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# ON THE UNIFORMIZATION OF THE $n$-PUNCTURED DISC

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By the Uniformization Theorem, every plane region $\Omega$ with at least two boundary points is conformally equivalent to the quotient of $U = \{\operatorname{Im} \tau > 0\}$ by a Fuchsian group of non-elliptic Möbius transformations. Although this is well-known, the actual quantitative relations involved in the equivalence are almost completely unknown, and the standard proofs of the Uniformization Theorem shed little light onto the matter. In this thesis a study is made of various parameters associated with the uniformization of regions of the form $\Omega = V\backslash\{p_1, \ldots, p_n\}$, where $V$ is a simply connected proper subdomain of $\mathbb{C}$ and $\{p_1, \ldots, p_n\}$ is a set of distinct points (the *punctures*) of $V$. By the Riemann Mapping Theorem, $V$ is conformally equivalent to $D = \{|z| < 1\}$, and so it can be assumed that $\Omega$ is the $n$-punctured unit disc $\Omega = D\backslash\{p_1, \ldots, p_n\}$.

Associated with the uniformization of $\Omega$ are three problems of particular interest. The first of these is to determine the $n$ parabolic generators of the (unique to within conjugation) Fuchsian covering group of $\Omega$. Secondly, if $\tau(z)$ denotes the (multi-valued) inverse of any universal cover of $\Omega$ by $U$, and 0 is not one of the punctures of $\Omega$, the Schwarzian derivative $\{\tau, z\}$ has the simple form

$$(1) \qquad \{\tau, z\} = \sum_{k=1}^{n} \left( \frac{1}{2(z - p_k)^2} + \frac{1}{2\left(z - \frac{1}{\overline{p_k}}\right)^2} + \frac{m_k}{z - p_k} - \frac{\overline{p_k}(1 + \overline{m_k p_k})}{z - \frac{1}{\overline{p_k}}} \right)$$

where the $m_k$ are constants, depending only on $\Omega$, which are partly determined by $\{\tau, z\} = O\left(\frac{1}{z^4}\right)$ as $z \to \infty$ (if 0 is one of the punctures, a slight modification is necessary). The $m_k$ are known as the *accessory parameters*, and their determination is the problem of interest. Finally, the hyperbolic density $\rho(z)$ on $\Omega$ satisfies

$$\rho(z) = \frac{1}{|z - p_k| \log \frac{1}{R_k |z - p_k|}} + o(1)$$

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as $z \to p_k$. Here $R_k$ is a positive constant which depends only on $\Omega$. The problem is to determine these so-called *proximity parameters* $R_k$.

It is the intention to present a largely self-contained thesis, and so in Chapter 1 a proof is provided of the Uniformization Theorem for plane regions $W$ with at least two boundary points. However, the approach is somewhat novel because the existence of a universal cover $z = \lambda(\tau)$ of $W$ by $U$ is established as a consequence of the existence of the hyperbolic density $\rho(z)$ on $W$. All of the well-known proofs of the theorem take the opposite approach, and establish the existence of $\rho(z)$ from the existence of $\lambda(\tau)$. Of course, the method requires $\rho(z)$ to be defined in a manner which is independent of the existence of a universal cover. To do this, $\rho(z)$ is defined pointwise on $W$ by $\rho(z) = \max_F f(z)$, where $F$ is the family of all $C^2$ functions of constant curvature $-1$ on $W$. By using a method due to Heins [**1**], the existence of $\rho(z)$ is established, and this enables a universal cover of $W$ by $U$ to be constructed.

In Chapter 2 a general description of the universal cover of $\Omega = D\backslash\{p_1, \dots, p_n\}$ by $U$ is presented. In particular, the construction of a certain fundamental domain for the covering group of $\Omega$ is described, and the validity of (1) established.

Chapters 3 and 4 contain the bulk of the thesis. In these chapters significant progress is made towards answering the three problems described above for the twice-punctured unit disc, the simplest punctured disc for which the problems are non-trivial. Note that by a Möbius transformation of $D$ onto itself, the two punctures can be mapped to $\pm p$, where $0 < p < 1$, and so the study can be restricted to that of the regions $\Omega_p = D\backslash\{\pm p\}$. Typical of the results obtained are those for $m = m(p)$, the accessory parameter at $p$ for $\Omega_p$, which is shown to be a strictly increasing function on $(0,1)$ which is implicitly defined in terms of $p$ by an infinite continued fraction (so $m$ can be determined by successive approximation). Further, $m$ becomes infinite as $p$ approaches $0$ or $1$. Very precise details of the behaviour of $m$ as $p \to 0, 1$ are obtained, including complete descriptions of the infinite parts of $m$ in these two cases. The methods in Chapters 3 and 4 can be broadly described as being an interplay between techniques in hyperbolic geometry and differential equations. The connection between the uniformization of $\Omega_p$ and differential equations is provided by the observation that if $\tau(z)$ denotes the inverse of any universal cover of $\Omega_p$ by $U$, then $(\tau'(z))^{-\frac{1}{2}}$ and $\tau(z)(\tau'(z))^{-\frac{1}{2}}$ are independent solutions to $\eta'' + \frac{1}{2}\{\tau, z\}\eta = 0$.

The methods of Chapters 3 and 4 depend considerably on the symmetry of $\Omega_p$, and so do not readily extend to the study of the unit disc punctured at three or more points. However, in Chapter 5, some answers to one particular aspect of the problem of the uniformization of $\Omega = D\backslash\{p_1, \dots, p_n\}$, where $n \geqslant 3$, are provided. To be specific, the problem of *confluence* is discussed, and estimates obtained for the behaviour of various parameters as two punctures of $\Omega$ approach one another, or as one puncture approaches the unit circle.

## References

[1] M. Heins, 'On a class of conformal metrics', *Nagoya Math. J.* **21** (1962), 1–60.

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