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ON THE LOWER RADICAL CONSTRUCTION OF TANGEMAN AND KREILING

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It is shown that the lower radical construction of Tangeman and Kreiling need not terminate at any ordinal.

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1. Introduction

Tangeman and Kreiling [7] have given a construction for the smallest radical class containing a given class C which differs from the classical construction of Kurosh [3]. Their construction is the easier one to use to show that this lower radical class $l(C)$ inherits certain properties from C. They point out that it need not terminate at the first infinite ordinal ω, in the case of associative rings, unlike the Kurosh construction. In a recent book Gardner asks for an investigation of its termination [1, Problem 7]. We shall show that it need not terminate at any ordinal. The proof involves only cardinality arguments and while we give it in the setting of any universal class of rings, as was done originally by Tangeman and Kreiling, it is clear that the same arguments apply also in other cases. This contrasts sharply with the Kurosh construction which terminates at ω for associative rings [6], for alternative rings [2] and for groups [5]. For non-associative rings neither construction need terminate [4].

We shall denote the cardinality of set A by $|A|$ and of the set of ordinals less than a given ordinal μ by $|\mu|$. With the Axiom of Choice assumed, as is usual in ring theory, we denote the successor of a cardinal α by α^+.

We recall that the Tangeman–Kreiling construction is given as follows:

Let W be any universal class of not necessarily associative rings. If C_0 is any class of rings in W then C_1 is its closure under the taking of homomorphic images. If for some ordinal $\mu \geqq 1$, C_μ has already been defined then $C_{\mu+1} = \{R \in W \mid \exists\, A \lhd R \text{ with } A, R/A \in C_\mu\}$. If λ is a limit ordinal and C_μ has been defined for all ordinals $\mu < \lambda$ then $C_\lambda = \{R \in W \mid \exists$ an ascending chain of ideals A_i in R such that $\bigcup A_i = R$ and there exists, for each i, an ordinal $\mu_i < \lambda$ with $A_i \in C_{\mu_i}\}$. Then the lower radical class $l(C_0)$ is the union of these classes C_μ taken over all ordinals μ.

As regards the ascending chain of ideals A_i it need only be assumed that it is indexed by a totally ordered set I such that for $i, j \in I$, $A_i \subseteq A_j$ if and only if $i \leqq j$. However it is convenient to assume that it is indexed by an initial segment of ordinals and that it is strictly increasing. This may be done as follows. Choose any A_j and call it B_1. If $B_1 = R$, we are done. Assume that for each ordinal $v \leqq \mu$, ideals $A_{i_v} = B_v$ have been chosen such that $B_v \subset B_\rho$ if and only if $v < \rho$ and that if λ is a limit ordinal $\leqq \mu$ than $\bigcup_{v<\lambda} B_v \subset B_\lambda$. If $B_\mu = R$ we are done. Otherwise there exists A_i with $B_\mu \subset A_i$. Then we denote A_i as $B_{\mu+1}$. If σ is a limit ordinal and B_μ is defined for all $\mu < \sigma$, with the properties described above, then if $\bigcup_{\mu<\sigma} B_\mu = R$ we are done. If not there exists A_k not contained in $\bigcup_{\mu<\sigma} B_\mu$, which implies that each $B_\mu \subset A_k$ and so that $\bigcup_{\mu<\sigma} B_\mu \subset A_k$. We denote A_k by B_σ. Since I is a set this process must terminate and so the desired chain of ideals exists. We shall assume all ascending chains in the sequel to be of this type.

2. Results

Theorem. *Corresponding to a class of rings C_0 let there be a cardinal α such that $R \in C_0$ implies $|R| \leqq \alpha$. Then, in the Tangeman–Kreiling construction, corresponding to each class C_μ there is a cardinal α_μ such that $R \in C_\mu$ implies $|R| \leqq \alpha_\mu$.*

