CHARACTERIZATION OF CERTAIN RIEMANN SPACES BY DEVELOPMENT

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The purpose of this paper is to characterize the Riemann space whose line element is given by

(1) $$ds^2 = a(x^i)^2 g_{\alpha\beta}(x^\tau) dx^\alpha dx^\beta + g_{\lambda\mu}(x^i) dx^\lambda dx^\mu.$$

Through the whole description we use the indices i, j, α, β, γ, λ, μ, ν when they run as follows

$$i, j \quad = 1, 2, \ldots, n$$
$$\alpha, \beta, \gamma = 1, 2, \ldots, k$$
$$\lambda, \mu, \nu = k+1, k+2, \ldots, n$$

and $a(x^i)$ and $g_{\lambda\mu}(x^i)$ shall mean to be functions of $x^1, \ldots, x^n$, $g_{\alpha\beta}(x^\tau)$ a function of $x^1, \ldots, x^k$ and $g_{\lambda\mu}(x^\nu)$ a function of $x^{k+1}, \ldots, x^n$. This Riemann space (1) contains as special cases many important spaces such as a directly decomposable space, a conformally separable space and a space with torse-forming vector field. K. Yano [1] characterized a conformally separable Riemann space by umbilical surfaces contained in it and the proof of Theorem 1 of [1] leads to the following:

THEOREM. *The necessary and sufficient condition for an n-dimensional Riemann space to have an arc-element given by (1) is that it has n−k-parametric family of k-dimensional totally umbilic surfaces and k-parametric family of n−k-dimensional surfaces which are orthogonal to the former.*

We characterize this space (1) from another point of view.

1. We begin by determining the Riemannian connection, namely the euclidean connection without torsion, attached to (1). We can take k Pfaffians

(2) $$\omega_\alpha = p_{\alpha\beta}(x^\tau) dx^\beta$$

such that

$$g_{\alpha\beta}(x^\tau) dx^\alpha dx^\beta = \sum (\omega_\alpha)^2.$$

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Then, according to E. Cartan, Pfaffians $\omega_{\alpha\beta}$ satisfying the relations

(3) $\omega_\alpha' = [\omega_\beta \omega_{\beta\alpha}], \qquad \omega_{\alpha\beta} = -\omega_{\beta\alpha}$

are uniquely determined.   Next we take

(4) $\pi_\lambda = p_{\lambda\mu}(x^i) dx^\mu$

such that

$$g_{\lambda\mu}(x^i) dx^\lambda dx^\mu = \sum (\pi_\lambda)^2.$$

Putting

(5) $\pi_\alpha = a(x^i)\omega_\alpha$

we get from (1)

$$ds^2 = \sum (\pi_i)^2.$$

For these π_i there exist uniquely determined Pfaffians π_{ij} such that

(6) $\pi_i' = [\pi_j, \pi_{ji}], \qquad \pi_{ij} = -\pi_{ji}.$

We get by (3) and (5)

$$\pi_\alpha' = (a\omega_\alpha)' = [da, \omega_\alpha] + a\omega_\alpha' = [da, \omega_\alpha] + a[\omega_\beta, \omega_{\beta\alpha}].$$

Then by (6)

$$[da, \omega_\alpha] + a[\omega_\beta, \omega_{\beta\alpha}] = [\pi_\beta, \pi_{\beta\alpha}] + [\pi_\lambda, \pi_{\lambda\alpha}].$$

Taking (5) into consideration we have

$$\left[\delta_{\alpha\beta}\frac{da}{a} + \omega_{\alpha\beta} - \pi_{\alpha\beta}, \ \pi_\beta\right] + [-\pi_{\alpha\lambda}, \ \pi_\lambda] = 0.$$

As π_β's and π_λ's are linearly independent, we can put

(7) $\delta_{\alpha\beta}\dfrac{da}{a} + \omega_{\alpha\beta} - \pi_{\alpha\beta} = C_{\alpha\beta i}\pi_i$

