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On Gap Properties and Instabilities of *p*-Yang–Mills Fields

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Abstract. We consider the p-Yang–Mills functional ($p \geq 2$) defined as $YM_p(\nabla) := \frac{1}{p}\int_M \|R^\nabla\|^p$. We call critical points of $YM_p(\cdot)$ the p-Yang–Mills connections, and the associated curvature R^∇ the p-Yang–Mills fields. In this paper, we prove gap properties and instability theorems for p-Yang–Mills fields over submanifolds in $\mathbb{R}^{n+k}$ and $\mathbb{S}^{n+k}$.

1 Introduction

Let M be a compact Riemannian manifold and E a Riemannian vector bundle over M with structure group G. Denote the space of E-valued p-forms by

$$\Omega^p(E) = \Gamma(\Lambda^p T^*M \otimes E).$$

A connection ∇ on E is $\nabla\colon \Omega^0(E) \to \Omega^1(E)$ which satisfies

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma, \quad \forall f \in C^\infty(M), \sigma \in \Omega^0(E).$$

The space of connections on E is denoted by $\mathcal{C}_E$. For each $\nabla \in \mathcal{C}_E$, the curvature 2-form $R^\nabla \in \Omega^2(\mathfrak{g}_E)$ is defined by $R^\nabla_{X,Y} := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, where $\mathfrak{g}_E$ is the bundle of the Lie algebra of G over M on which there is an invariant metric, and this induces a metric in $\Omega^2(\mathfrak{g}_E)$. For $p \geq 2$, we define the p-Yang–Mills functional as

$$YM_p(\nabla) := \frac{1}{p}\int_M \|R^\nabla\|^p. \tag{1.1}$$

We call critical points of $YM_p(\cdot)$ the p-Yang–Mills connections, and the associated curvature R^∇ the p-Yang–Mills fields. When $p = 2$, (1.1) is the usual Yang–Mills functional.

At each minimizer ∇ of the p-Yang–Mills functional, the second variation is non-negative:

$$\frac{d^2}{dt^2} YM_p(\nabla^t)|_{t=0} \geq 0 \tag{1.2}$$

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for any smooth family of connections ∇^t, with $|t| < \varepsilon$, $\nabla^0 = \nabla$. In general, we call a connection $\nabla \in \mathcal{C}_E$ satisfying (1.2) *weakly stable*. Otherwise, we call ∇ is *unstable*.

The case $p = 2$, *i.e.*, the usual Yang–Mills functional, has been intensively studied. In the well-known papers [1,2], Bourguignon and Lawson obtained a series of results on the stability and gap phenomena of Yang–Mills fields over $\mathbb{S}^n$ and other locally homogeneous spaces. Among other things, they proved the following.

Theorem A (**[2]**) There are no weakly stable Yang–Mills fields over the Euclidean sphere $\mathbb{S}^n$ for $n \geq 5$.

Theorem B (**[2]**) Let R^∇ be a Yang–Mills field over $\mathbb{S}^n$ ($n \geq 5$) which satisfies the pointwise condition

$$\|R^\nabla\|^2 \leq \frac{1}{2}\binom{n}{2}.$$

Then $R^\nabla \equiv 0$.

Xin [8] generalized the above instability result to Yang–Mills fields over compact submanifold M^n of the Euclidean space $\mathbb{R}^{n+k}$ under an assumption on the second fundamental form. Namely, he proved the following.

Theorem C (**[8]**) Let M^n be an n-dimensional compact submanifold in $\mathbb{R}^{n+k}$ with the second fundamental form $h(\cdot,\cdot)$ satisfying the pointwise condition

$$\sum_t \big[2\langle h(e_t,e_i), h(e_t,e_j)\rangle - \langle h(e_t,e_t), h(e_i,e_j)\rangle\big]\,\delta_{kl} + 2\langle h(e_i,e_j), h(e_k,e_l)\rangle \leq b\delta_{ij}\delta_{kl}$$

for $1 \leq i,j,k,l \leq n$, where $\{e_i\}$ is local orthonormal frame on M and $b < 0$ is a constant. Then any Yang–Mills field over M is unstable.

Instability of Yang–Mills fields over submanifolds of spheres $\mathbb{S}^{n+k}$ was obtained by Shen [4], and by Kobayashi, Ohnita and Takeuchi [3]. Results for the case of convex hypersurfaces in $\mathbb{R}^{n+1}$ and compact symmetric spaces can also be found in [3].

