T \in \mathcal{I}_m(\mathbb{R}^n)$ such that $T_{j_i}(\omega) \to T(\omega)$ for all $\omega \in \mathcal{D}^m(\mathbb{R}^n)$.*

The main point here is that the limit current is rectifiable. By the lower semicontinuity of the mass, $N(T) < \infty$. To apply Theorem 13.4, we need to know that the upper density of μ_T is positive, which roughly means that T should not be scattered around a set of non-σ-finite $\mathcal{H}^m$ measure. Another thing that needs checking is that the weight function of T is integer valued. The proof of the theorem is by induction on m, so we also ought to know to which $(m-1)$-dimensional currents we should apply the induction hypothesis. All this is dealt with in the following slicing criterion for rectifiability:

Lemma 13.6 *If $T \in \mathcal{D}_m(\mathbb{R}^n)$ is normal, $\partial T = 0$ and we have $\partial(T \llcorner B(a,r)) \in \mathcal{R}_{m-1}(\mathbb{R}^n)$ for every $a \in \mathbb{R}^n$ and for almost all $r \in (0,\infty)$, then $T \in \mathcal{R}_m(\mathbb{R}^n)$.*

To prove this, first the isoperimetric theorem with some covering arguments is used to prove that $\Theta^{*m}(\mu_T, x) > 0$ for μ_T almost all $x \in \mathbb{R}^n$. Then by Theorem 13.4 we know that $T \in R_m(\mathbb{R}^n)$. That the weight function of T is integer valued follows, for example, by a rather simple blow-up argument. Now we have reduced the problem one dimension lower and the compactness theorem follows by an induction argument. As an easy consequence it has the interesting boundary rectifiability theorem:

Theorem 13.7 *If $T \in \mathcal{R}_m(\mathbb{R}^n)$ with $M(\partial T) < \infty$, then $\partial T \in \mathcal{R}_{m-1}(\mathbb{R}^n)$.*

The above was essentially a very rough sketch of the original proof of Federer and Fleming in [205] (and also in [203]). Later other proofs were given which avoided the use of the Besicovitch–Federer projection Theorem 4.17, first by Solomon [404] using multivalued functions, then by White [436] by more classical analysis, and later by Ambrosio and Kirchheim in [17] in the metric space setting relying on BV-functions, as we shall see in the next section.

Above the coefficients of rectifiable currents have been real numbers or integers. Other coefficient groups also have been studied. In particular for integers modulo p, where $p \geq 2$ is an integer, the same rectifiability and compactness theorems are valid. White characterized in [439] the normed coefficient groups for which they hold. For this, he proved a rectifiability criterion with zero-dimensional slices. A similar criterion in the metric space setting was proved independently by Ambrosio and Kirchheim, which we shall discuss in the next section.

The currents modulo p exhibit many interesting new phenomena. They are extensively discussed in [161]. Simple illustrative examples are presented in [351, Section 11.1].

Instead of using mass to represent the area one can use size: for a rectifiable current T as in (13.2), $\mathrm{Size}(T) = \mathcal{H}^m(\{x \in E\colon \theta(x) \neq 0\})$. In some cases this is better than mass, but the existence of minimizers is harder to prove, see [139].

13.2 Currents in Metric Spaces

Based on an idea of De Giorgi, Ambrosio and Kirchheim [17] developed the theory of currents in complete metric spaces. It might seem that currents, as linear forms on differential forms, would need a differential structure. But a differential form ω can be written as a linear combination of $f_0 df_1 \wedge \cdots \wedge df_m$ with smooth functions f_i. For the geometric theory of currents we could also consider Lipschitz functions, and this would make sense in metric spaces.

Let X be a complete metric space and let $\mathcal{D}^m(X)$ be the set of all $(m+1)$-tuples $(g, \pi_1, \dots, \pi_m)$ of real-valued Lipschitz functions on X with g bounded. Then Ambrosio and Kirchheim defined an m-dimensional current on X to be any multilinear, positively homogeneous and continuous (in a suitable weak sense) functional T on $\mathcal{D}^m(X)$ such that $T(g, \pi_1, \dots, \pi_m) = 0$ whenever some π_i is constant in a neighbourhood of $\{g \neq 0\}$. In this simple setting, they were able to develop and generalize the Euclidean theory to a surprising extent. In particular they proved compactness and boundary rectifiability theorems, results that seemed to rely heavily on Euclidean tools such as the deformation theorem. Thus this work gives much new insight also to the classical theory. We skip the rest of the definitions, but let us see how boundary is defined. First the exterior derivative of $\omega = (g, \pi_1, \dots, \pi_m)$ is $d\omega = (1, g, \pi_1, \dots, \pi_m)$ and then, as before, $\partial T(\omega) = T(d\omega)$.