(8) $-\pi_{\alpha\lambda} = C_{\alpha\lambda i}\pi_i$

where

(9) $C_{\alpha ij} = C_{\alpha ji}.$

As $\omega_{\alpha\beta}$'s and $\pi_{\alpha\beta}$'s are skew symmetric with respect to indices α and β we get for $\alpha \neq \beta$

(10) $C_{\alpha\beta i} + C_{\beta\alpha i} = 0 \quad (\alpha \neq \beta).$

For $\alpha \neq \beta \neq \gamma$ we get by (9) and (10)

$$C_{\alpha\beta\gamma} = -C_{\beta\alpha\gamma} = -C_{\beta\gamma\alpha} = C_{\gamma\beta\alpha} = C_{\gamma\alpha\beta} = -C_{\alpha\gamma\beta} = -C_{\alpha\beta\gamma}.$$

Hence

(11) $C_{\alpha\beta\gamma} = 0 \quad (\alpha \neq \beta \neq \gamma).$

For $\alpha = \beta$ we obtain by (7)

$$\frac{da}{a} = C_{\alpha\alpha i}\pi_i.$$

As π_i's are linearly independent da/a can be written uniquely as a linear combination of π_i's. So $C_{\alpha\alpha i}$'s depend only on i and not on α. We put $C_{\alpha\alpha i} = C_i$ and we have

$$(12) \qquad \frac{da}{a} = C_i\pi_i.$$

By (9) and (10)

$$(13) \qquad C_{\alpha\beta\beta} = -C_{\beta\alpha\beta} = -C_{\beta\beta\alpha} = -C_\alpha = -C_{\alpha\alpha\alpha} \quad (\alpha \neq \beta),$$

and by (7)

$$\pi_{\alpha\beta} = \omega_{\alpha\beta} - C_{\alpha\beta i}\pi_i.$$

Taking (11) and (13) into consideration we have

$$(14) \qquad \pi_{\alpha\beta} = \omega_{\alpha\beta} - C_\beta\pi_\alpha + C_\alpha\pi_\beta - C_{\alpha\beta\lambda}\pi_\lambda.$$

Next by (6) and (8) we have

$$\begin{aligned}
\pi_\lambda' &= [\pi_\alpha, \pi_{\alpha\lambda}] + [\pi_\mu, \pi_{\mu\lambda}] \\
&= [\pi_\alpha, -C_{\alpha\lambda\beta}\pi_\beta - C_{\alpha\lambda\nu}\pi_\nu] + [\pi_\mu, \pi_{\mu\lambda}] \\
&= -C_{\alpha\lambda\beta}[\pi_\alpha, \pi_\beta] - C_{\alpha\lambda\nu}[\pi_\alpha, \pi_\nu] + [\pi_\mu, \pi_{\mu\lambda}].
\end{aligned}$$

Considering (2) and (4) and remarking that the terms $[dx^\alpha, dx^\beta]$ appear only in $C_{\alpha\lambda\beta}[\pi_\alpha, \pi_\beta]$ we get $C_{\alpha\lambda\beta} = C_{\beta\lambda\alpha}$. Then by (9) $C_{\alpha\beta\lambda} = C_{\beta\alpha\lambda}$, while according to (10) $C_{\alpha\beta\lambda} + C_{\beta\alpha\lambda} = 0$, and we get

$$C_{\alpha\beta\lambda} = 0 \quad (\alpha \neq \beta).$$

Then by (8)

$$\begin{aligned}
\pi_{\alpha\lambda} &= -C_{\alpha\lambda i}\pi_i = -C_{\alpha\lambda\beta}\pi_\beta - C_{\alpha\lambda\mu}\pi_\mu \\
&= -C_{\alpha\lambda\alpha}\pi_\alpha - C_{\alpha\lambda\mu}\pi_\mu \\
&= -C_{\alpha\alpha\lambda}\pi_\alpha - C_{\alpha\lambda\mu}\pi_\mu.
\end{aligned}$$

