BULL AUSTRAL. MATH. SOC. 18F15, 58A40
VOL. 34 (1986) 395-410

LIE GROUP VALUED INTEGRATION IN WELL-ADAPTED TOPOSES

ANDERS KOCK

In the context of synthetic differential geometry, we prove that group valued *1*-forms on the unit interval are exact, provided the group in question is a Lie group. This exactness is the basic assumption in a previous paper by the author on differential forms with values in groups.

0. Introduction.

We consider the standard well adapted topos models for synthetic differential geometry, and prove the validity here of a fundamental Theorem of differential geometry, namely that, for G a Lie Group,

* G-valued *1*-forms on R (or on $[0,1]$) are exact.

(the classical (well known) version of this Theorem has a less simple formulation, and is stated in the beginning of Chapter 3.) I have expounded the meaning of * in several articles [8], [9], [10].

The main technical tool for proving validity of * in the topos models is a generalization of a Theorem of O. Bruno [2] from the *1*-variable case to the n-variable case, and for this generalization, we

Received 6 January 1986. I want to thank Eduardo Dubuc for crucial stimulation and help in the present work, which was carried out while he was visiting Aarhus University on a Danish Natural Science Research Council grant.

https://doi.org/10.1017/S0004972700010285 Published online by Cambridge University Press

resort to convenient vector space theory [6], [14], [12].

The well-adapted models we consider are $\mathcal{Z}$, $\mathcal{F}$ and $\mathcal{G}$ of [4], [15], whose sites of definition have as objects C^∞-rings $C^\infty(\mathbb{R}^n)/I$ with I an arbitrary, respectively W-determined, respectively germ determined ideal (terminology of [7]). (The arguments and results we present are independent of which subcanonical Grothendieck topology we consider.) Any of the three toposes will be denoted $\mathcal{E}$. The category Mf of manifolds is embedded into $\mathcal{E}$, in the standard way, $M \in Mf$ being represented by the ring $C^\infty(M)$. We omit the embedding i from the notation, except that we write R for $i(\mathbb{R})$.

1. Congruence modulo ideals.

Let $I \subseteq C^\infty(\mathbb{R}^p)$ be an ideal, fixed for this section. Let M be a manifold (or any other set structured with a C^∞-ring $C^\infty(M)$ of functions $M \longrightarrow \mathbb{R}$, in particular, M may be a convenient vector space). We let $I(M)$ denote the equivalence relation on $C^\infty(\mathbb{R}^p, M)$ given by

$$f \equiv g \mod I(M) \text{ if and only if for all } \phi \in C^\infty(M), (\phi \circ f - \phi \circ g) \in I$$

or equivalently, if and only if

$$(1.1) \qquad C^\infty(M) \overset{f^*}{\underset{g^*}{\rightrightarrows}} C^\infty(\mathbb{R}^p) \longrightarrow C^\infty(\mathbb{R}^p)/I$$

commutes.

This we call <u>weak</u> congruence mod $I(M)$, or just mod I. If X is a convenient vector space, we let $I(X)$ denote the linear subspace of $C^\infty(\mathbb{R}^p, X)$ spanned (purely algebraically) by functions of the form

$$(1.2) \qquad h(t) \cdot k(t) \qquad t \in \mathbb{R}^p$$

with $h: \mathbb{R}^p \longrightarrow \mathbb{R}$ in I and $k: \mathbb{R}^p \longrightarrow X$ arbitary smooth.

Two maps $\mathbb{R}^p \rightrightarrows X$ will be called <u>strongly congruent</u> mod $I(X)$, or just mod I, if their difference belongs to $I(X)$.

To compare the two notions where it makes sense (Proposition 1.3 below), we shall use the following unsurprising Lemma from convenient vector space theory.

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LEMMA 1.1. *Let* $G: X \longrightarrow Y$ *be a smooth map between convenient vector spaces. Then there exists a smooth* $H: X \times X \times \mathbb{R} \longrightarrow Y$ *such that*

$$G(x + \lambda \cdot y) = G(x) + \lambda \cdot H(x,y,\lambda) \tag{1.3}$$

for all $x, y \in X$ *and* $\lambda \in \mathbb{R}$.

