

ON THE DISTINCTION OF IWAHORI-SPHERICAL DISCRETE SERIES REPRESENTATIONS

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Abstract Let E/F be a quadratic unramified extension of non-archimedean local fields and \mathbb{H} a simply connected semisimple algebraic group defined and split over F . We establish general results (multiplicities, test vectors) on $\mathbb{H}(F)$ -distinguished Iwahori-spherical representations of $\mathbb{H}(E)$. For discrete series Iwahori-spherical representations of $\mathbb{H}(E)$, we prove a numerical criterion of $\mathbb{H}(F)$ -distinction. As an application, we classify the $\mathbb{H}(F)$ -distinguished discrete series representations of $\mathbb{H}(E)$ corresponding to degree 1 characters of the Iwahori-Hecke algebra.

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1. Introduction

This work is a contribution to the study of distinguished complex representations for Galois symmetric spaces over a non-archimedean local field. It may be viewed as a continuation of [6] where the authors prove a particular case of Prasad’s conjecture on the distinction of the Steinberg representation ([15], Conjecture 3, p. 77). However, even though the Bruhat-Tits building will be used here as a geometrical tool as in [6], our fundamental techniques will be that of integration of matrix coefficients, in the spirit of [5].

We restrict ourselves to the easy case where the symmetric space is of the form $\mathbb{H}(E)/\mathbb{H}(F)$, where E/F is a quadratic unramified extension of non-archimedean local fields and \mathbb{H} a connected simple group defined and split over F . For simplicity sake, we shall assume that \mathbb{H} is simply connected. Indeed, in that case, the set of Iwahori subgroups of $\mathbb{H}(E)$ identifies with the set of chambers in the Bruhat-Tits building of $\mathbb{H}(E)$, and this makes our calculations less technical. However, our arguments should adapt easily to the non-simply connected case.

Recall that a smooth complex representation (π, \mathcal{V}) of $\mathbb{H}(E)$ is said to be $\mathbb{H}(F)$ -distinguished if the intertwining space $\text{Hom}_{\mathbb{H}(F)}(\mathcal{V}, \mathbb{C})$ is non-zero, where \mathbb{C} is a complex line with the trivial action of $\mathbb{H}(F)$. In this article, we study the distinction of *Iwahori-spherical* irreducible representations (i.e., representations (π, \mathcal{V}) such that $\mathcal{V}^I \neq 0$), where I is some fixed Iwahori subgroup of $\mathbb{H}(E)$. If (π, \mathcal{V}) is such a representation, the Iwahori-Hecke algebra \mathcal{H}_I of $\mathbb{H}(E)$ relative to I acts on $M = \mathcal{V}^I$ via an irreducible representation (r, M) .

Our first result is the following.

Theorem 1.1 (Theorem 4.1). *Let (π, \mathcal{V}) be an Iwahori-spherical irreducible representation of $\mathbb{H}(E)$. Let J be any $\text{Gal}(E/F)$ -stable Iwahori subgroup of $\mathbb{H}(E)$. Then the obvious restriction map $\text{Hom}_{\mathbb{H}(F)}(\mathcal{V}, \mathbb{C}) \rightarrow \text{Hom}_{\mathbb{C}}(\mathcal{V}^J, \mathbb{C})$ is injective.*

As a consequence, if $\Lambda \in \text{Hom}_{\mathbb{H}(F)}(\mathcal{V}, \mathbb{C})$ is non-zero, then for any $\text{Gal}(E/F)$ -stable Iwahori subgroup J of $\mathbb{H}(E)$, there exists a vector $v \in \mathcal{V}^J$ such that $\Lambda(v) \neq 0$.

The proofs of these results are based on a transitivity property of the action of $\mathbb{H}(F)$ on the set of chambers of the Bruhat-Tits building of $\mathbb{H}(E)$, which was a crucial ingredient of [6].

We next assume, moreover, that (π, \mathcal{V}) is a discrete series representation. Using the fact due to Zhang [19] that $\mathbb{H}(E)/\mathbb{H}(F)$ is a very strongly discrete symmetric space, we establish a second result.

Theorem 1.2 (Theorem 5.3). *Let (π, \mathcal{V}) be an irreducible Iwahori-spherical discrete series representation of $\mathbb{H}(E)$. Let J be some $\text{Gal}(E/F)$ -stable Iwahori subgroup of $\mathbb{H}(E)$ and ν denote a Haar measure on $\mathbb{H}(F)$.*

Then (π, \mathcal{V}) is $\mathbb{H}(F)$ -distinguished if, and only if, there exist $v \in \mathcal{V}^J$, $\tilde{v} \in \tilde{\mathcal{V}}^J$, such that

$$\int_{\mathbb{H}(F)} c_{v, \tilde{v}}(h) d\nu(h) \neq 0,$$

where $\tilde{\mathcal{V}}$ is the contragredient representation and $c_{v, \tilde{v}}$ is the coefficient of π attached to v and \tilde{v} .

To turn the previous theorem into an effective criterion, we use the notion of *generalized Poincaré series* attached to an Iwahori-Hecke algebra introduced by Gyoja [11]. Let (W, S) be a Coxeter system with associated Hecke algebra \mathcal{H} . Let $S_i, i = 1, \dots, m$ be the intersections of the conjugacy classes of W with S . For each representation (r, M) of \mathcal{H} , Gyoja attaches a power series $L(t_1, \dots, t_m, r) \in \text{End}(M) \otimes_{\mathbb{C}} \mathbb{C}[[t_1, \dots, t_m]]$. Now assume that $\mathcal{H} = \mathcal{H}_I$ is the Iwahori-Hecke algebra of $\mathbb{H}(E)$ relative to I and (r, M) is the representation associated to the \mathcal{H}_I -module \mathcal{V}^I , for a representation (π, \mathcal{V}) of $\mathbb{H}(E)$.