As $C_{\alpha\alpha\lambda} = C_\lambda$ we have

$$(15) \qquad \pi_{\alpha\lambda} = -C_\lambda\pi_\alpha - C_{\alpha\lambda\mu}\pi_\mu.$$

Thus we obtain the following relations by (12), (14), (15) and $C_{\alpha\beta\lambda} = 0$;

$$(A) \qquad \begin{cases} \pi_{\alpha\beta} = \omega_{\alpha\beta} - C_\beta\pi_\alpha + C_\alpha\pi_\beta \\ \pi_{\alpha\lambda} = -C_\lambda\pi_\alpha - C_{\alpha\lambda\mu}\pi_\mu \\ \dfrac{da}{a} = C_i\pi_i. \end{cases}$$

2. Now we introduce a Riemannian connection into our space by

$$d\mathbf{A} = \pi_i\mathbf{e}_i, \qquad d\mathbf{e}_i = \pi_{ij}\mathbf{e}_j.$$

and develop the tangent euclidean spaces on a euclidean space. Then for a point $\overline{A} = A + t_\lambda e_\lambda$ we have

(16)
$$d\overline{A} = dA + dt_\lambda e_\lambda + t_\lambda de_\lambda = \pi_i e_i + dt_\lambda e_\lambda + t_\lambda \pi_{\lambda i} e_i$$

$$= (\pi_\alpha + t_\lambda \pi_{\lambda\alpha}) e_\alpha + (\pi_\lambda + dt_\lambda + t_\mu \pi_{\mu\lambda}) e_\lambda.$$

When we develop along the k-dimensional surfaces $x^\lambda = $ const, we have by (4) $\pi_\lambda = 0$. Then $dA = \pi_\alpha e_\alpha$ and

$$\pi_\alpha + t_\lambda \pi_{\lambda\alpha} = \pi_\alpha + t_\lambda (C_\lambda \pi_\alpha + C_{\alpha\lambda\mu} \pi_\mu) = (1 + C_\lambda t_\lambda) \pi_\alpha.$$

Now we assume $a(x^i)$ contains at least one x^λ. In this case, owing to (12), there exists among C_λ's at least one C_λ which is not zero, and consequently there exist t_λ's such that

(17)
$$1 + C_\lambda t_\lambda = 0.$$

When we develop along k-dimensional surfaces $x^\lambda = $ const, we get for the point $\overline{A}$ satisfying (17)

(18)
$$d\overline{A} = (dt_\lambda + t_\mu \pi_{\mu\lambda}) e_\lambda.$$

The points $\overline{A} = A + t_\lambda e_\lambda$ for t_λ's satisfying (17) generate $n-k-1$-dimensional plane P_0 in the tangent euclidean space at A of the Riemann space (1). (18) indicates that when we develop along the k-dimensional surfaces $x^\lambda = $ const A is a fixed point or describes a curve touching to the k-dimensional plane P spanned by the vectors $e_{k+1}, \ldots, e_n$ with A as their origin. Now we take e_n on the perpendicular from A to P_0. Then any point $\overline{A}$ on P_0 can be represented by

$$\overline{A} = A + t_p e_p + t e_n$$

where p runs from $k+1$ to $n-1$ and t_p's are arbitrary numbers, while t is a fixed function of x^i's. For the frame thus chosen the relations (A) hold and along surfaces $x^\lambda = $ const we have (18) and (17), and so

$$1 + C_\lambda t_\lambda = 1 + C_p t_p + C_n t = 0$$

holds for any t_p's $(p = k+1, \ldots, n-1)$. Thus

$$C_p = 0, \qquad C_n = -\frac{1}{t}.$$

Hence we get from (A)

(B)
$$\begin{cases} \pi_{\alpha p} = -C_{\alpha p\mu} \pi_\mu, \qquad \pi_{\alpha n} = \dfrac{\pi_\alpha}{t} - C_{\alpha n\mu} \pi_\mu \\ \dfrac{da}{a} = -\dfrac{\pi_n}{t} + C_\alpha \pi_\alpha. \end{cases}$$