Proof. Since C_1 is the homomorphic closure of C_0 it is clear that we may take $\alpha_1 = \alpha$. Suppose that a cardinal α_μ exists such that $R \in C_\mu$ implies $|R| \leqq \alpha_\mu$. Let $R \in C_{\mu+1}$. Then there exists an ideal A of R with $A, R/A \in C_\mu$. Hence $|R| = |A||R/A| \leqq \alpha_\mu^2$. So we may choose $\alpha_{\mu+1} = \alpha_\mu^2$. Now suppose that λ is a limit ordinal and that for each ordinal $\mu < \lambda$ an appropriate cardinal α_μ exists. Let $R \in C_\lambda$. Then there is an ordinal ρ and a strictly ascending chain of ideals A_i of $R, i < \rho$, such that $R = \bigcup A_i$ and such that there exist ordinals $\mu_i < \lambda$, with $A_i \in C_{\mu_i}$. Let $\beta = \sum_{\mu<\lambda} \alpha_\mu$. Then $|A_i| \leqq \alpha_{\mu_i} \leqq \beta$. Now suppose, if possible, that $\beta^+ < |\rho|$. Then there is an ordinal $v < \rho$ with $|v| = \beta^+$. Since we have a strictly ascending chain of ideals it follows that $|A_v| \geqq |v| = \beta^+$. However we have $|A_v| \leqq \alpha_{\mu_v} \leqq \beta < \beta^+$. This contradiction implies that $|\rho| \leqq \beta^+$. Since $R = \bigcup_{i<\rho} A_i$, it follows from $|A_i| \leqq \beta$, $|\rho| \leqq \beta^+$ that $|R| \leqq \beta . \beta^+$. So we may choose $\alpha_\lambda = \beta . \beta^+$. The existence of these cardinal numbers α_μ for all ordinals μ now follows by transfinite induction.

Since radical classes are closed under the taking of arbitrary direct sums it follows that if such a cardinal α exists then, with the trivial exception of C_0 consisting only of the zero ring 0, the Tangeman–Kreiling construction cannot terminate at any ordinal μ.

We should note that in the above argument it is not possible, in general, to replace β^+ by β, so that the cardinal numbers α_μ grow rather quickly in size. If one starts with α a finite cardinal, say $C = \{F\}$, where F is a finite field, then one can take $\alpha_\omega = \aleph_0$, rather than $\aleph_1$, as given above, since a ring $R \in C_\omega$ can be shown to be a countable union of finite rings. However at the next limit ordinal, $\omega + \omega$, uncountable rings do arise. Let Ω be the first uncountable ordinal and let $R = \oplus B_i$, $i < \Omega$, with each $B_i \cong F$. Then letting A_j be the ideal of R corresponding to $\oplus B_i$, $i < j$, for each $j < \Omega$ we have $R = \bigcup A_j$ and each A_j, being a countable direct sum of copies of F, is isomorphic to A_ω which is clearly in C_ω. Hence R is in $C_{\omega+\omega}$ and so $\alpha_{\omega+\omega}$ cannot be taken less than $\aleph_0 . \aleph_0^+ = \aleph_1$.

As has been mentioned already in the introduction the arguments used in the proof are given purely in terms of ordinals and cardinals. Thus they apply not only to universal classes of rings but also to groups, abelian groups, etc. The above arguments depend only on cardinality and on the taking of direct sums of rings from C_0. So it is reasonable to ask about the termination of the construction whenever C_0 is assumed to be closed under the taking of arbitrary direct sums of rings from C_0. We do not know the answer here but give one example working in the class of abelian groups.*

Let $Z(p)$ denote the cyclic group of prime order p. Then, as above, taking $C_0=\{Z(p)\}$, the Tangeman–Kreiling construction will not terminate. However if we take for C_0 all direct sums of copies of $Z(p)$ then it terminates at ω. The radical class consists of all p-primary abelian groups. If G is such a group then G is a union of the ascending chain of subgroups $G[p^n]$, where $G[p^n]=\{g\in G|p^n g=0\}$. $G[p^n]$ is a direct sum of cyclic groups of orders $\leqq p^n$ and so belongs to C_{m_n} for some finite ordinal m_n. Hence $G\in C_\omega$, as required.

*P. N. Stewart has now given an example, in a private communication, showing that this question has a negative answer in the class of associative rings. The question remains open for abelian groups and so for A-radicals of associative rings.

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