Proof. Consider the function H defined by

$$H(x,y,\lambda) = \int_0^1 df_{x+s\lambda y}(y)\, ds .$$

It will serve in (1.3), by the standard (Hadamard) calculation. It depends smoothly on (x,y,λ); for, $df_x(y)$ depends smoothly on (x,y) (see [14], Satz p.299, or [6], Theorem 6.2), and integration preserves smoothness (see for example [12], Proposition 2.6).

Let X and Y again denote convenient vector spaces; then

PROPOSITION 1.2. *Let* $f,g: \mathbb{R}^p \longrightarrow X$ *be strongly congruent mod* I, *and let* $G: X \longrightarrow Y$ *be smooth. Then* $G \circ f$ *and* $G \circ g$ *are strongly congruent mod* I.

Proof. By assumption, $g(t) = f(t) + \Sigma\, h_i(t)k_i(t)$ with h_i and k_i as in (1.2). We may remove one $h_i(t) \cdot k_i(t)$ summand at a time, so it suffices to consider the case

$$g(t) = f(t) + h(t) \cdot k(t) .$$

Let H be as in Lemma 1.1. Then since $h(t) \in \mathbb{R}$, we have

$$\begin{aligned} G(g(t)) &= G(f(t) + h(t) \cdot k(t)) \\ &= G(f(t)) + h(t) \cdot H(f(t),k(t),h(t)), \end{aligned}$$

and the last term is in $I(Y)$ due to the factor $h(t)$.

PROPOSITION 1.3. *Let* X *be a convenient vector space; then strong congruence mod* I *of maps* $\mathbb{R}^p \longrightarrow X$ *implies weak congruence. For* X *finite dimensional, the converse holds.*

Proof. The first part is immediate from Proposition 1.2 (let $G = \phi \in C^\infty(M)$). For the second, let f and $g: \mathbb{R}^p \longrightarrow \mathbb{R}^n$ be weakly congruent mod $I(\mathbb{R}^n)$. For each of the n coordinate projections $\mathrm{proj}_i\colon \mathbb{R}^n \longrightarrow \mathbb{R}$, we therefore have

$$\mathrm{proj}_i \circ g - \mathrm{proj}_i \circ f \in I .$$

Denote this map by $h_i\colon \mathbb{R}^p \longrightarrow \mathbb{R}$. So

$$g(t) = f(t) + \Sigma\, h_i(t) \cdot \underline{e}_i$$

where $\underline{e}_i$ is the constant function $\mathbb{R}^p \longrightarrow \mathbb{R}^n$ with value $\underline{e}_i \in \mathbb{R}^n$. Since $h_i \in I$, this proves strong congruence.

It is clear that strong congruence behaves well with respect to products: for maps $\mathbb{R}^p \longrightarrow X_1 \times \ldots \times X_n$ (X_i convenient), congruence mod $I(X_1 \times \ldots \times X_n)$ is tested coordinatewise, that is by testing congruence mod $I(X_i)$ $(i = 1,\ldots,n)$. As a corollary of Proposition 1.2, we therefore derive

PROPOSITION 1.4. *Let* $G\colon X_1 \times \ldots \times X_n \longrightarrow Y$ *be smooth, and let* f_j, g_j *be maps* $\mathbb{R}^p \longrightarrow X_j$. *If (strongly)*

$$f_j \equiv g_j \mod I(X_j) \quad j = 1,\ldots,n$$

then (strongly)

$$G \circ (f_1,\ldots,f_n) \equiv G \circ (g_1,\ldots,g_n) \mod I(Y).$$

If K is a manifold, the ideal $I \subseteq C^\infty(\mathbb{R}^p)$ defines an ideal I^* in $C^\infty(\mathbb{R}^p \times K)$, namely the one spanned by functions $h(t)k(t,x)$ ($t \in \mathbb{R}^p$, $x \in K$, and $h \in I$). Clearly, under the isomorphism

(1.4) $$C^\infty(\mathbb{R}^p, C^\infty(K)) \simeq C^\infty(\mathbb{R}^p \times K) ,$$

$I(C^\infty(K)) \subseteq C^\infty(\mathbb{R}^p, C^\infty(K))$ corresponds to I^* .