If we put $x^\alpha = $ const, we get $\pi_\alpha = 0$ and so $dA = \pi_\lambda e_\lambda$ and $\dfrac{da}{a} = -\dfrac{\pi_n}{t}$. Thus we obtain

THEOREM 1. *Let the square of arc-element of a Riemann space be given by* (1), *where $a(x^i)$ actually contains at least one x^λ. Then there exist in the tangent euclidean space at any point A of the Riemann space an $n-k$-dimensional plane P through A and an $n-k-1$-dimensional plane P_0 lying on P and not passing through A, which have the following property;*

1°. *If we develop along the k-dimensional surfaces $x^\lambda = const$, A describes an arc perpendicular to P at every instant and any point on P_0 is either a fixed point or describes an arc touching to P.*

2°. *If we develop along $n-k$-dimensional surfaces $x^a = const$, A describes an arc touching to P, and when we denote the length of the normal from A to P_0 by t and the orthogonal component of an arc-element of A in the direction of this normal by π_n, π_n/t is a total differential of a certain function of x^λ's.*

3. Before stating the inverse of theorem 1 we prove a lemma.

LEMMA. Let ω_α ($\alpha = 1, \ldots, k$) be linearly independent Pfaffians with n variables. If there exist Pfaffians $\omega_{\alpha\beta}$ such that

$$\omega'_\alpha = [\omega_\beta, \omega_{\beta\alpha}], \qquad \omega_{\alpha\beta} = -\omega_{\beta\alpha},$$

then by a suitable choice of variables x^i ($i = 1, \ldots, n$) we have $\omega_\alpha = p_{\alpha\beta}(x^i)dx^\beta$, and for these variables $\sum(\omega_\alpha)^2$ does not contain x^λ ($\lambda = k+1, \ldots, n$), namely $\sum(\omega_\alpha)^2 = g_{\alpha\beta}(x^\tau)dx^\alpha dx^\beta$.

The proof, which is quite natural, runs as follows. By Frobenius's theorem [2] p. 193 variables stated in the lemma exist. Writing the condition $\omega'_\alpha = [\omega_\beta, \omega_{\beta\alpha}]$ more fully we have

$$d\omega_\alpha(\delta) - \delta\omega_\alpha(d) = \omega_\beta(d)\omega_{\beta\alpha}(\delta) - \omega_\beta(\delta)\omega_{\beta\alpha}(d).$$

Let the symbols d and δ be such that

$$(dx^1, \ldots, dx^k, 0, \ldots, 0) \quad \text{and} \quad (0, \ldots, 0, \delta x^{k+1}, \ldots, \delta x^n).$$

Then

$$-\delta\omega_\alpha(d) = \omega_\beta(d)\omega_{\beta\alpha}(\delta).$$

And we have

$$\delta\left(\sum(\omega_\alpha)^2\right) = 2\sum\omega_\alpha\delta\omega_\alpha = -2\sum\omega_\alpha\omega_\beta\omega_{\beta\alpha} = 0.$$

This shows $\sum(\omega_\alpha)^2$ does not contain x^λ.

Now the inverse of theorem 1 is

THEOREM 2. *If an n-dimensional Riemann space has coordinate system x^i in which the conditions 1° and 2° in theorem 1 hold, then the square of the arc-element of the space is of the form* (1), *$a(x^i)$ containing at least one of x^λ's.*

Proof. In the tangent euclidean space at A of the Riemann space we take a component e_n of an orthonormal system of vectors $e_1, \ldots, e_n$ on the perpendicular from A to P_0, and components e_λ ($\lambda = k+1, \ldots, n$) on P. Let the Riemannian connection be given by $dA = \pi_i e_i$, $de_i = \pi_{ij} e_j$. By the condition 1° we have $dA = \pi_\alpha e_\alpha$, which leads to $\pi_\lambda = 0$. So in general

$$(19) \qquad \pi_\lambda = p_{\lambda\mu}(x^i) dx^\mu.$$

For $\overline{A} = A + t_\lambda e_\lambda$ we have by (16)

$$d\overline{A} = (\pi_\alpha + t_\lambda \pi_{\lambda\alpha}) e_\alpha + (\pi_\lambda + dt_\lambda + t_\mu \pi_{\mu\lambda}) e_\lambda.$$

