BISIMPLE ω-SEMIGROUPS

by N. R. REILLY

(Received 19 July, 1965)

The structure of a bisimple inverse semigroup with an identity has been related by Clifford [2] to that of its right unit subsemigroup. In this paper we give an explicit structure theorem for bisimple inverse semigroups in which the idempotents form a simple descending chain

$$e_0 > e_1 > e_2 \ldots.$$

We call such a semigroup a *bisimple ω-semigroup*. The structure of a semigroup of this kind is shown to be determined entirely by its group of units and an endomorphism of its group of units.

These semigroups occur as subsemigroups of 0-simple semigroups with non-primitive idempotents [3, Theorem 2.54] and, since Rees [5] has obtained a structure theorem for 0-simple semigroups with primitive idempotents (that is, for completely 0-simple semigroups), the study of bisimple ω-semigroups seems a natural next step.

The results of Sections 2 and 3 of this paper can be obtained by combining the results of Clifford [2] with those of Rees [6], while the isomorphism theorem of the last section can be deduced from Warne's homomorphism theorem for bisimple inverse semigroups with an identity [7, Theorem 1.1]. However, we have favoured a more direct approach throughout.

Warne has also informed us that he had an equivalent form of Theorems 2.2 and 3.5 in terms of ordered quadruples at the time of submitting his paper [7].

1. Definitions and preliminaries. We shall use the terminology and notation of Clifford and Preston [3].

Two elements of a semigroup S are said to be $\mathscr{L}$- [$\mathscr{R}$-] equivalent if and only if they generate the same principal left [right] ideal of S. We write $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$ and $\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$. Then $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$ and $\mathscr{D}$ are equivalence relations on S such that $\mathscr{H} \subseteq \mathscr{L} \subseteq \mathscr{D}$ and $\mathscr{H} \subseteq \mathscr{R} \subseteq \mathscr{D}$. We call a semigroup *bisimple* if it contains only one $\mathscr{D}$-class.

If we denote by L_a the $\mathscr{L}$-class of a semigroup S containing the element a then we can define a partial ordering on the $\mathscr{L}$-classes by writing, for any two elements a, b of S, $L_a \leqq L_b$ [$L_a < L_b$] if and only if the principal left ideal generated by a is contained [strictly contained] in that generated by b. We can similarly denote by R_a the $\mathscr{R}$-class of S containing the element a and define a partial ordering of the $\mathscr{R}$-classes.

A *regular* semigroup is a semigroup S such that $a \in aSa$ for all elements a of S. An *inverse* semigroup is a semigroup S such that for every element a of S there exists a unique element x of S, called the *inverse* of a, such that $axa = a$ and $xax = x$. Then the following three conditions on a semigroup S are equivalent [3, Theorem 1.17]:

(1) S is regular and any two idempotents of S commute;

(2) every $\mathscr{L}$- [$\mathscr{R}$-]class of S contains a unique idempotent;

(3) S is an inverse semigroup.

https://doi.org/10.1017/S2040618500035346 Published online by Cambridge University Press

We denote the unique inverse of the element a of an inverse semigroup S by a^{-1}. Then aa^{-1} and $a^{-1}a$ are idempotents such that $(a^{-1}a, a) \in \mathscr{L}$ and $(aa^{-1}, a) \in \mathscr{R}$. If e is an idempotent, then $e^{-1} = e$. Also, for any elements a, b in S, we have [3, Lemma 1.18]

$$(a^{-1})^{-1} = a \quad \text{and} \quad (ab)^{-1} = b^{-1}a^{-1}.$$

Thus, if we write a^{-n} for $(a^{-1})^n$, then $(a^n)^{-1} = (a^{-1})^n = a^{-n}$.

2. Bisimple ω-semigroups. For any semigroup S we shall denote by E_S the set of idempotents of S. We define a partial ordering $\leqq$ on E_S by the rule that $e \leqq f$ if and only if $ef = e = fe$. If S is an inverse semigroup, then E_S is a commutative subsemigroup of S and any two elements of E_S have a greatest lower bound under the partial ordering. Let N denote the set of all non-negative integers. Then we say that a semigroup S is an *ω-semigroup* if and only if there exists a one-to-one mapping ϕ of E_S onto N such that, for any elements e, f of E_S, $e\phi \leqq f\phi$ if and only if $f \leqq e$. Thus, if S is an ω-semigroup, then we can write

$$E_S = \{e_m : m \in N\},$$

where $e_m \leqq e_n$ if and only if $m \geqq n$. In particular, E_S is totally ordered.