Theorem 1.3 (Theorem 7.1). *Let (π, \mathcal{V}) be an irreducible Iwahori-spherical discrete series representation of $\mathbb{H}(E)$. Then π is $\mathbb{H}(F)$ -distinguished if, and only if, $L(1/q_o, \dots, 1/q_o, r) \in \text{End}(M)$ is non-zero, where q_o is the size of the residue field of F .*

Our last result is concerned with the case where the discrete series (π, \mathcal{V}) corresponds to a degree 1 character of \mathcal{H}_I . Those representations were classified by A. Borel in [2]. It is by using techniques borrowed from *loc. cit* that we prove the following: the value of $L(t_1, \dots, t_m, r)$ occurring in Theorem 1.3 can be calculated using a Macdonald’s Poincaré series (cf. [14]). Using Macdonald’s formulas, we obtain the following:

Theorem 1.4 (Theorem 7.4). *Let (π, \mathcal{V}) be an irreducible Iwahori-spherical discrete series representation of $\mathbb{H}(E)$ such that \mathcal{V}^I is 1-dimensional. If \mathbb{H} is not of type G_2 , then π is $\mathbb{H}(F)$ -distinguished if, and only if, it is the Steinberg representation. If \mathbb{H} has type G_2 , there are two representations satisfying the assumptions of the theorem: the Steinberg representation and another one; they are both $\mathbb{H}(F)$ -distinguished.*

In other words, the Steinberg representation of $\mathbb{H}(E)$ is the unique representation satisfying the assumption of the theorem and which is $\mathbb{H}(F)$ -distinguished, except in type G_2 where there is another one.

The case of the Steinberg representation (originally D. Prasad’s conjecture) was already established in [6] (where \mathbb{H} is not necessarily assumed simply connected) and in characteristic 0, Prasad’s conjecture was entirely proved by Beuzart-Plessis [1].

Macdonald’s formulas [14] allow only to calculate Gyoja L -functions $L(t_1, \dots, t_m, r)$ when r is a degree 1 character; this is why we are not able to give a classification of distinguished discrete series representations when $\dim \mathcal{V}^I > 1$.

In [16], Dipendra Prasad states a conjecture predicting the distinction of irreducible smooth complex representations in the case of a general Galois symmetric space $\mathbb{H}(E)/\mathbb{H}(F)$ (Conjecture 2 of *loc. cit.*). An irreducible smooth representation π of $\mathbb{H}(E)$ conjecturally has a Langlands parameter $\phi_\pi: W_E \times \text{SL}_2(\mathbb{C}) \rightarrow {}^L G$, where the Langlands dual ${}^L G$ is a semidirect product $W_E \rtimes \widehat{G}$, \widehat{G} being the complex dual of $G = \mathbb{H}(E)$. Roughly speaking, the point of Prasad’s conjecture is to relate the $\mathbb{H}(F)$ -distinction of π to properties of the parameter φ_π . A precise formulation of this conjecture in our case may be stated as follows (this formulation as well as the two following remarks are due to Prasad himself).

Conjecture 1.5 (Prasad). *Let E/F be a quadratic unramified extension of non-archimedean local fields. Let \mathbb{H} be a simply connected split simple group over F if -1 belongs to the Weyl group of \mathbb{H} , and the unique quasi-split (but not split) group over F*

which splits over E otherwise. (This ensures that the opposition group H^{op} of $H = \mathbb{H}(F)$, defined in [16], is split over F .) A Iwahori-spherical discrete series representation π of $G = \mathbb{H}(E)$ has, by the work of Kazhdan-Lusztig, cf. [13], an L-parameter $\phi: W_E \times \text{SL}_2(\mathbb{C}) \rightarrow {}^L G$ which on W_E factors through an unramified extension of E . Assume that the group of connected components of the centralizer of the corresponding unipotent element in \tilde{G} is an elementary abelian 2-group (which is known to be the case for all groups of type B_n, C_n, D_n). Then π is distinguished by $H = \mathbb{H}(F)$ if and only if

1. The restriction of ϕ to W_E is trivial,
2. The representation π is generic. (Under condition (1), genericity should not depend on the choice of the Whittaker datum.)

Further, if the conditions (1) and (2) are satisfied, then $\dim \text{Hom}_H(\pi, \mathbb{C})$ is the order of the centralizer of $\phi(\text{SL}_2(\mathbb{C}))$ in \tilde{G} , which by assumption, is a finite elementary abelian 2-group.

Remark 1. We relate the above conjecture to that in [16]. In the present case, $W_F \times \text{SL}_2(\mathbb{C})$ and $W_E \times \text{SL}_2(\mathbb{C})$ can be replaced by $\langle \text{Fr}^{\mathbb{Z}} \rangle \times \text{SL}_2(\mathbb{C})$ and $\langle \text{Fr}^{2\mathbb{Z}} \rangle \times \text{SL}_2(\mathbb{C})$, where Fr is a Frobenius element of W_F . Because of the hypothesis in the conjecture, $\pi_0(Z_{\tilde{G}}(\phi(\text{SL}_2(\mathbb{C}))))$ being an elementary abelian 2-group, $\langle \text{Fr}^{2\mathbb{Z}} \rangle$ must land inside the connected component of identity of $Z_{\tilde{G}}(\phi(\text{SL}_2(\mathbb{C})))$. Hence, by the discreteness of the parameter ϕ , $Z_{\tilde{G}}(\phi(\text{SL}_2(\mathbb{C})))$ must be finite and therefore an elementary abelian 2-group, and the image of $\langle \text{Fr}^{2\mathbb{Z}} \rangle$ under ϕ must be trivial, which is the conclusion in part (1) of the conjecture. Further, the natural map from the component group of a lifted parameter $\tilde{\phi}$ for H^{op} to the component group for ϕ is clearly an isomorphism. Hence, by the conjecture of [16], the only π corresponding to the trivial character of the component group can be distinguished, which is the conclusion in part (2) of the conjecture.

Remark 2. Here is the example of G_2 , which appears in this work but for which the hypothesis in the conjecture above does not hold. There is a discrete series representation π of $G_2(E)$ which has a one dimensional fixed vector for an Iwahori subgroup, which by Reeder [17], page 480, corresponds to a subregular unipotent element with component group isomorphic to S_3 with the representation of the component group, the 2-dimensional irreducible representation ρ of S_3 . It is thus a non-generic representation of G_2 which is distinguished by $G_2(F)$ (by Theorem 7.4 of this paper). In this case, the representation ρ of S_3 has a fixed vector under any transposition in S_3 , thus confirming the general conjecture of [16] in an example with a non-abelian component group.

It would be nice to check Prasad's conjecture in our case. In this aim, one would have to determine the Galois parameters of our representations, which are not known in general.

The article is organized as follows. After introducing the general notation in §1, in §2 we recall the models of Iwahori-spherical representations introduced in [2] and explain how to deduce a very simple formula for the Iwahori-spherical coefficients. The bound on the dimension of the space of $\mathbb{H}(F)$ -invariant linear forms is given in §3, and the fact that test-vectors can be chosen Iwahori-spherical is proved in §4. In §5, we recall the definition of Gyoja's generalized Poincaré series, and we give Macdonald's formulas. Finally, §6 is

devoted to establishing the numerical criterion in terms of Gyoja’s Poincaré series values and to applying it to the case of degree 1 characters of the Iwahori-Hecke algebra.