The latter half of 1° shows that $d\overline{A}$ does not contain e_α for $x^\lambda = $ const and so

$$\pi_\alpha + t_\lambda \pi_{\lambda\alpha} = 0.$$

As $\overline{A}$ moves on P_0, $t_n = t$ is constant, while $t_{k+1}, \ldots, t_{n-1}$ vary freely. So $\pi_\alpha + t\pi_{n\alpha} = 0$, $\pi_{\alpha p} = 0$ ($p = k+1, \ldots, n-1$). In general owing to (19) we obtain for suitable $C_{\alpha\lambda\mu}$

$$(20) \qquad \pi_{\alpha\lambda} = \delta_{n\lambda} \frac{\pi_\alpha}{t} - C_{\alpha\lambda\mu} \pi_\mu.$$

From the condition 2° $dA = \pi_\lambda e_\lambda$ along the surfaces $x^\alpha = $ const, and so $\pi_\alpha = 0$. In general

$$(21) \qquad \pi_\alpha = p_{\alpha\beta}(x^i) dx^\beta.$$

By the latter half of 2° for $x^\alpha = $ const there exists a function $a = a(x^i)$ such that $\pi_n / t = -da/a$. So in general

$$(22) \qquad \frac{\pi_n}{t} = -\frac{da}{a} + C_\beta \pi_\beta$$

where $a = a(x^i)$ and C_β's are suitable functions.

Let $\omega_\alpha = \pi_\alpha / a$. Then

$$\pi'_\alpha = [da, \ \omega_\alpha] + a\omega'_\alpha = \left[da, \ \frac{\pi_\alpha}{a} \right] + a\omega'_\alpha$$

$$= \left[\frac{da}{a}, \ \pi_\alpha \right] + a\omega'_\alpha = \left[-\frac{\pi_n}{t} + C_\beta \pi_\beta, \ \pi_\alpha \right] + a\omega'_\alpha.$$

By (20)

$$[\pi_i, \ \pi_{i\alpha}] = [\pi_\beta, \ \pi_{\beta\alpha}] + [\pi_\lambda, \ \pi_{\lambda\alpha}]$$

$$= [\pi_\beta, \ \pi_{\beta\alpha}] + \left[\pi_\lambda, \ C_{\alpha\lambda\mu} \pi_\mu - \delta_{n\lambda} \frac{\pi_\alpha}{t} \right]$$

$$= [\pi_\beta, \ \pi_{\beta\alpha}] + C_{\alpha\lambda\mu} [\pi_\lambda, \ \pi_\mu] - \frac{1}{t} [\pi_n, \ \pi_\alpha].$$

As $\pi'_\alpha = [\pi_i \pi_{i\alpha}]$, we have

$$(23) \qquad a\omega'_\alpha = [\pi_\beta, \ \pi_{\beta\alpha}] - C_\beta [\pi_\beta, \ \pi_\alpha] + C_{\alpha\lambda\mu} [\pi_\lambda, \ \pi_\mu].$$

Here by (21) and $\pi_\alpha = a\omega_\alpha$ there are not terms $[dx^\lambda, dx^\mu]$ on the left side $a\omega'_\alpha$ $=q_{ij}[dx^i, dx^j]$, while on the right side $C_{\alpha\lambda\mu}[\pi_\lambda, \pi_\mu] = r_{\lambda\mu}[dx^\lambda, dx^\mu]$. So the last terms must be zero, namely $C_{\alpha\lambda\mu} = C_{\alpha\mu\lambda}$. We have by (23)

$$\omega'_\alpha = [\omega_\beta, \pi_{\beta\alpha}] - C_\beta[\omega_\beta, \pi_\alpha].$$