Suppose now that we have a smooth map ('operator')

(1.5) $$C^\infty(K)^n \xrightarrow{\ G\ } C^\infty(L)$$

with L a manifold. The composite

(1.6) $$C^\infty(\mathbb{R}^p \times K)^n \simeq C^\infty(\mathbb{R}^p, C^\infty(K))^n \xrightarrow{G_*} C^\infty(\mathbb{R}^p, C^\infty(L)) \simeq C^\infty(\mathbb{R}^p \times L) ,$$

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(where G_*, modulo the identification $C^\infty(\mathbb{R}^p, C^\infty(K))^n \simeq C^\infty(\mathbb{R}^p, C^\infty(K)^n)$, is just "composing with G") should be considered as "applying G parameterwise in $t \in \mathbb{R}^p$". Let us denote it by $G/\mathbb{R}^p$ (or just G). Let f_i and g_i be elements of $C^\infty(\mathbb{R}^p \times K)$ for $i = 1,\ldots,n$, and let G be as above (1.5). We then have

THEOREM 1.5. *Let* $f_i - g_i \in I^* \subset C^\infty(\mathbb{R}^p \times K)$, *for* $i = 1,\ldots,n$. *Then*

$$G/\mathbb{R}^p(f_1,\ldots,f_n) - G/\mathbb{R}^p(g_1,\ldots,g_n) \in I^* \subseteq C^\infty(\mathbb{R}^p \times L) .$$

(For $n = 1$, this is implicit in Bruno's Theorem 8, [2].)

Proof. The assumption means $\hat{f}_i \equiv \hat{g}_i \mod I(C^\infty(K))\ \forall i$; by Proposition 1.4,

$$G \circ (\hat{f}_1,\ldots,\hat{f}_n) = G \circ (\hat{g}_1,\ldots \hat{g}_n) \mod I(C^\infty(L))$$

and again this implies congruence mod $I^* \subseteq C^\infty(\mathbb{R}^p \times L)$ for the exponential adjoints, which are the terms appearing in the Theorem.

Even when the ideal $I \subseteq C^\infty(\mathbb{R}^p)$ is W-determined, respectively germ-determined, the ideal $I^* \subseteq C^\infty(\mathbb{R}^p \times K)$ may not be, so to get results about the models F and G (see the introduction), we need to take the 'W-radical', respectively 'germ-radial' of I^* (terminology of [7]).

It is known (see, for example [15]) that the W-radical $\bar{J}$ of an ideal $J \subseteq C^\infty(\mathbb{R}^k)$ is its closure in the Frechet space topology on $C^\infty(\mathbb{R}^k)$. An unpublished result of Penon says that the germ-radical $\tilde{J}$ similarly is the closure of J in a finer topology on $C^\infty(\mathbb{R}^k)$, called the Stone-topology in [2], where this topology is described, and a sketch of Penon's result is given.

We shall need the following important result. Let K and L be manifolds, and let $G\colon C^\infty(K,\mathbb{R}^n) \longrightarrow C^\infty(L)$ be a smooth operator. Then

THEOREM. (Frölicher [5]). *G is continuous with respect to the Frechet topologies.*

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THEOREM. (Bruno [2]). *G is continuous with respect to the Stone topologies.*

(Frölicher in fact proves that any (plot-) smooth map between Frechet spaces is continuous. Bruno proves the Theorem quoted only when K and L are coordinate spaces and $n = 1$, but his proof carries over immediately.)

Using these Theorems in conjunction with Theorem 1.5 leads to the following result (with notation as in Theorem 1.5):

THEOREM 1.7. *Let* $(f_i - g_i) \in \overline{I}^*$ *(respectively* $\tilde{I}^*) \subseteq C^\infty(\mathbb{R}^p \times K)$ *for* $i = 1,\ldots, n$. *Then*
$G/\mathbb{R}^p(f_1,\ldots,f_n) - G/\mathbb{R}^p(g_1,\ldots,g_n) \in \overline{I}^*$ *(respectively* $\tilde{I}^*) \subseteq C^\infty(\mathbb{R}^p \times L)$

(For $\tilde{I}^*$ and $n = 1$, this is Bruno's Theorem 8, [2] .

