An Introduction to Algebraic Topology by Andrew H. Wallace. Pergamon Press, New York, 1957. $6.80.

Algebraic Topology is generally considered a difficult subject, not only because of its nature, but also because of the lack of good up-to-date text-books on the subject. Even last year, when students asked me what is the best way to learn this subject, I had to confess that I do not know (adding half-seriously that they had better wait until I write a book). Today I would suggest Wallace's book in spite of the remarks at the end of this review. I am very enthusiastic about its pedagogical virtues, and I feel that it will gain friends for its subject in Canadian universities.

To be specific, the book under review is intended for graduate students, who are beginning the study of Topology and who would like to specialize in Algebraic Topology as soon as possible. The author states that the material can be covered in three semesters. Of course, the book is also suitable for a more mature mathematician working in a different field, and anxious to learn some Algebraic Topology.

An important feature of the book is that it does not even presuppose a knowledge of General Topology. The first three chapters (I. Introduction; II. Topological Spaces; III. Topological Properties of Spaces) give the necessary material from General Topology. In some parts of modern mathematics, definitions are more important than theorems (though the reviewer does not believe that this is always the case, as is claimed sometimes). In such cases motivation and illustrations of the definitions are very important for the beginner, who still does not know the fact (rather successfully hidden on the more elementary level) that the first step toward good comprehension is to examine special cases and to study examples. In this book a careful and detailed motivation is given with every definition; the reader also may check his understanding of the basic notions by working out the numerous problems of the book. (Some of the motivations given in the book are questionable, but this is probably not the author's fault.)

In introducing the notion of topological space, the author first studies properties of continuous functions, and arrives at

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the axioms of neighborhoods by abstraction, (1. Continuity and neighborhood; 2. The abstract concept of neighborhood). The definition of open sets and closed sets comes next. This is a more classical line (going back to Hausdorff) than the one followed by Bourbaki and Kelley in their well-known books; being nearer to the historical development, Wallace's presentation is probably easier to grasp for the beginner. Of course, these fifty-odd pages of General Topology are rudimentary, but in fact one does not need much in Algebraic Topology, the main topic of the book.

The fourth chapter is a very elementary account of the important and classical notion of fundamental group. This notion was probably somehow familiar to Riemann and Abel as it plays an important role in function theory; in full generality, it was introduced by Poincaré (via the notion of covering space, not mentioned in the book). Roughly stated the elements α of the fundamental group $\pi_1(X, x_0)$ of the space X are classes of loops beginning and ending at $x_0 \in X$, two loops belonging to the same class if and only if one can be continuously deformed into the other. In order to define the product $\beta\alpha$ of $\alpha, \beta \in \pi_1(X, x_0)$ we choose loops $f \in \alpha$, $g \in \beta$, then we describe first f then g and take the class of this "composed" loop which will be by definition $\beta\alpha$. The book gives the most important definitions related to this concept, establishes that $\pi_1(X, x_0)$ is a group and a topological invariant of the (connected) space X. Apart from this not much can be done in twenty-six pages with such a careful and detailed exposition (it is not proved, for example, that a continuous f: $X \to Y$ induces a homomorphism $\pi_1(X, x_0) \to \pi_1(Y, fx_0)$, although this would not be much more difficult than the proof of the topological invariance of the fundamental group) but at least the reader is well prepared to read a more detailed text-book (like the corresponding chapter of Seifert and Threlfall or Pontrjagin).