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2. Notation

Throughout the article, we use the following notation.

If X is a set, we denote by $|X|$ its cardinal. If a group Γ acts on X and Γ' is a subset of Γ , $X^{\Gamma'}$ denotes the subset of those $x \in X$ that are fixed by Γ' .

We let F denote a non-archimedean, non-discrete, locally compact field. We denote by \mathfrak{o}_F its ring of integers, by \mathfrak{p}_F the maximal ideal of \mathfrak{o}_F , and we set $k_F = \mathfrak{o}_F/\mathfrak{p}_F$ (residue field), $q_0 = |k_F| = p^r$, p being the characteristic of k_F (residue characteristic).

We fix an *unramified quadratic extension* E/F and use the obvious pieces of notation $\mathfrak{o}_E, \mathfrak{p}_E, k_E$. We denote by θ the generator of the Galois group $\text{Gal}(E/F)$. We set $q = |k_E|$, so that $q = q_0^2$.

We fix a connected simple F -algebraic group \mathbb{H} assumed to be *split over F and simply connected*, and we write $H = \mathbb{H}(F)$ for the locally compact group of its F -rational points. We denote by $\mathbb{G} = \text{Res}_{E/F}\mathbb{H}$ the F -algebraic group obtained by Weil’s restriction of scalars. It is semisimple and simply connected. We set $G = \mathbb{G}(F) = \mathbb{H}(E)$.

The Galois group $\text{Gal}(E/F)$ acts on \mathbb{G} by F -rational automorphisms of algebraic group. We still denote by θ the action of θ on \mathbb{G} and G so that $\mathbb{H} = \mathbb{G}^\theta$ and $H = G^\theta$. The pair (G, H) (or the G -set G/H) is called a *(Galois) reductive symmetric space*.

We let X_F (resp. X_E) denote the Bruhat-Tits building of \mathbb{H} over F (resp. over E). Recall that X_E identifies canonically with the Bruhat-Tits building of $\text{Res}_{E/F}(\mathbb{H})$ over F . As simplicial complexes X_F and X_E have dimension d , the F -rank of \mathbb{H} . As a particular case of building functorialities, there is an injection $j: X_F \rightarrow X_E$ satisfying the following properties:

- (a) j is $\text{Gal}(E/F) \times H$ -equivariant,
- (b) the image of j is $X_E^{\text{Gal}(E/F)}$,
- (c) j is simplicial and maps chambers to chambers.

We identify X_F as a subsimplicial complex of X_E , viewing j as an inclusion.

We fix a maximal F -split torus \mathbb{T} of \mathbb{H} and set $T_F = \mathbb{T}(F)$, $T_E = \mathbb{T}(E)$. We denote by $\Phi = \Phi(\mathbb{T}, \mathbb{H})$ the root system of \mathbb{T} in \mathbb{H} . Let $W_F^{\text{Aff}} = N_H(T)/T_F^\circ$ be the affine Weyl group of H relative to \mathbb{T} , where $N_H(T_F)$ is the normalizer of \mathbb{T} in H and T_F° the maximal compact subgroup of T_F . The group W_E^{Aff} is defined in the same way, and the inclusion $N_H(T_F) \subset N_G(T_E)$ induces an isomorphism of groups $W_F^{\text{Aff}} \simeq W_E^{\text{Aff}}$. In the sequel, we abbreviate $W^{\text{Aff}} = W_F^{\text{Aff}} = W_E^{\text{Aff}}$.

Let \mathcal{A}_F (resp. \mathcal{A}_E) denote the apartment of X_F (resp. X_E) associated to \mathbb{T} . It follows from the construction of j that $\mathcal{A}_F = \mathcal{A}_E$. Fix a chamber C_o of \mathcal{A}_F , and let I_F° (resp. I_E°) denote the Iwahori subgroup of H (resp. the $\text{Gal}(E/F)$ -stable Iwahori subgroup of G) fixing C_o . Since \mathbb{H} is simply connected, these Iwahori subgroups are the (global) fixators

of C_o . Let $S \subset W_F^{\text{Aff}}$ be the set of involutions corresponding to reflections in \mathcal{A}_F relative to the walls of C_o . Then, since \mathbb{H} is simply connected, $(I_F^o, \mathcal{N}_H(T_F))$ (resp. $(I_E^o, \mathcal{N}_G(T_E))$) is a Tits system (or BN-pair) in H (resp. in G) with Coxeter system (W^{Aff}, S) . We denote by l the length function of W relative to the generative subset S . Recall that

$$|I_E^o w I_E^o / I_E^o| = q^{l(w)}, \quad w \in W^{\text{Aff}}.$$

Recall that the vertices of X_E are naturally labelled by the elements of S . Since the action of G preserves the labelling, it suffices to define the labels of vertices of C_o . Then if $s \in S$, the vertex fixed by the standard maximal parahoric subgroup $I_E^o \langle S \setminus \{s\} \rangle I_E^o$ has label s (here, $\langle S \setminus \{s\} \rangle$ is the subgroup of W_E^{Aff} generated by $S \setminus \{s\}$). If C is a chamber of X_E , the wall of C which does not contain the unique vertex of type s is said to be of type s . If C and D are two chambers of X_E , we write $D \sim_s C$ if D and C are adjacent and if the wall $D \cap C$ is of type s .

We let \mathcal{H}_I denote the Iwahori-Hecke algebra of G relative to our choice I_E^o of Iwahori subgroup. Recall that as a complex vector space, \mathcal{H}_I is the set of complex functions of G which are bi-invariant by I_E^o and have compact support. It is endowed with the convolution product defined by

$$u \star v(g) = \int_G u(gx^{-1})v(x) d\mu(x) = \sum_{x \in I_E^o \backslash G} u(gx^{-1})v(x), \quad g \in G, \quad u, v \in \mathcal{H}_I,$$

where μ is the Haar measure on G normalized by $\mu(I_E^o) = 1$. Recall that $(e_w)_{w \in W^{\text{Aff}}}$ is a \mathbb{C} -basis of \mathcal{H}_I , where for $w \in W^{\text{Aff}}$, e_w denotes the characteristic function of $I_E^o w I_E^o$. Recall that we have the classical relations

$$(e_s + 1) \star (e_s - q) = 0, \quad s \in S$$

$$e_w \star e_{w'} = e_{ww'} \text{ if } l(ww') = l(w) + l(w'), \quad w, w' \in W^{\text{Aff}}$$

and that the $e_s, s \in S$ generate \mathcal{H}_I as an algebra.

