On the Computation of a Lagrangian Interpolation.

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1. Interpolation is one of the most frequent processes in calculation, and yet it is the process in which most computers find the ordinary methods least satisfactory and most troublesome. Indeed, whenever linear interpolation is not practicable, it is usually worth while to find out a method depending on the nature of the functions involved in the calculation, and use it in preference to the ordinary difference or Lagrangian formulae. In interpolation by differences there is the want of adequate tables* of the coefficients, and worse than that, the necessity for watching the signs and the decimal points, a necessity which in these days of calculating machines is relatively a great trouble. There is usually, moreover, a lack of system about interpolation by differences that makes it peculiarly susceptible to slips of working. In this connection I might mention a useful and not too well-known arrangement of the work for Newton's formula which Legendre gives in his *Traité des fonctions elliptiques*.†

2. The other general method of interpolation—the Lagrangian —is described in all the text-books on Algebra, and used as the basis for difference formulae in the text-books on Calculation, but is not, as a rule, advocated as a weapon of much direct practical power. It is of course pointed out that, when there is no regularity in the intervals of the argument, there is no other feasible general method, if graphical interpolation be ruled out, but it is not generally stated that with regular intervals it is often of the greatest value.

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* A serviceable table is given by G. W. Jones, of Cornell, in his collection of Tables (London, Macmillan & Co.). It contains the first five Binomial and also Bessel coefficients to five decimal places for every ·01 of the argument.

† Tome II., p. 36.

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3. One particularly useful case* is where the required interpolate is a missing value in a sequence (say, n in number) of values of a function at regular intervals of the argument. In this case the coefficients are obtained by taking the $n-1^{\text{th}}$ line of the Pascal triangle (with alternate signs) and dividing the numbers in this line by the number (with sign reversed) corresponding to the required interpolate. The alternate signs are to be arranged in such a way that the sum of the coefficients is, as always, $+1$. Thus, if we want to fill the gap in the sequence of 7,

$$f(a),\ f(a+\omega),\ f(a+2\omega),\ f(a+3\omega),\ \ldots,\ f(a+5\omega),\ f(a+6\omega),$$

we take the line of the Pascal triangle

$$+1,\ \ -6,\ \ +15,\ \ -20,\ \ +(15),\ \ -6,\ \ +1,$$

and the Lagrangian interpolation formula is

$$\begin{aligned} f(a+4\omega) = &-\tfrac{1}{15} f(a) + \tfrac{6}{15} f(a+\omega) - \tfrac{15}{15} f(a+2\omega) + \tfrac{20}{15} f(a+3\omega) \\ &+ \tfrac{6}{15} f(a+5\omega) - \tfrac{1}{15} f(a+6\omega). \end{aligned}$$

One of the manifold uses of this case is in the extrapolation of an extra value at the end of a table if we have reasons for disregarding the oft-repeated warnings against the dangers of extrapolation.

4. Another useful formula is a modified Lagrange formula for inverse interpolation given by Karl Pearson in the Introduction to "Tables for Statisticians." † As there is an error in the formula as originally stated, I give here what appears to me to be the most useful form.

To interpolate for a value $f(a+\theta\omega)$ in the sequence $f(a-\omega)$, $f(a)$, $f(a+\omega)$, $f(a+2\omega)$ or, say, f_{-1}, f_0, f_1, f_2, take

$$\begin{aligned} A &= f_0 + f_1 - f_{-1} - f_2 \\ B &= 5f_1 - 3f_0 - f_{-1} - f_2 \\ C &= f(a+\theta\omega) - f_0, \end{aligned}$$

* Pointed out to me by my friend, Mr A. T. Doodson.

† Cambridge University Press, 1914. For another slightly different form of the rule see the "Errata" issued recently.

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and solve the quadratic

$$A\theta^2 - B\theta + 4C = 0.$$

As in all cases where the coefficient of the square term is small, this quadratic is most conveniently solved by the formula

$$\theta = \theta_0 \left(1 + \theta_0 \frac{A}{B} + 2\left(\theta_0 \frac{A}{B}\right)^2 + 5\left(\theta_0 \frac{A}{B}\right)^3 + \dots\right)$$

where $\theta_0 = \frac{4C}{B}$, the first approximation when A is neglected. The formula is obtained by taking the Lagrange formulae for interpolation between f_{-1}, f_0, f_1 and between f_0, f_1, f_2 and combining the two.