LEMMA 2.1. *Let S be a regular ω-semigroup. Then S is an inverse ω-semigroup with an identity. In particular, a bisimple ω-semigroup is a bisimple inverse ω-semigroup.*

Proof. Let S be a regular ω-semigroup and let e and f be any two idempotents of S. Then either $e \leqq f$ or $f \leqq e$; that is, either $ef = fe = e$ or $ef = fe = f$. In both cases $ef = fe$. Hence the idempotents of S commute and so S is an inverse semigroup. Let $E_S = \{e_m : m \in N\}$, where $e_m \geqq e_n$ if and only if $m \leqq n$. Let a be any element of S. Then $aa^{-1} = e_m$ for some idempotent e_m of S. Hence $e_0 a = e_0(e_m a) = (e_0 e_m)a = e_m a = a$. Similarly $ae_0 = a$. Thus e_0 is an identity. Now a bisimple semigroup containing an idempotent is a regular semigroup [3, Theorem 2.11 (i)] and so a bisimple ω-semigroup is necessarily a bisimple regular ω-semigroup, that is, a bisimple inverse ω-semigroup.

Example. We shall denote by B the bicyclic semigroup, which we can define to be $N \times N$ under the following multiplication. For any elements (m, n), (p, q) of $N \times N$,

$$(m, n)(p, q) = (m+p-r, n+q-r),$$

where $r = \min(n, p)$. Then B is a bisimple inverse semigroup [3, p. 45] and can readily be shown to be an ω-semigroup. The following theorem shows how we can generate bisimple ω-semigroups from any group and any endomorphism of that group.

THEOREM 2.2. *Let G be a group and α an endomorphism of G. Let*

$$S = S(G, \alpha) = \{[(m, n);\ g] \in B \times G:\ (m, n) \in B \text{ and } g \in G\}.$$

Define multiplication on S as follows:

$$[(m_1, n_1);\ g_1][(m_2, n_2);\ g_2] = [(m_1, n_1)(m_2, n_2);\ g_1\alpha^{m_2-r}g_2\alpha^{n_1-r}],$$

where $r = \min(n_1, m_2)$ and we take α^0 to be the identity automorphism of G. Then S is a bisimple ω-semigroup.

Proof. We first verify that the multiplication is associative. Let $[(m_i, n_i);\ g_i]$ $(i = 1, 2, 3)$ be any three elements of S. Then, with $r_1 = \min(n_1, m_2)$ and $r_2 = \min(n_1+n_2-r_1, m_3)$, we have

$$\begin{aligned}
&([(m_1, n_1);\ g_1][(m_2, n_2);\ g_2])[(m_3, n_3);\ g_3] \\
&\quad = [(m_1+m_2-r_1, n_1+n_2-r_1);\ g_1\alpha^{m_2-r_1}g_2\alpha^{n_1-r_1}][(m_3, n_3);\ g_3] \\
&\quad = [(m_1+m_2+m_3-r_1-r_2, n_1+n_2+n_3-r_1-r_2); \\
&\qquad\qquad g_1\alpha^{m_2+m_3-r_1-r_2}g_2\alpha^{n_1+m_3-r_1-r_2}g_3\alpha^{n_1+n_2-r_1-r_2}].
\end{aligned}$$

Similarly, if we write $r_3 = \min(n_2, m_3)$ and $r_4 = \min(n_1, m_2+m_3-r_3)$, then we have that

$$\begin{aligned}
&[(m_1, n_1);\ g_1]([(m_2, n_2);\ g_2][(m_3, n_3);\ g_3]) \\
&\quad = [(m_1+m_2+m_3-r_3-r_4, n_1+n_2+n_3-r_3-r_4); \\
&\qquad\qquad g_1\alpha^{m_2+m_3-r_3-r_4}g_2\alpha^{n_1+m_3-r_3-r_4}g_3\alpha^{n_1+n_3-r_3-r_4}].
\end{aligned}$$