If $f \in \mathcal{H}_I$, we define \check{f} by $\check{f}(x) = f(x^{-1}), x \in G$. The map $f \mapsto \check{f}$ is an automorphism of the vector space \mathcal{H}_I satisfying $(f \star g)^\check{ } = \check{g} \star \check{f}, f, g \in \mathcal{H}_I$. If (r, M) is a complex representation of \mathcal{H}_I , we define its contragredient (\tilde{r}, \tilde{M}) in the dual \tilde{M} of M by $\tilde{r}(f) = {}^t r(\check{f}), f \in \mathcal{H}_I$, where t denotes the transposition.

In this article, representations of G, H or \mathcal{H}_I are always assumed to be in complex vector spaces. If (π, \mathcal{V}) is a smooth representation of $G, (\tilde{\pi}, \tilde{\mathcal{V}})$ denotes its contragredient (i.e., its smooth dual). If W is a complex vector space, $W^* = \text{Hom}_{\mathbb{C}}(W, \mathbb{C})$ denotes its algebraic dual.

3. Iwahori-spherical representations: models and coefficients

In this section, we abbreviate $I = I_E^o$.

Let $\mathcal{S}(G)$ denote the abelian category of smooth representations of G and $\mathcal{S}(G)_I$ denote the full subcategory of those representations (π, \mathcal{V}) that are generated by the fixed vector set \mathcal{V}^I . Let $\mathcal{H}_I\text{-Mod}$ denote the category of left \mathcal{H}_I -modules. If (π, \mathcal{V}) is an object of

$\mathcal{S}(G)_I$, then \mathcal{V}^I is naturally a left \mathcal{H}_I -module, so that we have a well-defined functor

$$\mathbf{M} : \mathcal{S}(G)_I \longrightarrow \mathcal{H}_I\text{-Mod}, (\pi, \mathcal{V}) \mapsto \mathcal{V}^I.$$

It is now a well-known fact ([2],[8]) that \mathbf{M} is an equivalence of categories; in other words, the trivial character of I is a *type* for G in the sense of [7]. Moreover, if (π, \mathcal{V}) is admissible, we have $\mathbf{M}(\tilde{\mathcal{V}}) = \mathbf{M}(\mathcal{V})^\vee$: the functor commutes with the operation of taking the contragredient.

A pseudo-inverse \mathbf{V} for \mathbf{M} may be constructed as follows (cf. [2]§4). Let us fix a left \mathcal{H}_I -module M viewed as a representation (r, M) of \mathcal{H}_I . Let $C_c(G/I)$ be the space of functions on G which are right I -invariant and have compact support. This is naturally a right \mathcal{H}_I -module and a smooth representation of G (by left translation) and both structures are compatible. Hence, we may form the tensor product $\mathbf{V}(M) := C_c(G/I) \otimes_{\mathcal{H}_I} M$. Of course, we have an isomorphism of left \mathcal{H}_I -modules $M \simeq \mathbf{V}(M)^I$; it is explicitly given by $m \mapsto e_1 \otimes m$.

We shall need two other functorial constructions $\mathbf{P}, \mathbf{P}^\circ$, considered by Borel in loc. cit. §2. If (r, M) is a representation of \mathcal{H}_I , we set

$$\mathbf{P}^\circ(M) = \{f \in C(G/I, M) \mid f \star u = r(u).f, u \in \mathcal{H}_I\},$$

where $C(G/I, M)$ denotes the space of M -valued functions on G which are right I -invariant. This space is acted upon by G via left translation, but the obtained representation of G is not smooth in general; we denote by $\mathbf{P}(M) \subset \mathbf{P}^\circ(M)$ the subspace of smooth vectors. By [2] Proposition 2.4, the map

$$\nu_o : \mathbf{P}(M)^I = \mathbf{P}^\circ(M)^I \longrightarrow M, f \mapsto f(1)$$

is an isomorphism of vector spaces. Moreover, for $m \in M$, the element $f_m \in \mathbf{P}(M)^I$ mapped onto m by ν_o is the function f_e on G mapping any element of IwI , $w \in W^{\text{Aff}}$, to $q_w^{-1} r(\tilde{e}_w).m$, symbolically written by Borel as

$$f_m = \sum_{w \in W^{\text{Aff}}} e_w q_w^{-1} r(\tilde{e}_w).m. \tag{1}$$

The tensor product $C_c(G/I) \otimes_{\mathcal{H}_I} M$ may be viewed as the quotient of $C_c(G/I) \otimes_{\mathbb{C}} M$ by the G -invariant subspace generated by $(f \star u) \otimes m - f \otimes r(u).m$, $f \in C_c(G/I)$, $u \in \mathcal{H}_I$, $m \in M$. We may identify $C_c(G/I) \otimes_{\mathbb{C}} M$ with the space $C_c(G/I, M)$ of M -valued functions on G which are right I -invariant. The bilinear G -invariant map

$$\mathbf{P}^\circ(\tilde{M}) \times C_c(G/I, M) \longrightarrow \mathbb{C}, (f, g) \mapsto \sum_{x \in G/I} \langle f(x), g(x) \rangle_M \tag{2}$$

induces a well-defined G -invariant bilinear map

$$\mathbf{P}^\circ(\tilde{M}) \times \mathbf{V}(M) \longrightarrow \mathbb{C}$$

whence a G -intertwining operator

$$\psi_M : \mathbf{P}^\circ(\tilde{M}) \longrightarrow \mathbf{V}(M)^*,$$

where $*$ denotes an algebraic dual. From [2] Proposition 2.6, we have the following.

Proposition 3.1. *Let (r, M) be a representation of \mathcal{H}_I . Then the intertwining operator $\psi_M : \mathbf{P}^o(\tilde{M}) \rightarrow \mathbf{V}(M)^*$ is an isomorphism and restricts to an isomorphism $\mathbf{P}(\tilde{M}) \simeq \mathbf{V}(M)^\vee$.*

Observe that if M is finite dimensional, $\mathbf{V}(M)$ is admissible and we have $\mathbf{V}(M)^\vee \simeq \mathbf{V}(\tilde{M})$. So exchanging the roles of M and \tilde{M} , we obtain $\mathbf{P}(M) \simeq \mathbf{V}(M)$ as G -modules.

Let (r, M) be a finite dimensional representation of \mathcal{H}_I , and set $(\pi, \mathcal{V}) = \mathbf{V}(r, M)$. We make the canonical identifications $\mathcal{V}^I = M$ and $\tilde{\mathcal{V}}^I = \tilde{M}$. Recall that for $m \in M$ and $\tilde{m} \in \tilde{M}$, the corresponding coefficient $c_{m, \tilde{m}}$ of \mathcal{V} is the complex function on G defined by

$$c_{m, \tilde{m}}(g) = \langle \tilde{m}, \pi(g).m \rangle_{\mathcal{V}}, g \in G.$$

Let us observe that by construction, $c_{m, \tilde{m}}$ is I -bi-invariant.