5. With regard to the general use of Lagrange's Interpolation Formula, there is a distinct want of a form for computation that will get rid of the array of brackets and fractions in which algebraically the formula is of necessity clothed. The advantage of systematic forms can be more readily appreciated in more complicated calculations such as the solution of equations that arise in the method of least squares and in the practice of harmonic analysis, where we may follow the methods of Gauss and of Runge and Whittaker, but even in this less formidable problem of Lagrangian interpolation the use of some such form as I proceed to give should not be altogether despised.

Suppose it be required to get the value of a function for value x of the argument when the values of the function are known at values a_1, a_2, a_3, a_4, a_5, a_6 of the argument. The case of six values is chosen merely for explicitness. The coefficients of the Lagrangian formula are

$$\frac{(x - a_2)(x - a_3) \dots (x - a_4)}{(a_1 - a_2)(a_1 - a_3) \dots (a_1 - a_6)}, \quad \dots .$$

A matrix of the differences required is drawn up as in the form, and the continued products C_x, C_1, C_2, ..., C_6 are then calculated, C_x being $(x - a_1)(x - a_2) \dots (x - a_6)$, and C_1, for instance, being $(a_1 - a_2)(a_1 - a_3) \dots\dots (a_1 - a_6)(x - a_1)$. The coefficients of the Lagrangian formula are then $\frac{C_x}{C_1}$, ..., and are

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immediately obtained by division. In the form the row $x-a$ is placed at the top so that the subtractions might be on the same basis throughout as regards sign, but it is perhaps more convenient to disregard the possibility of a wrong sign running through the resulting formula and place the $(x-a)$ row below the matrix. Advantage may also be taken of the skew symmetry of the matrix, and only one half of the differences entered.

The computation form is of course of greater value when the quantities involved are decimal fractions, but the above example has been chosen as being clearer for purposes of illustration than one with a large number of significant figures. It remains only to be emphasised that the number of significant figures (as distinct from places after the decimal) should throughout be kept in excess of what is ultimately wanted.

COMPUTATION FORM FOR LAGRANGIAN INTERPOLATION.

x	$x-a_1$	$x-a_2$	$x-a_3$	$x-a_4$	$x-a_5$	$x-a_6$	$C_x =$
	a_1	a_2	a_3	a_4	a_5	a_6	
a_1	——	a_2-a_1	a_3-a_1	a_4-a_1	a_5-a_1	a_6-a_1	
a_2	a_1-a_2	——	a_3-a_2	a_4-a_2	a_5-a_2	a_6-a_2	
a_3	a_1-a_3	a_2-a_3	——	a_4-a_3	a_5-a_3	a_6-a_3	
a_4	a_1-a_4	a_2-a_4	a_3-a_4	——	a_5-a_4	a_6-a_4	
a_5	a_1-a_5	a_2-a_5	a_3-a_5	a_4-a_5	——	a_6-a_5	
a_6	a_1-a_6	a_2-a_6	a_3-a_6	a_4-a_6	a_5-a_6	——	
	C_1	C_2	C_3	C_4	C_5	C_6	
Lagrangian Coefficients =	$\frac{C_x}{C_1}$	$\frac{C_x}{C_2}$	$\frac{C_x}{C_3}$	$\frac{C_x}{C_4}$	$\frac{C_x}{C_5}$	$\frac{C_x}{C_6}$	**Check: sum = +1**

Example :

If we put the example of § 3 in this form, we have

5	4	3	2	1	-1	-2	$Cx=4!2!$
	1	2	3	4	6	7	
1	—	1	2	3	5	6	
2	-1	—	1	2	4	5	
3	-2	-1	—	1	3	4	
4	-3	-2	-1	—	2	3	
6	-5	-4	-3	-2	—	1	
7	-6	-5	-4	-3	-1	—	
$C=$	$-6!$	$+5!1!$	$-4!2!$	$+3!3!$	$-1!5!$	$-6!$	
Lagrangian Coefficients $=$	$-\frac{4!2!}{6!}$	$+\frac{4!2!}{5!1!}$	$-\frac{4!2!}{4!2!}$	$+\frac{4!2!}{3!3!}$	$+\frac{4!2!}{1!5!}$	$-\frac{4!2!}{6!}$	

The connection with the rule given in § 3 is evident when the appropriate binomial coefficients are written down :—

$$\frac{6!}{6!}-\frac{6!}{1!5!}+\frac{6!}{2!4!}-\frac{6!}{3!3!}\left(+\frac{6!}{4!2!}\right)-\frac{6!}{5!1!}+\frac{6!}{6!}\,.$$

Moreover, there is nothing in the general rule that is not really contained in this particular case, and further proof is unnecessary algebra.