Now, a straightforward verification, such as that used when establishing the associativity of the multiplication in B, will establish that $r_1+r_2 = r_3+r_4$. Hence the multiplication in $S(G, \alpha)$ is associative, Henceforth, for the sake of convenience, we shall adopt the more compact notation $(m;\ g;\ n)$ for the element $[(m, n);\ g]$ of S. For any element $(m;\ g;\ n)$ of S, we have

$$(m;\ g;\ n)(n;\ g^{-1};\ m)(m;\ g;\ n) = (m;\ g;\ n).$$

Hence S is a regular semigroup. The element $(m;\ g;\ n)$ will be an idempotent if and only if

$$(m;\ g;\ n) = (m;\ g;\ n)(m;\ g;\ n) = (2m-r;\ g\alpha^{m-r}g\alpha^{n-r};\ 2n-r),$$

where $r = \min(m, n)$. This is so if and only if $m = r = n$ and $g = g\alpha^0 g\alpha^0 = g^2$; that is, if and only if $m = n$ and $g = 1$, the identity of G. Thus $E_S = \{(m;\ 1;\ m):\ m \in N\}$. It is easily verified that $(m;\ 1;\ m) \leqq (n;\ 1;\ n)$ if and only if $m \geqq n$. Hence S is a regular ω-semigroup and so an inverse ω-semigroup. The inverse of the element $(m;\ g;\ n)$ is just the element $(n;\ g^{-1};\ m)$ and the unit group of S is $\{(0;\ g;\ 0) : g \in G\}$. From the multiplication it is readily verified that, for any elements $(m_1;\ g_1;\ n_1)$ and $(m_2;\ g_2;\ n_2)$ of S,

$$(m_1;\ g_1;\ n_1)\mathscr{R}(m_2;\ g_2;\ n_2) \text{ if and only if } m_1 = m_2;$$

https://doi.org/10.1017/S2040618500035346 Published online by Cambridge University Press

also

$$(m_1;\ g_1;\ n_1)\mathscr{L}(m_2;\ g_2;\ n_2) \text{ if and only if } n_1 = n_2.$$

Hence, if $(m;\ g;\ n)$ and $(p;\ h;\ q)$ are any elements of S, then

$$(m;\ g;\ n)\mathscr{R}(m;\ g;\ q) \quad \text{and} \quad (m;\ g;\ q)\mathscr{L}(p;\ h;\ q);$$

that is, $(m;\ g;\ n)\mathscr{D}(p;\ h;\ q)$. This completes the proof.

Note. If α is the zero endomorphism of G, that is, if α is such that $g\alpha = 1$ for all elements g of G, then $S(G, \alpha)$ is an extension of G of a type first discussed by Bruck [**1**]. Moreover, if in the above theorem we relax the conditions on G and allow G to be a [regular, inverse] semigroup then the construction in the theorem yields a [regular, inverse] semigroup.

3. The structure theorem. Let S be a bisimple ω-semigroup with $E_S = \{e_m;\ m \in N\}$, where $e_m \geqq e_n$ if and only if $n \geqq m$. Then, from Lemma 2.1, we know that e_0 is the identity of S. Let R_i [L_i] denote the $\mathscr{R}$- [$\mathscr{L}$-] class of S containing the idempotent e_i; that is, $R_i = R_{e_i}$ [$L_i = L_{e_i}$]. Since a bisimple ω-semigroup is an inverse semigroup, it follows that every $\mathscr{R}$- [$\mathscr{L}$-] class contains a unique idempotent. Hence the set of $\mathscr{R}$- [$\mathscr{L}$-] classes of S is just $\{R_i : i \in N\}$ [$\{L_i : i \in N\}$], where, since $e_iS \supset e_jS$ [$Se_i \supset Se_j$] if and only if $j > i$, we have $R_i > R_j$ [$L_i > L_j$] if and only if $j > i$. Let $H_{ij} = R_i \cap L_j$. Then H_{ij} is non-empty, for all i, j in N, since S is bisimple. In particular, $H_{0i} = R_0 \cap L_i$ is non-empty for all $i \in N$. Now R_0 is the right unit subsemigroup of S and so we can apply the following lemma ([**2**], Lemma 2.1) to R_0.