Then putting

(24) $$\omega_{\beta\alpha} = \pi_{\beta\alpha} - C_\beta\pi_\alpha + C_\alpha\pi_\beta$$

we have

$$\omega'_\alpha = [\omega_\beta, \omega_{\beta\alpha}].$$

As $\omega_{\alpha\beta}$ are shew symmetric with respect to indices α and β, we have by lemma $\sum(\omega_\alpha)^2 = g_{\alpha\beta}(x^\tau)dx^\alpha dx^\beta$. Thus

$$ds^2 = \sum(\pi_i)^2 = \sum(\pi_\alpha)^2 + \sum(\pi_\lambda)^2 = \sum(a\omega_\alpha)^2 + \sum(\pi_\lambda)^2$$
$$= a(x^i)^2 g_{\alpha\beta}(x^\tau)dx^\alpha dx^\beta + g_{\lambda\mu}(x^i)dx^\lambda dx^\mu.$$

By (22) $a(x^i)$ contains at least one of x^λ's.

4. Now we return to **2** and treat the case in which $a(x^i)$ in (1) does not contain any x^λ. This case reduces to the case $a(x^i) = 1$. When $a(x^i) = 1$, we have by (12) $C_i = 0$ and by (15) $\pi_{\alpha\lambda} = -C_{\alpha\lambda\mu}\pi_\mu$. Along surfaces $x^\lambda = $const we have $\pi_{\alpha\lambda} = 0$ and consequently $d\mathbf{e}_\alpha = \pi_{\alpha\beta}\mathbf{e}_\beta$. Hence

THEOREM 3. *If the square of arc element of an n-dimensional Riemann space is for a suitably chosen coordinate system x^i*

$$ds^2 = g_{\alpha\beta}(x^\tau)dx^\alpha dx^\beta + g_{\lambda\mu}(x^i)dx^\lambda dx^\mu,$$

then there exists in a tangent euclidean space at any point **A** *of the Riemann space an $n-k$-dimensional plane P satisfying the following conditions:*

*$1°$. If we develop along k-dimensional surfaces $x^\lambda = $const, **A** describes an arc which is perpendicular to P at any instant, and P moves parallel to a fixed $n-k$-dimensional plane.*

*$2°$. If we develop along $n-k$-dimensional surfaces $x^\alpha = $const, **A** describes an arc which is parallel to P.*

The inverse of this theorem is also true.

THEOREM 4. *If two conditions $1°$ and $2°$ in theorem 3 hold for suitably chosen coordinates x^i of an n-dimensional Riemann space, then the square of arc element of our space is of the form*

$$ds^2 = g_{\alpha\beta}(x^\tau)dx^\alpha dx^\beta + g_{\lambda\mu}(x^i)dx^\lambda dx^\mu.$$

Proof. In the tangent euclidean space at **A** of our Riemann space we take $\mathbf{e}_\lambda$ $(\lambda = k+1, \ldots, n)$ on the plane P. Then by the condition $1°$ we have $\pi_\lambda = 0$

along surfaces $x^\lambda = $ const, and in general

(25) $$\pi_\lambda = p_{\lambda\mu}(x^i)dx^\mu.$$

The latter half of 1° indicates $d\mathbf{e}_\lambda = \pi_{\lambda\mu}\mathbf{e}_\mu$ along $x^\lambda = $ const and consequently $\pi_{\alpha\lambda} = 0$. So in general

(26) $$\pi_{\alpha\lambda} = -C_{\alpha\lambda\mu}\pi_\mu.$$

By the condition 2°

(27) $$\pi_\alpha = p_{\alpha\beta}(x^i)dx^\beta.$$

By (26) and (6)

$$\pi'_\alpha = [\pi_\beta, \pi_{\beta\alpha}] + [\pi_\lambda, \pi_{\lambda\alpha}] = [\pi_\beta, \pi_{\beta\alpha}] + C_{\alpha\lambda\mu}[\pi_\lambda, \pi_\mu].$$

Taking (25) and (27) into consideration we get $C_{\alpha\lambda\mu}[\pi_\lambda, \pi_\mu] = 0$ and we have $\pi'_\alpha = [\pi_\beta, \pi_{\beta\alpha}]$ and by lemma $\sum(\pi_\alpha)^2 = g_{\alpha\beta}(x^\tau)dx^\alpha dx^\beta$.