Proof. For each $i = 1,\ldots,n$, let $(h^i_m)_{m \in \mathbb{N}}$ be a sequence in I^* converging in the relevant topology to $g_i - f_i$. For each m, Theorem 1.5 applies to the n-tuple

$$(f_i, f_i + h^i_m)_{i = 1,\ldots,n}$$

to give

$$G/\mathbb{R}^p(f_1,\ldots,f_n) - G/\mathbb{R}^p(f_1 + h^1_m,\ldots,f_n + h^n_m) \in I^* . \tag{1.7}$$

As $m \to \infty$, the right hand term converges to $G/\mathbb{R}^p(g_1,\ldots,g_n)$ by continuity of $G/\mathbb{R}^p$ (which is a smooth map $C^\infty(\mathbb{R}^p \times K, \mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^p \times L)$, hence continuous by the Theorems quoted). So the difference is the one in the Theorem, and is a limit of expressions (1.7) in I^*, hence in $\overline{I}^*$ (respectively $\tilde{I}^*$).

Consider more generally a smooth operator

$$C^\infty(K)^n \longrightarrow C^\infty(L,M),$$

with K,L and M manifolds. Replacing the codomain in (1.6) by

$$C^\infty(\mathbb{R}^p, C^\infty(L,M)) \simeq C^\infty(\mathbb{R}^p \times L, M)$$

yields a smooth map $G/\mathbb{R}^p$:

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$$C^\infty(\mathbb{R}^p \times K)^n \longrightarrow C^\infty(\mathbb{R}^p \times L, M) \; .$$

With $\tilde{\cdot}$ denoting closure for any of the three topologies under consideration (discrete, Frechet, Stone), and with $f_i, g_i \in C^\infty(\mathbb{R}^p \times K)$ as before we have

THEOREM 1.7'. *If* $f_i - g_i \in \tilde{I}^*$ $(i = 1,\ldots,n)$, *we have*

$$G/\mathbb{R}^p\ (f_1,\ldots,f_n) \equiv G/\mathbb{R}^p\ (g_1,\ldots g_n) \mod \tilde{I}^*(M) \; . \tag{1.8}$$

Proof. The conclusion (1.8) means 'weak congruence' of course. So let $\phi\colon M \to \mathbb{R}$ be smooth, and apply Theorem 1.5 (for the discrete case) or Theorem 1.7 to the smooth operator

$$C^\infty(K)^n \xrightarrow{\ G\ } C^\infty(L,M) \xrightarrow{\ \phi_*\ } C^\infty(L) \; .$$

2. The functor E.

Recall that $\mathcal{E}$ denotes any of the well adapted toposes $\mathcal{Z}$, $\mathcal{F}$ and $\mathcal{G}$ mentioned in the introduction. If M is a manifold and $I \subseteq C^\infty(M)$, we let $\tilde{I}$ denote its closure in any of the three topologies (discrete, Frechet, Stone), according to whether we read $\mathcal{Z}$, $\mathcal{F}$ or $\mathcal{G}$, for $\mathcal{E}$. Similarly, $\tilde{\otimes}$ denotes coproduct in the sites of definition of either; if A is in the site, $\bar{A} \in \mathcal{E}$ denotes the object it represents. Thus $\bar{A} \times \bar{B} = (A \mathbin{\tilde{\otimes}} B)^{-}$.

Let $J \subseteq C^\infty(\mathbb{R}^p)$ be a closed ideal, $J = \tilde{J}$. For any manifold K , we have

$$C^\infty(\mathbb{R}^n)/J \mathbin{\tilde{\otimes}} C^\infty(K) \simeq C^\infty(\mathbb{R}^p \times K)/\tilde{J}^* \; ; \tag{2.1}$$

this requires a small argument, which we shall not reproduce here (and I thank E. Dubuc for convincing me of its truth in the $\mathcal{G}$ case), since we shall only need the result for $K = \mathbb{R}^k$, where it is evidently true.