LEMMA 3.1. *Let P be the right unit subsemigroup of a bisimple semigroup S with an identity. Then two elements of P are $\mathscr{L}$-equivalent in P if and only if they are $\mathscr{L}$- equivalent in S.*

Hence the $\mathscr{L}$-equivalence classes of R_0 are just the sets of the form $R_0 \cap L_i$ $(i \in N)$. Let us denote the $\mathscr{L}$-class $R_0 \cap L_i$ of R_0 by $L(i)$. Then clearly $L(i) > L(j)$ if and only if $j > i$. Moreover e_0 is contained in $L(0)$. We have (as the left-right dual of [**6**], Lemma 3.2)

LEMMA 3.2. *Let T be a right cancellative semigroup with an identity and let $\{L(m) : m \in N\}$ be the set of $\mathscr{L}$-classes of T, where $L(m) > L(n)$ if and only if $n > m$. Let a be any element of $L(1)$. Then a^n is contained in $L(n)$.*

Now R_0 satisfies all the conditions of Lemma 3.2 and so, for any element a of $L(1) = R_0 \cap L_1$, we have that a^n is contained in $R_0 \cap L_n$. Moreover, in S, $(a^n, a^n a^{-n}) \in \mathscr{R}$ for all n in N. Also e_0 and a^n lie in R_0, an $\mathscr{R}$-class of S. Thus $e_0 = a^n a^{-n}$, since each $\mathscr{R}$-class of S contains a unique idempotent. Similarly, since a^n and e_n lie in L_n and $(a^{-n}a^n, a^n) \in \mathscr{L}$, it follows that $(a^{-n}a^n, e_n) \in \mathscr{L}$ and therefore that $e_n = a^{-n}a^n$. Thus we have

LEMMA 3.3. *For any element a in $L(1)$, $a^n a^{-n} = e_0$ and $a^{-n}a^n = e_n$.*

LEMMA 3.4. *Let a be any element of $L(1)$. Then every element of S can be written uniquely in the form $a^{-m}ga^n$, where m and n are elements of N and g is an element of H_{00}.*

https://doi.org/10.1017/S2040618500035346 Published online by Cambridge University Press

Proof. Let s be any element of S and suppose that $s \in H_{mn}$. Then $s \in R_m \cap L_n$ and so $ss^{-1} = e_m$ and $s^{-1}s = e_n$. Also, for $g = a^m s a^{-n}$, we have from Lemma 3.3 that

$$gg^{-1} = a^m s a^{-n} a^n s^{-1} a^{-m} = a^m s e_n s^{-1} a^{-m} = a^m s s^{-1} a^{-m} = a^m e_m a^{-m} = a^m a^{-m} = e_0;$$

similarly $g^{-1}g = e_0$. Hence $g \in H_{00}$. Moreover, $a^{-m} g a^n = a^{-m} a^m s a^{-n} a^n = e_m s e_n = s$.

Now suppose that, for $x = a^{-m} g a^n$ and $y = a^{-r} h a^s$, where m, n, r, s are elements of N and g, h are elements of H_{00}, we have $x = y$. Then $xx^{-1} = yy^{-1}$, where $xx^{-1} = a^{-m} g a^n a^{-n} g^{-1} a^m = a^{-m} g e_0 g^{-1} a^m = a^{-m} e_0 a^m = e_m$ and, similarly, $yy^{-1} = e_r$. Thus $e_m = e_r$ and so $m = r$. Similarly $n = s$. Now $a^{-m} g a^n = a^{-m} h a^n$ implies that $g = e_0 g e_0 = a^m a^{-m} g a^n a^{-n} = a^m a^{-m} h a^n a^{-n} = e_0 h e_0 = h$.

This completes the proof of the lemma.