With the last five chapters (V. The Homology Groups; VI. Continuous mappings and the Homology Groups; VII. Barycentric Subdivision and Excision; VIII. The Homology Sequence; IX. Simplicial Complexes) we arrive at the main topic of the book; here the reader is supposed to have a certain knowledge of n-dimensional affine geometry. Homology groups are presented as multidimensional generalizations of the fundamental group. In a given space X, we consider subspaces called chains which are finite unions of continuous images of affine simplexes of dimension p. The boundary of such a chain will be the union of the boundaries of its simplexes, not counting the ones which are with opposite orientation on two p-simplexes. If we try to make this definition more precise, we see that in reality we have to consider not sub-spaces of X but linear combinations of maps of rectilinear simplexes; such a linear combination is then called a chain. A chain is called a cycle if its boundary is empty (i.e. the null chain). Two cycles are called homologous if their difference is the boundary of a chain. The set of all p-cycles homologous to a given one is called a homology class; the totality of p-th homology classes is the p-dimensional homology group $H_p(X)$ of the space X. This is, of course, the Eilenberg-Steenrod approach to the singular homology (introduced by Lefschetz). Without supposing a certain knowledge of homology theory, it would be difficult to describe the topics of these chapters of the book. Technically speaking, the author proves that the singular homology theory satisfies the Eilenberg-Steenrod axioms, and gives the most immediate corollaries, like the determination of the homology groups of cells and spheres. The connection with the more classical simplicial theory is established in the last chapter, where the author shows that the homology groups of a simplicial complex can be computed using simplicial chains only; the simplicial chain group is defined as the homology group of the p-skeleton modulo the (p-1)-skeleton. In plain terms, a simplicial complex is a space which is decomposed in a given, definite way into a finite number of simplexes. The aforementioned result means then that the homology groups of such a space can be computed using simplexes of the given decomposition only. This is not trivial at all, because it means, in particular, that the homology groups derived from one decomposition of a space are topological invariants of that space. Also, this result shows that for a reasonable simple space the homology groups are generated by a finite number of elements.

As we have already said, we encourage everybody to read this book, but we add one word of caution for the beginner. A more complete development of the singular homology theory runs somehow into trouble for two reasons (at least): a) some of the most important applications of Algebraic Topology are not accesible to this method; b) many applications, even elementary ones, are very cumbersome. (For beginners, the reviewer would prefer an introduction to some other sort of homology theory. More in detail, see Fary, Bulletin de la Societe Mathematique de France, 82 (1954), pp. 94-135, and Annals of Mathematics, 65 (1957), p. 47.) Let us illustrate the points (a) and (b) by examples.

There are spaces for which the singular homology groups are defined but are completely useless. Brouwer constructed a compact set B in the plane R^2 such that: (1) the complement of B in R^2 has m+1 components $U_o, \ldots, U_m$ ($m \geqslant 3$);(2) every $x \in B$ is a boundarypoint of each U_i, $i = 0, \ldots, m$. The circle has these properties with m = 2, but for $m \geqslant 3$ Brouwer's construction is very complicated and ingenious. Now, using Leray's cohomology theory, it is easy to prove the following generalization of the famous Jordan curve theorem: If B' is homeomorphic to B and is a sub-space of R^2, then B' also has properties (1) and (2). In particular, if B is a circle, B' can be any Jordan curve and the theorem states that such a curve divides the plane in two domains and is the complete boundary of both domains. All this is trivial using Leray's theory, but it cannot be proved using singular homology theory (even the proof of the classical Jordan curve theorem is rather cumbersome in singular homology theory, so far as I know). The reason is that a space B is locally non-connected in a very strong sense (for $m \geqslant 3$). Now a beginner coming across such a situation may think that Algebraic Topology is only useful for spaces which are "locally nice"; this is definitely wrong, and suggested by singular homology only.

Even "very smooth" spaces X for which every homology theory gives the same groups $H_p(X)$ may cause trouble. It may happen that the computation of the homology groups is the most complicated in singular homology theory. For example, let us take Pontrjagin's method for computing the homology groups of a classical group G. The method is roughly the following. We single out sub-manifolds of G, treat them as cycles, compute the intersection matrix, and show that for a suitable choice of these sub-manifolds we have a homology basis. Of course, here we need intersection theory. However, this is very complicated in singular homology; even if we do this via the singular cohomology and cup-product theory, we have to develop as long a theory as the one given in the book.

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Introduction to Difference Equations by S. Goldberg. John Wiley and Sons, New York, 1958. 260 pages. \$7.10.

A very elementary introduction to finite differences and difference equations with illustrative examples from economics, psychology and sociology. This book is intended primarily for social scientists. Sections involving any knowledge of elementary calculus are starred. Generating functions and matrix methods