If M and L are manifolds, the exponential object $M^L \in \mathcal{E}$ goes by the global sections functor Γ to the set $C^\infty(L,M)$ of smooth maps. So a map ('operator')

$$N^K \xrightarrow{\ G\ } M^L$$

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in $\mathcal{E}$ goes by Γ to an operator

$$C^\infty(K,N) \xrightarrow{\Gamma(G)} C^\infty(L,M)$$

which evidently is (plot-) smooth. A main result in Bruno [2] is that this process can be reversed when $N = \mathbb{R}$ (he also has some inessential restrictions on K, L, M). From Theorem 1.7' we get a generalization of this result to the case $N = \mathbb{R}^n$ (obtained independently also by Moerdijk and Reyes):

THEOREM 2.1. *Let K, L and M be manifolds. To any smooth operator G:*

$$C^\infty(K,\mathbb{R}^n) \cong C^\infty(K)^n \xrightarrow{G} C^\infty(L,M) ,$$

there is a unique map in $\mathcal{E}$

$$(R^n)^K \xrightarrow{E(G)} M^L$$

with $\Gamma(E(G)) = G$.

Proof. Let $A = C^\infty(\mathbb{R}^p)/J$ be an object in the site of definition of $\mathcal{E}$ (so $J = \tilde{J}$). We must produce a set theoretic map

$$(R^n)^K(A) \xrightarrow{\varepsilon_A} M^L(A)$$

natural in A. An element b on the left corresponds, by Yoneda, exponential adjointness, and (2.1), to an n-tuple of elements $b_i \in C^\infty(\mathbb{R}^p \times K)/\tilde{J}^*$. Let $\beta_i \in C^\infty(\mathbb{R}^p \times K)$ be a representative of b_i, so that we have a smooth map $\beta = (\beta_1,\ldots,\beta_n)\colon \mathbb{R}^n \times K \longrightarrow \mathbb{R}^n$.

Consider

$$\gamma := G/\mathbb{R}^p\ (\beta) \in C^\infty(\mathbb{R}^p \times L,M) . \tag{2.2}$$

We get a C^∞-algebra map 'composing with γ'

$$C^\infty(M) \longrightarrow C^\infty(\mathbb{R}^p \times L) .$$

If we choose different representatives β_i' for b_i, (so $\beta_i' - \beta_i \in \tilde{J}^*$), we get immediately from Theorem 1.7' that $\gamma' \equiv \gamma \mod \tilde{J}^*(M)$ (here $\tilde{J}^* \subset C^\infty(\mathbb{R}^p \times L)$); expressing this fact in the style of (1.1), and then taking the corresponding 'dual' diagram in $\mathcal{E}$, yields commutativity of

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$$\bar{A} \times L \longrightarrow R^p \times L \underset{\gamma'}{\overset{\gamma}{\rightrightarrows}} M ,$$

so that b well-defines a map $\bar{A} \times L \longrightarrow M$, or, equivalenty, an element of $M^L(A)$, as desired. Naturality in A is straightforward (at least, the construction was not un-natural). So the map $E(G)$ in $\mathcal{E}$ is now declared to be the natural transformation with components ε_A .

It is clear that $\Gamma(E(G))$ is just G : put $A = C^\infty(\mathbb{R}^0) = \mathbb{R}$, and use $G/\mathbb{R}^0 = G$ and (2.2).

The uniqueness assertion yet to be proved we separate out as a separate, slightly more general 'faithfulness' assertion, Proposition 2.2 below.

Recall that the unit interval $[0,1]$ is represented in $\mathcal{E}$ by the ring $C^\infty(\mathbb{R})/H$, where H is the ideal of functions $\mathbb{R} \longrightarrow \mathbb{R}$ that vanish on $[0,1]$.

PROPOSITION 2.2. *Let* K *and* M *be representable (in particular they may be manifolds), and let* L *be a manifold, or* $[0,1]$. *Then any two maps* ψ_1, ψ_2 :

$$(R^n)^K \rightrightarrows M^L$$

with $\Gamma(\psi_1) = \Gamma(\psi_2)$ *are equal.*

Proof. Since M is a subobject of some R^m , we reduce immediately to the case $M = R$, and it suffices to prove that a map $\psi\colon (R^n)^K \longrightarrow R^L$ with $\Gamma(\psi) = 0$ is itself 0 . Let $\bar{A}$ be a representable object, and $b\colon \bar{A} \longrightarrow (R^n)^K$. Consideration of the exponential adjoint of b , and representability of K (and thus of $\bar{A} \times K$) leads to the extension of b to some $c\colon R^p \longrightarrow (R^n)^K$, and it suffices to prove $\psi \circ c = 0$. Now $\psi \circ c\colon R^p \longrightarrow R^L$ corresponds to a map $\phi\colon R^p \times L \longrightarrow R$, such that $\phi(x,-)\colon L \longrightarrow R$ is the zero map for all (global) points $x \in \mathbb{R}^p$, by assumption on $\Gamma(\psi)$. In particular, for any (global) point $y \in L$, $\phi(x,y) = 0$, so ϕ has $\Gamma(\phi) = 0$, and since manifolds are fully embedded into $\mathcal{E}$, ϕ has to be 0 , for the case when L is a manifold.