We now select an element a from $L(1)$ and keep this element fixed throughout the following discussion. Let $G = H_{00} = R_0 \cap L_0 = L(0)$. Then G is the unit group of S and also of R_0. Hence [**6**, p. 108] the equation

$$ag = (g\alpha)a \quad (g \in G)$$

defines an endomorphism α of G. Taking inverses we find that, for all elements g of G,

$$g^{-1}a^{-1} = a^{-1}(g\alpha)^{-1} = a^{-1}(g^{-1}\alpha);$$

that is, $ha^{-1} = a^{-1}(h\alpha)$ for all elements h of G.

We now define a mapping ϕ of S into $S(G, \alpha)$. For any element s of S we write

$$s\phi = (a^{-m} g a^n)\,\phi = (m;\ g;\ n),$$

where s is contained in H_{mn} and $g = a^m s a^{-n}$. From Lemma 3.4 it follows that ϕ is well-defined and is a bijection. To show that ϕ is a homomorphism, let $x = a^{-m} g a^n$ and $y = a^{-p} h a^q$ be any two elements of S.

Assume first that $n \geqq p$. Then

$$xy = a^{-m} g a^{n-p} \,.\, a^p a^{-p} \,.\, h a^q = a^{-m} g a^{n-p} e_0 h a^q$$
$$= a^{-m} g (a^{n-p} h) a^q = a^{-m} g (h\alpha^{n-p}) a^{n-p+q}.$$

Similarly, if $n \leqq p$, then

$$xy = a^{-m} g \,.\, a^n a^{-n} \,.\, a^{-(p-n)} h a^q = a^{-m-p+n} (g\alpha^{p-n}) h a^q.$$

Thus

$$(xy)\phi = \begin{cases} (m;\ g(h\alpha)^{n-p};\ n+q-p) & \text{if } n \geqq p, \\ (m+p-n;\ (g\alpha^{p-n})h;\ q) & \text{if } n \leqq p, \end{cases}$$
$$= (m;\ g;\ n)(p;\ h;\ q) = (x\phi)(y\phi).$$

Thus S is isomorphic with $S(G, \alpha)$.

Hence we have established the following theorem.

THEOREM 3.5. *Let S be a bisimple ω-semigroup with group of units G. Then there exists an endomorphism α of G such that S is isomorphic with $S(G, \alpha)$, where $S(G, \alpha)$ is defined as in the statement of Theorem* 2.2.

4. The isomorphism theorem.

THEOREM 4.1. *Let $S_1 = S(G_1, \alpha)$ and $S_2 = S(G_2, \beta)$, where α and β are endomorphisms of the groups G_1 and G_2 respectively. Then there exists an isomorphism ϕ of S_1 onto S_2 if and only if there exists an isomorphism θ of G_1 onto G_2 such that $\alpha\theta = \theta\beta\lambda_z$, where, for some element z of G_2, λ_z is the inner automorphism of G_2 defined by $g\lambda_z = zgz^{-1}$.*

Proof. Let ϕ be an isomorphism of S_1 onto S_2. Then ϕ must induce a one-to-one order-preserving mapping of E_{S_1} onto E_{S_2}. Thus $(m;\ 1;\ m)\phi = (m;\ 1;\ m)$, for all m in N, where we have denoted the identities of both G_1 and G_2 by 1. For any element $a = (m;\ g;\ n)$ of S_1, let $a\phi = (m;\ g;\ n)\phi = (p;\ h;\ q) = b$, say. Now, $a^{-1}\phi = (a\phi)^{-1}$ and so

$$(aa^{-1})\phi = a\phi a^{-1}\phi = a\phi(a\phi)^{-1} = bb^{-1}.$$

Thus $(m;\ 1;\ m)\phi = (p;\ 1;\ p)$. Hence, by the above, $m = p$. Similarly $n = q$. Thus, for any element $(m;\ g;\ n)$ of S_1, we have $(m;\ g;\ n)\phi = (m;\ h;\ n)$, for some element h of G_1.