Corollary 3.2. *For $m \in M$, $\tilde{m} \in \tilde{M}$, $w \in W^{\text{Aff}}$, the value of $c_{m, \tilde{m}}$ on IwI is $q_w^{-1} \langle \tilde{m}, r(e_w).m \rangle_M$.*

Proof. The vector $m \in \mathcal{V}$ corresponds to the function $v_m \in C_c(G/I, M)$ which has support I and constant value m on I . The linear form \tilde{m} corresponds to the function $f_{\tilde{m}} \in C(G/I, \tilde{M})$ whose value on IwI , $w \in W^{\text{Aff}}$, is given by $q_w^{-1} \tilde{r}(\check{e}_w).m$. Applying the pairing formula (2), for $w \in W^{\text{Aff}}$, we have $c_{m, \tilde{m}}(w) = \langle q_w^{-1} \tilde{r}(\check{e}_w).\tilde{m}, m \rangle_M = q_w^{-1} \langle \tilde{m}, r(e_w).m \rangle$, as required. \square

We now describe a geometric model for the space $\mathbf{P}^o(\tilde{M})$ for a given representation (r, M) of \mathcal{H}_I . We let $\text{Ch}(X_E)$ denote the G -set of chambers of X_E and $\mathcal{F}(\text{Ch}(X_E), \tilde{M})$ the space of \tilde{M} -valued functions on $\text{Ch}(X_E)$. We let $\mathcal{F}(\text{Ch}(X_E), \tilde{M})_I$ denote the subspace of $\mathcal{F}(\text{Ch}(X_E), \tilde{M})$ formed of those functions f satisfying for all $C \in \text{Ch}(X_E)$ and for all $s \in S$:

$$\tilde{r}(e_s).f(C) = \sum_{D \sim_s C} f(D). \tag{3}$$

Since G acts on $\text{Ch}(X_E)$ by preserving the labelling of vertices, $\mathcal{F}(\text{Ch}(X_E), \tilde{M})_I$ is a G -subspace of $\mathcal{F}(\text{Ch}(X_E), \tilde{M})$

Proposition 3.3. *The spaces $\mathbf{P}^o(\tilde{M})$ and $\mathcal{F}(\text{Ch}(X_E), \tilde{M})_I$ are isomorphic as G -modules.*

Proof. Since G acts transitively on $\text{Ch}(X_E)$ and the global stabilizer of C_o is I , we may identify $C(G/I, \tilde{M})$ with $\mathcal{F}(\text{Ch}(X_E), \tilde{M})$. Since $\{e_s \mid s \in S\}$ generates \mathcal{H}_I as an algebra, a function $f \in C(G/I, \tilde{M})$ is in $\mathbf{P}^o(\tilde{M})$ if and only if $f \star e_s = \tilde{r}(e_s).f$, for all $s \in S$. It follows that $\mathbf{P}^o(\tilde{M})$ identifies with the space of functions $f: \text{Ch}(X_E) \rightarrow \tilde{M}$ satisfying $f \star e_s = \tilde{r}(e_s).f$. Now our result follows from [2] §4 Equation (6):

$$f \star e_s(C) = \sum_{D \sim_s C} f(D), f \in \mathcal{F}(\text{Ch}(X_E), \tilde{M}), s \in S. \tag{4}$$

\square

4. Multiplicities

We continue to abbreviate $I = I_E^o$, a fixed $\text{Gal}(E/F)$ -stable Iwahori subgroup of G .

We fix an irreducible Iwahori-spherical representation (π, \mathcal{V}) of G . It is of the form $\mathbf{V}(M)$, for some irreducible representation (r, M) of \mathcal{H}_I . The aim of this section is to give a bound for the dimension of the intertwining space $\text{Hom}_H(\mathcal{V}, \mathbb{C})$, where \mathbb{C} is acted upon via the trivial representation of H .

We shall prove the following.

Theorem 4.1. *The obvious restriction map $\text{Hom}_H(\pi, \mathbb{C}) \rightarrow \text{Hom}_{\mathbb{C}}(\pi^I, \mathbb{C})$ is injective. In particular, we have*

$$\dim_{\mathbb{C}} \text{Hom}_H(\mathbf{V}(M), \mathbb{C}) \leq \dim_{\mathbb{C}} M.$$

By Propositions 3.1 and 3.3, we have an isomorphism of vector spaces:

$$\text{Hom}_H(\mathbf{V}(M), \mathbb{C}) = (\mathbf{V}(M)^*)^H \simeq \mathbf{P}^o(\tilde{M})^H \simeq \mathcal{F}(\text{Ch}(X_E), \tilde{M})_r^H.$$

Through this isomorphism, the restriction map $\text{Hom}_H(\pi, \mathbb{C}) \rightarrow \text{Hom}_{\mathbb{C}}(\pi^I, \mathbb{C})$ corresponds to the map $\mathcal{F}(\text{Ch}(X_E), \tilde{M})_r^H \rightarrow \tilde{M}$, $f \mapsto f(C_o)$. Indeed, recall that $\pi = \mathbf{V}(M)$, where $\mathbf{V}(M)$ is the image of $C_c(G/I, M) \simeq C_c(G/I) \otimes M$ in the quotient $C_c(G/I) \otimes_{\mathcal{H}_I} M$. Then $\pi^I \simeq M$ corresponds to the images \bar{v}_m , $m \in M$, where $v_m \in C_c(G/I, M)$ is the function with support I and value m . For $f \in \text{Hom}_H(\pi, \mathbb{C}) \simeq \mathbf{P}^o(\tilde{M})^H$ and $m \in \pi^I \simeq M$, we have

$$f(\bar{v}_m) = \sum_{x \in G/I} \langle f(xC_o), v_m(x) \rangle_M = f(C_o)(m).$$

Hence, we are reduced to proving the following result.