The theory of rectifiable currents in metric spaces is based on the theory of rectifiable sets in [16], recall Chapter 7. The proofs of the compactness and boundary rectifiability theorems are again by induction, and one of the main tools is slicing. But now the rectifiability criterion for m-dimensional currents is given in terms of the slices with Lipschitz maps $f\colon X \to \mathbb{R}^m$, which are zero-dimensional currents, that is, measures, and when rectifiable, sums of point masses. Another basic tool is provided by a theory of metric space-valued BV-functions which Ambrosio developed in [13] and which was mentioned in Section 12.3. A key fact is that $y \mapsto \langle T, f, y\rangle, y \in \mathbb{R}^m$, is a BV-function. In the Euclidean setting this was proved by Jerrard in [261]. Since the BV-functions are Lipschitz on large subsets, one can, essentially, conclude that if T is a normal current, then the set of those $x \in \operatorname{spt} T$ which are atoms of $\langle T, f, f(x)\rangle$

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is rectifiable. Using this one then characterizes rectifiable m-currents T by the property that for every Lipschitz map $f\colon X \to \mathbb{R}^m$ for $\mathcal{L}^m$ almost all $y \in \mathbb{R}^m$ the slice $\langle T, f, y\rangle$ is a rectifiable 0-current. The compactness and boundary rectifiability theorems then follow by induction arguments employing slicing with real-valued functions.

As mentioned before this gives new proofs also in the Euclidean setting.

Lang developed in [285] another approach which applies to local currents, not necessarily having finite mass. In [17], it is essential that the currents have finite mass.

Currents in Heisenberg groups is a fairly complicated issue, even from the point of view of definitions, since they are based on a difficult (at least to me) concept of Rumin's complex. Their theory is much less developed than the Euclidean and metric theories, see [430].

13.3 Varifolds

Varifolds, like currents, are generalized surfaces, better in some aspects and worse in some. In particular, there is no concept of boundary and no need for orientation. In a way they are more general than currents; any current T with finite mass induces a varifold via the formula (13.1). They were introduced by Almgren in the 1960s in unpublished notes and his little book [8], and the basic results were presented by Allard in [7]. They are discussed in [297] and [397].

For any $A \subset \mathbb{R}^n$ we set $G_m(A) = A \times G(n, m)$.

Definition 13.8 Any Radon measure on $G_m(\mathbb{R}^n)$ is called an *m-varifold*. To each $\mathcal{H}^m$ measurable and m-rectifiable set E and non-negative $\mathcal{H}^m$ measurable function θ on E with $\int_E \theta\, d\mathcal{H}^m < \infty$, we associate the rectifiable m-varifold $v(E, \theta)$ defined by

$$v(E,\theta)(B) = \int_{\{x\in E\colon\ (x,\mathrm{apTan}(E,x))\in B\}} \theta(x)\, d\mathcal{H}^m x,\ \ B \subset G_m(\mathbb{R}^n) \text{ a Borel set.}$$

When $\theta = 1$ we write $v(E) = v(E, \theta)$.

To any m-varifold v we associate the Radon measure μ_v on $\mathbb{R}^n$ by $\mu_v(A) = v(G_m(A))$. The *mass* of v is $M(v) = \mu_v(\mathbb{R}^n)$. The image $f_\# v$ of v under a smooth map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$f_\# v(A) = \int_{F^{-1}(A)} J_V f(x)\, dv(x, V),\ \ A \subset G_m(\mathbb{R}^n) \text{ a Borel set,}$$

where $F(x, V) = (f(x), Df(x)(V))$ and $J_V f$ is a Jacobian of f along V. Then

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for a rectifiable m-varifold $v(E,\theta)$, $M(v(E,\theta)) = \int_E \theta \, d\mathcal{H}^m$ and the image of $v(E,\theta)$ under a diffeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is the rectifiable m-varifold given by $f_\# v(E,\theta) = v(f(E), \theta \circ f^{-1})$.

There is no natural boundary operator, but in order to study Plateau-type problems we can use the classical approach from the calculus of variations. Let $h_t\colon \mathbb{R}^n \to \mathbb{R}^n, t \geq 0$ be a one-parameter family of diffeomorphisms with h_0 the identity and with each h_t the identity outside a fixed compact set K. Then a computation shows that

$$\frac{d}{dt} M\left(h_{t\#}(v \llcorner G_m(K))\right)_{|t=0} = \int \operatorname{div}_V X(x) \, dv(x, V),$$

where $X(x) = \frac{d}{dt} h_t(x)_{|t=0}$ and $\operatorname{div}_V X(x)$ is the divergence along V. Motivated by this we define

Definition 13.9 The *first variation* δv of a varifold v is defined by

$$\delta v(X) = \int \operatorname{div}_V X(x) \, dv(V, x)$$

for any smooth vector-field $X\colon \mathbb{R}^n \to \mathbb{R}^n$ with compact support. If $\delta v(X) = 0$ for all such X with support in an open set U, then v is called *stationary* in U.

The first variation is a very interesting operator on vector fields for many reasons, not only that it defines stationary varifolds. If $v = v(M)$ is a rectifiable varifold over a smooth manifold M, then δv can be expressed with the mean curvature of M, which leads to a concept of generalized mean curvature.