We define a mapping θ of G_1 into G_2 by $(0;\ g;\ 0)\phi = (0;\ g\theta;\ 0)$. Since ϕ is an isomorphism and must clearly map the unit group of S_1 onto the unit group of S_2, it follows that θ is a bijection. It is straightforward to verify that θ is also a homomorphism. Now suppose that $(0;\ 1;\ 1)\phi = (0;\ z;\ 1)$, for some element z of G_2. Then, for all g in G_1,

$$(0;\ g\alpha;\ 1)\phi = ((0;\ g\alpha;\ 0)(0;\ 1;\ 1))\phi = (0;\ g\alpha;\ 0)\phi(0;\ 1;\ 1)\phi$$

$$= (0;\ g\alpha\theta;\ 0)(0;\ z;\ 1) = (0;\ (g\alpha\theta)z;\ 1).$$

Also

$$(0;\ g\alpha;\ 1)\phi = ((0;\ 1;\ 1)(0;\ g;\ 0))\phi = (0;\ 1;\ 1)\phi(0;\ g;\ 0)\phi$$

$$= (0;\ z;\ 1)(0;\ g\theta;\ 0) = (0;\ z(g\theta\beta);\ 1).$$

Hence, for all elements g of G, we have $(g\alpha\theta)z = z(g\theta\beta)$; that is, $g\alpha\theta = z(g\theta\beta)z^{-1} = g\theta\beta\lambda_z$. Thus $\alpha\theta = \theta\beta\lambda_z$.

Conversely, suppose that there exists an isomorphism θ of G_1 onto G_2 such that $\alpha\theta = \theta\beta\lambda_z$ for some element z of G_2. Then $\alpha^p\theta = \theta(\beta\lambda_z)^p$, for all p in N. We define a mapping ϕ of S_1 into S_2 as follows: for any element $(m;\ g;\ n)$ of S_1 we write

$$(m;\ g;\ n)\phi = (1;\ z^{-1};\ 0)^m(0;\ g\theta;\ 0)(0;\ z;\ 1)^n.$$

Now, in S_2, the element $(0;\ z;\ 1)$ is contained in $R_0 \cap L_1 = L(1)$ and so if we write $a = (0;\ z;\ 1)$ and apply Lemma 3.4 to S_2 then we see that ϕ is necessarily a bijection.

Now let $(m;\ g;\ n)$ and $(p;\ h;\ q)$ be any two elements of S_1. Then, for $n \geqq p$, we have

$$(m;\ g;\ n)\phi(p;\ h;\ q)\phi = (1;\ z^{-1};\ 0)^m(0;\ g\theta;\ 0)(0;\ z;\ 1)^n(1;\ z^{-1};\ 0)^p(0;\ h\theta;\ 0)(0;\ z;\ 1)^q$$
$$= (1;\ z^{-1};\ 0)^m(0;\ g\theta;\ 0)(0;\ z;\ 1)^{n-p}(0;\ h\theta;\ 0)(0;\ z;\ 1)^q$$

and

$$\begin{aligned}(0;\ z;\ 1)^{n-p}(0;\ h\theta;\ 0) &= (0;\ z;\ 1)^{n-p-1}(0;\ z(h\theta\beta);\ 1)\\ &= (0;\ z;\ 1)^{n-p-1}(0;\ z(h\theta\beta)z^{-1}z;\ 1)\\ &= (0;\ z;\ 1)^{n-p-1}(0;\ h\theta\beta\lambda_z;\ 0)(0;\ z;\ 1)\\ &= \dots\dots\dots\dots\dots\dots\\ &= (0;\ h\theta(\beta\lambda_z)^{n-p};\ 0)(0;\ z;\ 1)^{n-p}.\end{aligned}$$

Thus

$$(m;\ g;\ n)\phi(p;\ h;\ q)\phi = (1;\ z^{-1};\ 0)^m(0;\ g\theta;\ 0)(0;\ h\theta(\beta\lambda_z)^{n-p};\ 0)(0;\ z;\ 1)^{n-p}(0;\ z;\ 1)^q$$
$$= (1;\ z^{-1};\ 0)^m(0;\ (g\theta)(h\theta(\beta\lambda_z)^{n-p});\ 0)(0;\ z;\ 1)^{n+q-p}.$$