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If L is the unit interval, we argue as follows: extend $\phi: R^P \times L \longrightarrow R$ into a $\Phi: R^P \times R \longrightarrow R$. The smooth function $\Phi: \mathbb{R}^P \times \mathbb{R} \longrightarrow \mathbb{R}$ has the property that for each $x \in \mathbb{R}^P$, $\Phi(x,-)$ belongs to the ideal H, that is $\Phi(x,t) = 0 \quad \forall x \in \mathbb{R}^P, \forall t \in [0,1]$. But by a deep result of Calderón-Quê-Reyes [16], this implies that $\Phi \in H^*$, so in particular $\Phi \in \tilde{H}^*$, which is equivalent to saying $\phi = 0$.

(The Proposition holds (with the same proof) for any L which is represented by an ideal with line determined extensions in the sense of Bruno [3], which by [3] is a Frechet closed ideal I such that also all I^* are Frechet closed)

3. Application to integration in the topos models.

Let G be a Lie group, and LG its tangent space at the neutral element $e \in G$. Consider the pair of operators

$$C^\infty(\mathbb{R},LG) \underset{S}{\overset{T}{\leftrightarrows}} C^\infty(\mathbb{R},G) ,$$

where T is the 'differentiation' operator which to $g: \mathbb{R} \longrightarrow G$ associates f given by

$$f(t) = \frac{d}{ds}\Big|_{s=0} g(t+s) \cdot g(t)^{-1} , \tag{3.1}$$

and where S to $f: \mathbb{R} \longrightarrow LG$ associates the unique g satisfying (3.1) and $g(0) = e$. (It is a classical result that this S exists and is smooth in parameters. In fact, if G is a matrix group, g is the solution of a linear homogeneous differential (matrix-) equation with variable coefficients f.)

The result of the previous section apply to S since $LG \simeq \mathbb{R}^n$ (but not to T). By Theorem 2.1, we get a map $\sigma = E(S)$ with $\Gamma(\sigma) = S$. We also have a τ in the other direction

$$LG^R \underset{\sigma}{\overset{\tau}{\leftrightarrows}} G^R ,$$

namely 'synthetic differentiation', given by the synthetic analogue of (3.1), that is $\tau(g) = f$ with f given by

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(3.2) $$f(t)(d) = g(t+d) \cdot g(t)^{-1} \qquad \forall d \in D .$$

A standard argument, as in [7] III Theorem 3.2, shows that $\Gamma(\tau) = T$. Thus

$$\Gamma(\tau \circ \sigma) = \Gamma(\tau) \circ \Gamma(\sigma) = T \circ S = id ,$$

By the 'Faithfulness' Proposition 2.2, $\tau \circ \sigma$ is the identity map on LG^R.

Let us identify the Kernel pair of τ by a synthetic argument. Suppose $g, h \in G^R$ have $\tau(g) = \tau(h)$. Define $g^{-1} \cdot h \in G^R$ by

(3.3) $$(g^{-1} \cdot h)(t) = g(t)^{-1} \cdot h(t) .$$

Then

$$\begin{aligned}\tau(g^{-1} \cdot h)(t)(d) &= g(t+d)^{-1} \cdot h(t+d) \cdot h(t)^{-1} \cdot g(t)\\ &= g(t+d)^{-1} \cdot \tau(h)(t)(d) \cdot g(t)\\ &= g(t+d)^{-1} \cdot \tau(g)(t)(d) \cdot g(t)\\ &= g(t+d)^{-1} \cdot g(t+d) \cdot g(t)^{-1} \cdot g(t) = e ,\end{aligned}$$

so $\tau(g^{-1} \cdot h) \equiv e$ or

$$(g^{-1} \cdot h)(t+d) = (g^{-1} \cdot h)(t) \qquad \forall t \in R ,\ d \in D .$$

From Proposition 3.1 below it follows that $g^{-1} \cdot h$ is constant, that is there is a unique $c \in G$ so that

(3.4) $$h(t) = g(t) \cdot c .$$

(Conversely, if (3.4) holds, then clearly $\tau(g) = \tau(h)$.)