Allard [7] proved the following rectifiability theorem:

Theorem 13.10 *Suppose that the m-varifold v satisfies $|\delta v(X)| \lesssim \|X\|_\infty$ for all smooth $X\colon \mathbb{R}^n \to \mathbb{R}^n$ with compact support, that is, (the total variation of) δv is a Radon measure. If $\Theta^{*m}(\mu_v, x) > 0$ for μ_v almost all $x \in \mathbb{R}^n$, then v is rectifiable.*

The two main tools to prove this are the monotonicity formula and tangent cones. Both of these have analogues for currents and they are important in many ways.

The monotonicity formula for stationary varifolds, and analogously for mass minimizing currents, is the following: if v is a stationary m-varifold in U, then for $x \in U$ and $0 < r < s$ with $B(x,s) \subset U$,

$$s^{-m}\mu_v(B(x,s)) - r^{-m}\mu_v(B(x,r)) = \int_{y \in B(x,s) \setminus B(x,r)} |y-x|^{-m-2} |P_{V^\perp}(y-x)|^2 \, dv(y, V).$$

This is proved using test vector fields of the type $y \mapsto \varphi(|y-x|)(y-x)$ in the definition of the first variation. In particular, the finite density $\Theta^m(\mu_v, x)$

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exists. Assuming its positivity, as in Theorem 13.10, we could conclude by Preiss's theorem 4.11 the rectifiability of μ_v. This was not available for Allard, but it would anyway be unnecessarily heavy machinery; the monotonicity formula gives much more than the mere existence of the density. A variant of the monotonicity formula also holds when δv is Radon measure, not necessarily stationary, and gives the existence of density.

Tangent cones are important in particular for studying singularities of mass minimizing currents and stationary varifolds. Let, for example, v be a stationary m-varifold and, as in the case of tangent measures, $T_{x,r}(y) = (y - x)/r$. We say that a varifold C is a *tangent cone* of v at x if $T_{0,r\#}C = C$ for $r > 0$, that is, C is a cone with vertex at 0, and there is a sequence $r_i > 0, \lim_{i\to\infty} r_i = 0$ such that $\lim_{i\to\infty} T_{x,r_i\#}v = C$. The monotonicity of the density ratios together with an easy compactness theorem imply that such limits exist, and further, still based on the monotonicity formula, they are cones. But can there be more than one tangent cone at a point? The uniqueness of tangent cones is a central open problem and known only in some cases. White proved it for two-dimensional currents in [435], this paper also gives references to other cases. In [437], he constructed a counterexample for harmonic maps, see also [161, 346].

We now briefly explain how the monotonicity formula and tangent cones can be used to prove Theorem 13.10 when v is stationary. The same ideas work in the general case. As for rectifiable sets and measures, it suffices to show that for μ_v almost all x the varifold v has a unique tangent cone which is an m-plane. Let C be some tangent cone at a typical point x. Then C is stationary and $0 \in \operatorname{spt}\mu_C$. As for tangent measures, μ_C is an m-uniform measure:

$$\mu_C(B(y,r)) = \alpha(m)\Theta^m(\mu_v, x)r^m \text{ for } y \in \operatorname{spt}\mu_C, r > 0. \tag{13.3}$$

From this we see by the monotonicity formula that $P_{V^\perp}(y - x) = 0$ for C almost all (y, V). This implies that when $(0, V) \in \operatorname{spt} C$, then $\operatorname{spt}\mu_C \subset V = V_C$, and further, by stationarity and a constancy theorem, that $C = \Theta^m(\mu_v, x)v(V_C)$. So we have left to show that V_C is unique, but this is now fairly easy: a general differentiation theorem implies that for any continuous function φ on $G(n,m)$ and v almost all (x, V) the limit $\lim_{r\to 0}\int_{G_m(B(x,r))}\varphi(V)\,dv(x,V)/\mu_v(B(x,r))$ exists. But using the definition of the tangent varifold one quickly checks that this equals $\varphi(V_C)$; go to zero through the sequence defining C. This implies that V_C is unique.

Allard's main result was a regularity theorem for stationary varifolds, and more generally for varifolds with L^p conditions for the generalized mean curvature, see [397] and [297].

De Philippis, De Rosa and Ghiraldin extended Allard's rectifiability theorem in [173], replacing mass by more general integrands. Their proof was different.

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Instead of the monotonicity formula, they used tangent measures and results of Preiss from [382]. They showed that generically tangent measures are translation invariant in at least m directions while the positivity of the lower density, which they assumed, ensures that there exists at least one tangent measure that is invariant along at most m directions. This leads to unique flat tangent measures. Still another proof, based on PDE operators, and a more general result is provided by [29], see Section 15.5.

In [20], Ambrosio and Soner introduced generalized varifolds and applied them to gradient flows; $G(n,m)$ considered as a class of matrices is replaced by a larger subclass of symmetric matrices. They proved a rectifiability theorem, but relied on Allard's theorem. Brakke [80] developed mean curvature flow with varifolds. We shall come to it in Section 15.3.

Moser proved a general result in [353] related to the above as well as to harmonic maps and Yang–Mills connections discussed in Chapter 15.