On the other hand, for $n \geqq p$, we have

$$\begin{aligned}((m;\ g;\ n)(p;\ h;\ q))\phi &= (m;\ g(h\alpha^{n-p});\ n+q-p)\phi\\ &= (1;\ z^{-1};\ 0)^m(0;\ g\theta(h\alpha^{n-p}\theta);\ 0)(0;\ z;\ 1)^{n+q-p}\\ &= (1;\ z^{-1};\ 0)^m(0\ g\theta h\theta(\beta\lambda_z)^{n-p};\ 0)(0;\ z;\ 1)^{n+q-p}\\ &= ((m;\ g;\ n)\phi)((p;\ h;\ q)\phi).\end{aligned}$$

A similar argument holds for $n \leqq p$. Thus ϕ is an isomorphism. This completes the proof.

Note. Congruences on a bisimple ω-semigroup are considered in [**4**], and a generalisation of Theorem 4.1 is stated.

Let G be any group. Then we shall denote by $B \dot{\times} G$ the direct product of B and G; that is, $B \dot{\times} G = \{((m, n), g) : (m, n) \in B \text{ and } g \in G\}$ under componentwise multiplication. However, to conform with our present notation, we shall write $B \dot{\times} G = \{(m;\ g;\ n) : m, n \in N \text{ and } g \in G\}$. Then, for any elements $(m;\ g;\ n)$ and $(p;\ h;\ q)$ of $B \dot{\times} G$, we have

$$(m;\ g;\ n)(p;\ h;\ q) = (m+p-r;\ gh;\ n+q-r),$$

where $r = \min(n, p)$. Thus $B \dot{\times} G = S(G, \iota)$ where ι denotes the identity automorphism of G.

Corollary 4.2. *$S = S(G, \alpha)$ is isomorphic with $B \dot{\times} G$ if and only if α is an inner automorphism of G.*

Proof. Let S be isomorphic with $B \dot{\times} G = S(G, \iota)$. Then, by Theorem 4.1, there exists an automorphism θ of G and an element z of G such that

$$\alpha\theta = \theta\iota\lambda_z = \theta\lambda_z.$$

Thus, for all elements g of G, $g\alpha\theta = g\theta\lambda_z = z(g\theta)z^{-1}$. Hence

$$g\alpha = (z\theta^{-1})g(z^{-1}\theta^{-1}) = (z\theta^{-1})g(z\theta^{-1})^{-1} = g\lambda_{z\theta^{-1}}$$

for all elements of g of G. Thus $\alpha = \lambda_{z\theta^{-1}}$, an inner automorphism of G.

Conversely, if $\alpha = \lambda_z$ then, with $\theta = \iota$, we have $\alpha\theta = \lambda_z\iota = \lambda_z = \theta\iota\lambda_z$, and so, by Theorem 4.1, S is isomorphic with $S(G, \iota) = B \dot{\times} G$.

This paper is part of a Ph.D. thesis submitted in 1965 to the University of Glasgow. I would particularly like to thank Dr W. D. Munn for his invaluable advice and encouragement and also the Department of Scientific and Industrial Research for financial support.

REFERENCES

1. R. H. Bruck, *A survey of binary systems*, Ergebnisse der Math., Neue Folge, Vol. 20 (Berlin, 1958).

2. A. H. Clifford, A class of *d*-simple semigroups, *Amer. J. Math.* **75** (1953), 547–556.

3. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, American Mathematical Society Mathematical Surveys No. 7, Vol. 1 (Providence, R. I., 1961).

4. W. D. Munn and N. R. Reilly, Congruences on a bisimple ω-semigroup, *Proc. Glasgow Math. Assoc.* (to appear).

5. D. Rees, On semigroups, *Proc. Cambridge Philos. Soc.* **36** (1940), 387–400.

6. D. Rees, On the ideal structure of a semigroup satisfying a cancellation law, *Quart. J. Math. Oxford Ser.* (2) **19** (1948), 101–108.

7. R. J. Warne, Homomorphisms of *d*-simple inverse semigroups with identity, *Pacific J. Math.* **14** (1964), 1111–1122.

UNIVERSITY OF GLASGOW
GLASGOW, W. 2