Proposition 4.2. *A map $f \in \mathcal{F}(\text{Ch}(X_E), \tilde{M})_r^H$ is entirely determined by the value $f(C_o)$ (i.e., the mapping $\mathcal{F}(\text{Ch}(X_E), \tilde{M})_r^H \rightarrow \tilde{M}$, $f \mapsto f(C_o)$ is injective).*

To prove the proposition, we use the same tool that allowed us to establish the multiplicity 1 property of the main result of [6] (Theorem 2):

Proposition 4.3 (François Courtès). *Let d be a non-negative integer and C be a chamber of X_E at combinatorial distance d from X_F . Let D^+ be a chamber of X_E , adjacent to C and at combinatorial distance $d+1$ from X_F . Consider the wall $A = D^+ \cap C$. The set \mathcal{C}_A of chambers of X_E containing A splits into two subsets: the set \mathcal{C}_A^+ (resp. \mathcal{C}_A^-) formed of those chambers at combinatorial distance $d+1$ (resp. d) from X_F .*

Then the stabilizer H_A of A in H acts transitively of \mathcal{C}_A^+ et \mathcal{C}_A^- . Moreover, we have

$$(|\mathcal{C}_A^-|, |\mathcal{C}_A^+|) = \begin{cases} (1, q_o^2) \\ \text{or} \\ (q_o + 1, q_o^2 - q_o). \end{cases}$$

Proof. This follows from [6] Proposition A1 and the proof of [9] Theorem 6.1.

Proof of Proposition 4.2. Let $f \in \mathcal{F}(\text{Ch}(X_E), \tilde{M})_r^H$. For a non-negative integer d , we let $\text{Ch}(X_E)_d$ denote the set of chambers in X_E at combinatorial distance from X_F less than

or equal to d . In particular, $\text{Ch}(X_E)_0 = \text{Ch}(X_F)$, the set of chambers in X_F . We prove by induction on d that for all $d \geq 0$, the restriction of f to $\text{Ch}(X_E)_d$ depends only on $f(C_o)$.

Our assertion holds for $d = 0$ since H acts transitively on $\text{Ch}(X_F)$. Assume it holds for an integer $d \geq 0$. Let D be a chamber a combinatorial distance $d + 1$ from X_F . By definition, there exists a chamber C adjacent to D and at combinatorial distance d from X_F . Use the notation $A = D \cap C$, \mathcal{C}_A^- and \mathcal{C}_A^+ as in Proposition 4.3. Let s be the type of A . By Relation 3, we have

$$\tilde{r}(e_s).f(C) = \sum_{E \in \mathcal{C}_A^- \setminus \{C\}} f(E) + \sum_{E \in \mathcal{C}_A^+} f(E) = (|\mathcal{C}_A^-| - 1)f(C) + |\mathcal{C}_A^+|f(D), \tag{5}$$

where we used the fact that H acts transitively on \mathcal{C}_A^+ and \mathcal{C}_A^- , and that f is H -invariant. Hence, we have

$$f(D) = \frac{1}{|\mathcal{C}_A^+|} \tilde{r}(e_s - (|\mathcal{C}_A^-| - 1)e_1).f(C) \tag{6}$$

so that $f(D)$ depend only on $f(C_o)$, as required.

Theorem 4.1 has the following nice consequence. □

Theorem 4.4. *Let (π, \mathcal{V}) be an irreducible Iwahori-spherical representation satisfying $\dim \mathcal{V}^I = 1$. Then π has the multiplicity 1 property*

$$\dim_{\mathbb{C}} \text{Hom}_H(\mathcal{V}, \mathbb{C}) \leq 1.$$

5. Test vectors

Let us fix an irreducible square integrable (or discrete series) representation (π, \mathcal{V}) of G . By definition, for $v \in \mathcal{V}$ and $\tilde{v} \in \tilde{\mathcal{V}}$, the coefficient $c_{v, \tilde{v}}: g \mapsto \langle \tilde{v}, \pi(g).v \rangle_{\mathcal{V}}$, $g \in G$, lies in $L^2(G)$. Fix a Haar measure ν on H .

Proposition 5.1. *For all $v \in \mathcal{V}$, $\tilde{v} \in \tilde{\mathcal{V}}$, the integral*

$$\int_H c_{v, \tilde{v}}(h) d\nu(h)$$

is absolutely convergent.

Proof. Indeed, by [10] Corollary 1.2, a Galois symmetric space as G/H is strongly discrete: the restriction to H of a coefficient of an irreducible discrete series representation of G lies in $L^1(H)$. □

Following Zhang [19], for $v \in \mathcal{V}$ and $\tilde{v} \in \tilde{\mathcal{V}}$, we set

$$\mathcal{L}(v, \tilde{v}) = \int_H c_{v, \tilde{v}}(h) d\nu(h).$$

Then $\mathcal{L}: \mathcal{V} \times \tilde{\mathcal{V}} \rightarrow \mathbb{C}$ is a bi- H -invariant bilinear form. For $\tilde{v} \in \tilde{\mathcal{V}}$, we set

$$\mathcal{L}_{\tilde{v}} : \mathcal{V} \rightarrow \mathbb{C}, v \mapsto \mathcal{L}(v, \tilde{v}).$$

By construction, $\mathcal{L}_{\tilde{v}} \in \text{Hom}_H(\mathcal{V}, \mathbb{C})$ for all $\tilde{v} \in \tilde{\mathcal{V}}$, and we set $\mathcal{H}(\pi) = \{\mathcal{L}_{\tilde{v}} \mid \tilde{v} \in \tilde{\mathcal{V}}\} \subset \text{Hom}_H(\mathcal{V}, \mathbb{C})$.

By [19] Proposition 3.2, G/H is very strongly discrete in the sense of *loc. cit.* Definition 1.3; this means that the linear form

$$\mathcal{C}(G) \longrightarrow \mathbb{C}, f \mapsto \int_H f(h) d\nu(h)$$

is well defined and continuous, where $\mathcal{C}(G)$ denotes the Harish-Chandra’s Schwartz space of G (cf. *loc. cit.* §2). As a consequence, by *loc. cit.* Theorem 1.4, we have the following result.

Theorem 5.2 (Zhang). *We have the equality $\mathcal{H}(\pi) = \text{Hom}_H(\mathcal{V}, \mathbb{C})$ (i.e., any H -invariant linear form on \mathcal{V} is obtained by integration on H against a smooth linear form on \mathcal{V}).*

The following result relates the distinction of π to the integral of a bi- I -invariant coefficient.