Remark. In place of 1° and 2° in theorem 4 we take the following two conditions:

1°. If we develop along k-dimensional surfaces $x^\lambda = $ const, P moves parallel to a fixed $n-k$-dimensional plane.

2°. If we develop along $n-k$-dimensional surfaces $x^\alpha = $ const, P moves parallel to a fixed $n-k$-dimensional plane.

Then by 1° $\pi_{\alpha\lambda} = 0$ for $x^\lambda = $ const and by 2° $\pi_{\alpha\lambda} = 0$ for $x^\alpha = $ const. So in general $\pi_{\alpha\lambda} = 0$, and by (6)

$$\pi'_\alpha = [\pi_i, \pi_{i\alpha}] = [\pi_\beta, \pi_{\beta\alpha}] + [\pi_\lambda, \pi_{\lambda\alpha}] = [\pi_\beta, \pi_{\beta\alpha}]$$
$$\pi'_\lambda = [\pi_i, \pi_{i\lambda}] = [\pi_\alpha, \pi_{\alpha\lambda}] + [\pi_\mu, \pi_{\mu\lambda}] = [\pi_\mu, \pi_{\mu\lambda}].$$

Then by lemma $\sum(\pi_\alpha)^2 = g_{\alpha\beta}(y^\tau)dy^\alpha dy^\beta$ for suitably chosen coordinates $y^i = y^i(x^j)$ and $\sum(\pi_\lambda)^2 = g_{\lambda\mu}(z^\nu)dz^\lambda dz^\mu$ for suitably chosen coordinates $z^i = z^i(x^j)$. If we take $u^\alpha = y^\alpha(x^j)$, $u^\lambda = z^\lambda(x^j)$ as coordinates we get

$$ds^2 = \sum(\pi_i)^2 = g_{\alpha\beta}(u^\tau)du^\alpha du^\beta + g_{\lambda\mu}(u^\nu)du^\lambda du^\mu.$$

This space has been treated by many authors.

5. Now we treat the special case $k = n-1$. In this case the condition 2° in theorem 2 is unnecessary. The second part of 2°, namely the condition that π_n/t is a total differential for $x^\alpha = $ const, is always satisfied. We will prove the first part of 2° is also satisfied under the condition 1°. By 1 the equations (20) hold, which in this case reduce to $\pi_{\alpha n} = \pi_\alpha/t - C_{\alpha nn}\pi_n$. Consequently

$$\pi'_\alpha = [\pi_\beta, \pi_{\beta\alpha}] + [\pi_n, \pi_{n\alpha}] = [\pi_\beta, \pi_{\beta\alpha}] + \left[\pi_n, -\frac{\pi_\alpha}{t} + C_{\alpha nn}\pi_n\right]$$

$$= \left[\pi_\beta, \pi_{\beta\alpha} + \delta_{\beta\alpha}\frac{\pi_n}{t}\right]$$

$$\pi'_n = [\pi_\alpha,\ \pi_{\alpha n}] = \left[\pi_\alpha,\ \frac{\pi_\alpha}{t} - C_{\alpha n n}\pi_n\right] = C_{\alpha n n}[\pi_n,\ \pi_\alpha].$$

By Frobenius's theorem [2] p. 193 for suitably chosen coordinates $y^\alpha = y^\alpha(x^j)$ and $y^n = y^n(x^j)$ we have

$$\pi_\alpha = p_{\alpha\beta}(y^i)dy^\beta, \qquad \pi_n = p(y^i)dy^n.$$

As $\pi_n = 0$ for $x^n = $ const by (19), y^n is a function of only one variable x^n. For coordinate system y^i thus chosen the first part of 2° is satisfied, and we have