PROPOSITION 3.1. *Let M be a manifold. If $f \in M^R$ has $f(t+d) = f(t)$ $\forall t \in R$ $\forall d \in D$, then $f(t) = f(0)$ $\forall t \in R$.*

Proof. Since there exists a monic $M \rightarrowtail R^m$ for some m, one quickly reduces the question to the case $M = R$. Let $b: \bar{A} \longrightarrow R^R$ be an element at stage $\bar{A}$, and extend it, as in the proof of Proposition 2.2 to an element $c: R^P \longrightarrow R^R$. Taking exponential adjoints gives an actual map in

$$R^P \times R \longrightarrow R,$$

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and the assumption gives $\frac{\partial \gamma}{\partial t}(x,t) \equiv 0$. The external map $\Gamma(\gamma)$ corresponding to γ then has the same property, so $\Gamma(\gamma) = \Gamma(\gamma)(x,0)$. But Γ is faithful on the subcategory $Mf \subset E$, so $\gamma(x,t) = \gamma(x,0)$ $\forall t$ holds internally.

With G a Lie group, and LG its Lie algebra, as above, we derive the following Theorem about G-valued integration:

THEOREM 3.2. *In* E *we have*

$$\forall f \in LG^R \quad \exists! g \in G^R \text{ with } g(0) = e \text{ and}$$

$$(3.5) \qquad g(t+d) \cdot g(t)^{-1} = f(t)(d) \quad \forall t \in R, \quad d \in D .$$

Proof. The (internal) map $E(S) = \sigma$, together with the fact that $\tau \circ \sigma$ is the identity, gives the existence. The uniqueness is immediate from the above identification of the kernel pair of τ .

The Theorem can be reformulated in terms of differential forms with values in the group G , in the sense of [8]:

THEOREM 3.2. *In* E *we have that any* G*-valued 1-form on* R *is exact (with primitive unique modulo right multiplication by a unique constant from* G*).*

Proof. The 1-form ω associates with any neighbour pair (x,y) of R an element $\omega(x,y) \in G$, with $\omega(x,x) = e$ $\forall x$. Now (x,y) is of the form $(x,x+d)$ for a unique $d \in D$, so

$$d \longmapsto \omega(x,x+d)$$

defines for each $x \in R$ a tangent vector at $e \in G$. So the information of ω is equivalent to that of a curve $R \longrightarrow LG$. The equation (3.5) equivalent to

$$g(y) \cdot g(x)^{-1} = \omega(x,y)$$

for x and $y = x+d$ any neighbour pair of R . So g is the primitive, witnessing exactness of ω . The uniqueness assertion is clearly equivalent to the previous identification of the kernel pair of τ . This proves the Theorem.

We next consider the more important case of Lie group valued integration of functions, defined on the unit interval $[0,1]$. Let G and LG be a Lie group and its Lie algebra, as in Theorem 3.2.

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THEOREM 3.3. *In* $\mathcal{E}$ *we have*

$$\forall f \in LG^{[0,1]} \quad \exists ! g \in G^{[0,1]} \quad with \quad g(0) = e \quad and$$

$$g(t+d) \cdot g(t)^{-1} = f(t)(d) \quad \forall t \in [0,1] \quad \forall d \in D;$$

equivalently, G-valued 1-forms on $[0,1]$ *are exact, with primitive unique modulo right multiplication by a unique constant from* G.