Theorem 5.3. *The discrete series representation (π, \mathcal{V}) is H -distinguished if, and only if, there exist $v \in \mathcal{V}^I$ and $\tilde{v} \in \tilde{\mathcal{V}}^I$ such that $\int_H c_{v, \tilde{v}}(h) d\nu(h) \neq 0$.*

Proof. One implication in Theorem 5.3 being obvious, let us assume that (π, \mathcal{V}) is H -distinguished, and let us fix a non-zero $\Lambda \in \text{Hom}_H(\mathcal{V}, \mathbb{C})$. By Theorem 5.2, there exists $\tilde{v}_1 \in \tilde{\mathcal{V}}$ such that $\Lambda = \mathcal{L}_{\tilde{v}_1}$. By Theorem 4.1, pick $v \in \mathcal{V}^I$ such that $\Lambda(v) = \mathcal{L}(v, \tilde{v}_1) \neq 0$. The map $\tilde{\Lambda}: \tilde{v} \mapsto \mathcal{L}(v, \tilde{v})$ is a non-zero element of $\text{Hom}_H(\tilde{\mathcal{V}}, \mathbb{C})$. Applying Theorem 4.1 again to $(\tilde{\pi}, \tilde{\mathcal{V}})$, there exists $\tilde{v} \in \tilde{\mathcal{V}}^I$ such that $\mathcal{L}(v, \tilde{v}) \neq 0$, and we are done. \square

6. Generalized Poincaré series

Recall that we abbreviate $W = W_{\tilde{F}}^{\text{Aff}} = W_E^{\text{Aff}}$.

The Poincaré series of the Coxeter system (W, S) is the formal power series $W(t) = \sum_{w \in W} t^{l(w)} \in \mathbb{Z}[[t]]$. This is the generating function of the length of elements of W . It is known ([3],[18]) to be a rational function of t . We shall need a generalization of the Poincaré series due to Macdonald and Gyoja.

We let S_1, \dots, S_m be the non-empty intersections of S with the conjugacy classes of W . By [4] Proposition 5, page 16, for $w \in W$ the number of elements of S_i occurring in a reduced expression of w depends only on w , we denote it by $l_i(w)$. We have $l(w) = l_1(w) + \dots + l_m(w)$. Let t_1, \dots, t_m be indeterminates. For $w \in W$, we write $\mathbf{t} = (t_1, \dots, t_m)$ and $\mathbf{t}^{l(w)} := t_1^{l_1(w)} \dots t_m^{l_m(w)}$.

By [2] §3.3, the $S_i, i = 1, \dots, m$ are the connected components of the subgraph of the Coxeter graph $\text{Cox}(W, S)$ obtained by erasing the multiple edges. The number m is so given as follows:

- $m = 1$ if G is of type $A_n (n \geq 2), D_n (n \geq 3)$ and $E_i (i = 6, 7, 8)$,
- $m = 2$ if G is of type $A_1, B_n (n \geq 3), G_2$ and F_4 ,
- $m = 3$ if G is of type $C_n (n \geq 2)$.

Remark. There is a mistake in *loc. cit.*. Indeed, Borel claims that $m = 2$ when G is of type $B_2 = C_2$.

We choose the indexing in $(t_i)_{i=1, \dots, m}$ as in [11], pages 173, 174.

Following Gyoja [11], for a representation (r, M) of \mathcal{H}_1 , define an element of $\text{End}_{\mathbb{C}}(M) \otimes_{\mathbb{C}} \mathbb{C}[[t_1, \dots, t_m]]$ by

$$L(\mathbf{t}, r) = \sum_{w \in W} r(e_w) \mathbf{t}^{l(w)}. \tag{7}$$

Theorem 6.1 ([11] Theorem 1.8, page 90). *In any basis of M , the matrix components of $L(\mathbf{t}, r)$ are rational functions of t_1, \dots, t_m .*

When (r, M) is the trivial representation of \mathcal{H}_1 , we set $W(\mathbf{t}) = L(\mathbf{t}, r)$, that is $W(\mathbf{t}) = \sum_{w \in W} \mathbf{t}^{l(w)}$. This rational function has been computed by Macdonald ([14] Theorem 3.3).

When $m = 1$, then $W(\mathbf{t}) = W(t)$ is the ordinary Poincaré series, and Bott’s formula gives

$$W(t) = \prod_{i=1, \dots, l} \frac{1 - t_i^{m_i+1}}{(1 - t_i)(1 - t_i^{m_i})},$$

where m_1, \dots, m_l are the exponents of the spherical Coxeter group attached to W (cf. [4] Chap. V, §6, Définition 2).

If $m = 2$, the formulas are the following.

Type A_1 , $W(t_1, t_2) = \frac{(1 + t_1)(1 + t_2)}{1 - t_1 t_2}$.

Type B_n ($n \geq 3$), $W(t_1, t_2) = \frac{1 - t_1^n}{(1 - t_1)^n} \prod_{i=1}^{n-1} (1 - t_1^{2i}) \prod_{i=0}^{n-1} \frac{1 + t_1^i t_2}{1 - t_1^{n-1+i} t_2}$.

Type G_2 , $W(t_1, t_2) = \frac{(1 + t_1)(1 + t_1 + t_1^2)(1 + t_2)(1 + t_1 t_2 + t_1^2 t_2^2)}{(1 - t_1^2 t_2)(1 - t_1^3 t_2^2)}$.

Type F_4 , $W(t_1, t_2) =$

$$\left(\prod_{i=1}^3 \frac{(1 - t_1^{i+1})(1 + t_1^i t_2)(1 - t_2^i)}{(1 - t_1)(1 - t_2)} \right) \frac{(1 + t_1 t_2^2)(1 + t_1^2 t_2^2)(1 + t_1^3 t_2^3)}{(1 - t_1^3 t_2^2)(1 - t_1^4 t_2^3)(1 - t_1^5 t_2^3)(1 - t_1^6 t_2^5)}.$$

Finally, for $m = 3$, we have the following formula.

Type C_n ($n \geq 2$), $W(t_1, t_2, t_3) = \prod_{i=0}^{n-1} \frac{(1 - t_1^{i+1})(1 + t_1^i t_2)(1 + t_1^i t_3)}{(1 - t_1)(1 - t_1^{n-1+i} t_2 t_3)}$.

Remark. These formulas are taken from [14], page 173.

7. A numerical criterion with application to degree 1 characters

Let (π, \mathcal{V}) be an irreducible discrete series representation of G , and let us assume that it is Iwahori-spherical: $\mathcal{V}^I \neq 0$. Let (r, M) denote the representation of \mathcal{H}_1 in $M = \mathcal{V}^I$. The following result is a numerical criterion to decide whether or not (π, \mathcal{V}) is H -distinguished.