THEOREM 5. *The necessary and sufficient condition for an n-dimensional Riemann space to have an arc element which can be writen in the form*

$$ds^2 = a(x^i)^2 g_{\alpha\beta}(x^\tau)dx^\alpha dx^\beta + g_{nn}(x^i)(dx^n)^2$$

$$(\alpha,\ \beta,\ \gamma = 1, 2,\ \ldots,\ n-1)$$

is that there exists a point P_0 in the tangent euclidean space at any point A of the Riemann space which has the property that, if we develop along any hypersurface of one-parametric family of hypersurfaces, A describes an arc perpendicular to AP_0 and P_0 describes an arc which touches the straight line AP_0 or is a fixed point.

Especially we treat the case $g_{nn}(x^i) = g_{nn}(x^n)$, which reduces to $g_{nn}(x^i) = 1$. Then in 2 we have $\pi_n = dx^n$ and consequently $\pi'_n = [\pi_\alpha, \pi_{\alpha n}] = 0$.
By (B) we have $\pi_{\alpha n} = \pi_\alpha/t - C_{\alpha n n}\pi_n$. So $-C_{\alpha n n}[\pi_\alpha, \pi_n] = 0$ and hence $C_{\alpha n n} = 0$. Thus

$$(28) \qquad\qquad \pi_{\alpha n} = \frac{\pi_\alpha}{t}.$$

By (16) we have

$$(29) \qquad\qquad d(A + te_n) = (\pi_\alpha - t\pi_{\alpha n})e_\alpha + (\pi_n + dt)e_n.$$

So on account of (28)

$$(30) \qquad\qquad d(A + te_n) = (\pi_n + dt)e_n.$$

Thus by development the point $P_0 = A + te_n$ is a fixed point or describes an arc touching to AP_0.

Conversely we assume there is a point P_0 in the tangent euclidean space of any point A of Riemann space which has the property that by development P_0 is a fixed point or describes an arc touching to AP_0. Then (30) holds and (28) holds by virtue of (29). Consequently $\pi'_n = [\pi_{n\alpha}, \pi_\alpha] = 0$ and π_n is a total differential of a certain function. Putting $\pi_n = dx^n$ we proceed as in the proof of theorem 2. Hence

THEOREM 6. *The necessary and sufficient condition for a Riemann space to have an arc-element such that*

$$ds^2 = a(x^i)^2 g_{\alpha\beta}(x^\tau) dx^\alpha dx^\beta + (dx^n)^2 \qquad (\alpha, \beta, \gamma = 1, 2, \ldots, n-1)$$

is that there is a point P_0 in the tangent euclidean space at any point $\mathbf{A}$ of the Riemann space such that by development P_0 is a fixed point or describes an arc touching to $\mathbf{A}P_0$.

This case has been proved by K. Yano (the so-colled Riemann space with torse-forming vector field). If P_0 is a fixed point we have by (30) $\pi_n + dt = 0$ and then by 2 (B) $da/a = dt/t + C_\alpha \pi_\alpha$, namely $a = ct$, c being a function of x^α. So this case reduces to the case $a(x^i) = x^n$. The inverse is also true.

If $k = n-1$ in theorem 4, the condition $2°$ is also unnecessary. Hence

THEOREM 7. *The necessary and sufficient condition for the Riemann space to have an arc-element given by*

$$ds^2 = g_{\alpha\beta}(x^\tau) dx^\alpha dx^\beta + g_{nn}(x^i)(dx^n)^2 \qquad (\alpha, \beta, \gamma = 1, 2, \ldots, n-1)$$

is that there exists a direction P in the tangent euclidean space at any point of the Riemann space such that if we develop along any hypersurface of one-parametric family of hypersurfaces P moves parallel to a fixed direction.

The special case in which $g_{nn}(x^i) = g_{nn}(x^n)$ holds reduces to the case $g_{nn}(x^i) = 1$, which is treated in the remark of theorem 4.

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