Proof. The restriction map $LG^R \longrightarrow LG^{[0,1]}$ is epic, since $LG \simeq R^n$ and $[0,1] \rightarrowtail R$ is a representable subobject. Equivalently, in a synthetic argument, we may assume that every $[0,1] \longrightarrow LG$ may be extended to $R \longrightarrow LG$, and then the existence assertion follows immediately from the existence assertion in Theorem 3.2. To prove the uniqueness, it suffices to prove that if $g,h \in G^R$ have $\tau(g)\big|_{[0,1]} = \tau(h)\big|_{[0,1]}$, with τ the differentiation process of (3.2), then the function $g \cdot h^{-1}$, defined in (3.3) is constant on $[0,1]$. The same calculation as before yields

$$\tau(g^{-1} \cdot h)\big|_{[0,1]} \equiv e \,.$$

so the result will follow from the analogue of Proposition 3.1:

PROPOSITION 3.4. *Let* M *be a manifold. If* $f \in M^R$ *has* $f(t+d) = f(t) \;\; \forall t \in [0,1] \;\; \forall d \in D$, *then* $f(t) = f(0) \; \forall t \in [0,1]$.

Proof. As in the proof of Proposition 3.1, it suffices to consider the case $M = R$, and again, to consider a generalized element f of R^R at stage $\bar{A} = R^p$, $f \colon R^p \longrightarrow R^R$. The exponential adjoint $\gamma \colon R^p \times R \longrightarrow R$ satisfies by assumption

$$\frac{\partial \gamma}{\partial t}(x,t) = 0 \qquad \forall x, \; \forall t \in [0,1] \,,$$

so for $\Gamma(\gamma) \colon \mathbb{R}^p \times \mathbb{R} \longrightarrow \mathbb{R}$, we have, for all $x \in \mathbb{R}^p$,

$$\Gamma(\gamma)(x,t) - \Gamma(\gamma)(x,0) = 0 \quad \forall t \in [0,1] \,.$$

This means that the composite of f with the restriction map

(3.6) $$R^p \longrightarrow R^R \longrightarrow R^{[0,1]}$$

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has the property that Γ takes it to the zero map. By Proposition 2.2 (which here is really the Calderón-Quê-Reyes result!), the map (3.6) itself is the zero map, and the validity of $f(t) = f(0)$ $\forall t \in [0,1]$ follows.

We remark that specializing Theorem 3.3 to the case $G = (R,+)$ gives the validity of the usual "integration axiom" of [13]. The validity of this for the topos F was first proved in Belair [1], and for the topos G was known to Reyes, Dubuc and Penon. For the "Cahiers topos" C (terminology of [7]), the arguments of the present article require some modifications, since $[0,1]$ in no longer representable; but the category of manifolds with boundary is neverthless fully embedded in C which should make the modification of the arguments easy. Anyway, for C, we gave an independent proof of R-valued integration in [13], and this argument may be extended to give G-valued integration for C, as pointed out in [8].

Let us also remark that the 'lifting" of smooth operators

$$C^{\infty}(K,N) \longrightarrow C^{\infty}(L,M)$$

to the Cahiers topos, in the case where N and M are vector spaces $\mathbb{R}^n$ (or convenient vector spaces) alternatively may be seen as an immediate consequence of the full embedding of convenient vector spaces into C, [11].

We finish by proving the validity of a simple comprehensive form of the Frobenius Theorem. Recall [8] that a G-valued 1-form ω on a manifold M is a law which to each neighbour pair x,y of M associates an element $\omega(x,y) \in G$ with $\omega(x,x) = e$, and that ω is called closed if

$$\omega(y,z) \cdot \omega(x,y) = \omega(x,z)$$

whenever x,y and z are mutual neighbours. If $f\colon M \longrightarrow G$ is a function, df is the 1-form on M given by

$$df(x,y) = f(y) \cdot f(x)^{-1},$$

and df is clearly closed; 1-forms of form df are called exact, and f is called a primitive of f.

Let E be any of the well adapted topos models mentioned in the introduction.

THEOREM 3.5. *Let G be a Lie group, and M a connected, simply connected manifold*. Then any closed G-valued 1-form on M is exact (with primitive unique modulo a unique constant $\in G$).*

Proof. By Theorem 3.3, the group G "admits integration" in the sense of [8] (6.1). The result then follows from loc.cit Theorem 7.2.

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* These connectedness conditions should hold internally in E. When this follows from the external condition has still to be investigated.

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Matematisk Institut
Aarhus University
Ny Munkegade
8000 AARHUS C. DANMARK

Added in Proof: The arguments for Propositions 3.1 and 3.4 are not quite complete.