Theorem 7.1. (a) *The representation π is H -distinguished if, and only if, the endomorphism $L(\mathbf{t}_o, r) \in \text{End}_{\mathbb{C}}(M)$ is non-zero, where $\mathbf{t}_o = (\frac{1}{q_o}, \dots, \frac{1}{q_o})$.*

(b) *We have $\dim \text{Hom}_H(\pi, \mathbb{C}) \geq \text{rank } L(\mathbf{t}_o, r)$.*

Proof. (a) We abbreviate $J = I_F^o$. Recall that $JwJ \subset IwI$, for all $w \in W$. Let ν the Haar measure on H normalized by $\nu(J) = 1$. Let $m \in M$ and $\tilde{m} \in \tilde{M}$. We first calculate the integral $\int_H c_{m, \tilde{m}}(h) d\nu(h)$. By the Bruhat-Tits decomposition $H = \sqcup_{w \in W} JwJ$, we have

$$\int_H c_{m, \tilde{m}}(h) d\nu(h) = \sum_{w \in W} \int_{JwJ} c_{m, \tilde{m}}(h) d\nu(h) \tag{8}$$

$$= \frac{\nu(JwJ)}{q^{l(w)}} \sum_{w \in W} \langle \tilde{m}, r(e_w) \cdot m \rangle_M \tag{9}$$

$$= \frac{q_o^{l(w)}}{q^{l(w)}} \sum_{w \in W} \langle \tilde{m}, r(e_w) \cdot m \rangle_M \tag{10}$$

$$= \langle \tilde{m}, \sum_{w \in W} r(e_w) \left(\frac{1}{q_o}\right)^{l(w)} \cdot m \rangle_M \tag{11}$$

$$= \langle \tilde{m}, L(\mathbf{t}_o, r) \cdot m \rangle_M, \tag{12}$$

where in (9), we used Corollary 3.2 and where we set $\mathbf{t}_o = (\frac{1}{q_o}, \dots, \frac{1}{q_o})$. □

By Proposition 5.3, π is H -distinguished if and only if there exist $m \in M$ and $\tilde{m} \in \tilde{M}$ such that $\int_H c_{m, \tilde{m}}(h) d\nu(h) \neq 0$. Thanks to the last calculation, this is indeed equivalent to $L(\mathbf{t}_o, r) \neq 0$, as required.

(b) It follows from (a), using the notation of §5, that for $m \in M$ and $\tilde{m} \in \tilde{M}$, we have $\mathcal{L}_{\tilde{m}}(m) = \langle \tilde{m}, L(\mathbf{t}_o, r) \cdot m \rangle_M$. Let us put $s = \text{rank } L(\mathbf{t}_o, r)$. Then there exist $\tilde{m}_1, \dots, \tilde{m}_s \in \tilde{M}$ such that the $(\mathcal{L}_{\tilde{m}_i})_{|M}$, $i = 1, \dots, s$, are linearly independent. It follows that the $\mathcal{M}_{\tilde{m}_i} \in \text{Hom}_H(\pi, \mathbb{C})$, $i = 1, \dots, s$, are linearly independent, and we are done.

We now assume that the discrete series representation (π, \mathcal{V}) satisfies $\dim M = \dim \mathcal{V}^I = 1$, so that we may view r as an algebra homomorphism $\mathcal{H}_I \rightarrow \mathbb{C}$. Such discrete series were classified in [2] §5 by A. Borel. We recall this classification.

The character r is constant on each $\{e_s \mid s \in S_i\}$, $i = 1, \dots, m$ and because of the quadratic relations, we have $r(e_s) \in \{-1, q\}$, $s \in S$. Following Borel *loc. cit.*, for $i = 1, \dots, m$, we set

$$\epsilon_i = \begin{cases} 1 & \text{if } r(e_s) = q \\ -1 & \text{if } r(e_s) = -1 \end{cases}$$

so that the representation π is entirely characterized by the m -uple $(\epsilon_1, \dots, \epsilon_m)$; we write $\pi = \pi(\epsilon_1, \dots, \epsilon_m)$. Recall that $\mathbf{St}_G = \pi(-1, \dots, -1)$ is the Steinberg representation of G .

Proposition 7.2 (Borel [2] §5.8). *The discrete series representations of G of the form $\pi(\epsilon_1, \dots, \epsilon_m)$ are the Steinberg representation and*

$\pi(-1,1)$ for types B_n ($n \geq 3$), F_4 , G_2 ,
 $\pi(-1, -1,1)$, $\pi(-1,1, -1)$ for types C_2 , C_3 ,
 $\pi(-1, -1,1)$, $\pi(-1,1, -1)$, $\pi(-1,1,1)$ for type C_n ($n \geq 4$).

Lemma 7.3. *Assume that $\pi(\epsilon_1, \dots, \epsilon_m)$ is a discrete series representations of G . We have $L(\mathbf{t}_o, r) = W(\epsilon_1 q_o^{\epsilon_1}, \dots, \epsilon_m q_o^{\epsilon_m})$. In particular, $\pi(\epsilon_1, \dots, \epsilon_m)$ is H -distinguished if, and only if, $W(\epsilon_1 q_o^{\epsilon_1}, \dots, \epsilon_m q_o^{\epsilon_m}) \neq 0$.*

Proof. Observe that for $i = 1, \dots, m$ and $s \in S_i$, we have $r(e_s) = \epsilon_i q_o^{\epsilon_i+1}$, so that for $w \in W$, we have $r(e_w) = \prod_{i=1}^m (\epsilon_i q_o^{\epsilon_i+1})^{l_i(w)}$. Hence, we have

$$\begin{aligned} L(\mathbf{t}_o, r) &= \sum_{w \in W} r(e_w) \left(\frac{1}{q_o}\right)^{l(w)} \\ &= \sum_{w \in W} \prod_{i=1}^m \left(\epsilon_i \frac{q_o^{\epsilon_i+1}}{q_o}\right)^{l_i(w)} \\ &= W(\epsilon_1 q_o^{\epsilon_1}, \dots, \epsilon_m q_o^{\epsilon_m}) \end{aligned}$$

as required. □

We may now state the main result of this article.

Theorem 7.4. *The discrete series representation $\pi(\epsilon_1, \dots, \epsilon_m)$ is H -distinguished if and only if it is a Steinberg representation, or if H is of type G_2 and $(\epsilon_1, \epsilon_2) = (-1, 1)$. When it is distinguished, it has the multiplicity 1 property $\dim \text{Hom}_H(\pi(\epsilon_1, \dots, \epsilon_m), \mathbb{C}) = 1$.*

Proof. The multiplicity 1 property follows from Theorem 4.1. The first assertion is proved by a case-by-case calculation based on Macdonald’s formulas given in §6. Note that in the case of the Steinberg representation, the distinction property was originally D. Prasad’s conjecture ([15] Conjecture 3, p. 77) and was proved in [6] (cf. [1] for a proof of Prasad’s conjecture in the general case). □

Competing interest. None.

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