Actually, the p-Yang–Mills functional (1.1) was first considered by Uhlenbeck [6] who proved a weak compactness theorem for sequences of connections $\{\nabla_n\}$ with uniformly bounded $YM_p(\nabla_n)$. As a geometric variational model, the p-Yang–Mills functional is a natural generalization of the usual Yang–Mills functional and has interests in its own right. Recall the similar case of p-harmonic maps, where a satisfactory theory of representing homotopy classes is established, and new simple proofs of many well-known theorems in geometry such as the Cartan–Hadamard theorem, the Preisman theorem, the Gromoll–Wolf (or Lawson–Yau) theorem and the Bochner–Frankel theorems can be given by using the tools of p-harmonic maps, *cf.* [7]. On the other hand, a good understanding of the p-Yang–Mills functionals should be helpful for the study of the usual Yang–Mills functionals, as we have seen in [6], and similarly in the well-known work [5] of Sacks and Uhlenbeck who used p-harmonic maps to deduce significant results on the usual harmonic maps. Therefore, it seems natural and interesting to investigate the p-Yang–Mills functional (1.1). In this paper, we focus on instability and gap phenomena of p-Yang–Mills fields over submanifolds M^n of the Euclidean spaces $\mathbb{R}^{n+k}$ and the spheres $\mathbb{S}^{n+k}$.

Suppose M^n is a submanifold of N^{n+k}, and denote the second fundamental form by $h(\cdot,\cdot)$. Set the index ranges $1 \le i, j \le n$; $n+1 \le \mu \le n+k$, and choose local orthonormal frames $\{e_1, e_2, \dots, e_{n+k}\}$ on N such that $\{e_i \mid i = 1, 2, \dots, n\}$ is tangent to M and $\{e_\mu \mid \mu = n+1, \dots, n+k\}$ is normal to M. Let $h(e_i, e_j) := h^\mu_{ij} e_\mu$ and $H^\mu := \sum_i h^\mu_{ii}$; here we use the Einstein summation convention. We will prove the following results.

Theorem 3.1 *Let $M^n (n \ge 5)$ be a submanifold of $\mathbb{R}^{n+k}$ satisfying either*

$$(H^\mu h^\mu_{jl} - h^\mu_{jm} h^\mu_{ml})\delta_{ki} - h^\mu_{ik} h^\mu_{jl} \le (2-n)\delta_{jk}\delta_{il}$$

or

$$(H^\mu h^\mu_{jl} - h^\mu_{jm} h^\mu_{ml})\delta_{ki} - h^\mu_{ik} h^\mu_{jl} \le -(2-n)\delta_{ik}\delta_{jl}.$$

If a p-Yang–Mills field R^∇ over M satisfies

$$\|R^\nabla\|^2 \le \frac{1}{2}\binom{n}{2},$$

then $R^\nabla \equiv 0$.

If $M^n = \mathbb{S}^n \subset \mathbb{R}^{n+1}$, then M satisfies the condition in the above theorem. Therefore when $p = 2$, we obtain Theorem B. Thus Theorem 3.1 is a generalization of a result in [2]. For the case of submanifolds of the Euclidean spheres we have the following.

Theorem 3.2 *Let $M^n (n \ge 5)$ be a submanifold of $\mathbb{S}^{n+k}$ satisfying either*

$$(H^\mu h^\mu_{jl} - h^\mu_{jm} h^\mu_{ml})\delta_{ki} - h^\mu_{ik} h^\mu_{jl} \le b\delta_{jk}\delta_{il}$$

or

$$(H^\mu h^\mu_{jl} - h^\mu_{jm} h^\mu_{ml})\delta_{ki} - h^\mu_{ik} h^\mu_{jl} \le -b\delta_{ik}\delta_{jl}$$

for some $b \le 0$. If a p-Yang–Mills field R^∇ over M satisfies

$$\|R^\nabla\|^2 \le \frac{1}{2}\binom{n}{2},$$

then $R^\nabla \equiv 0$.

For the instability of p-Yang–Mills fields, we will prove the following results in the cases of submanifolds of $\mathbb{R}^{n+k}$ and $\mathbb{S}^{n+k}$.

Theorem 4.1 *Let M^n be a submanifold of $\mathbb{R}^{n+k}$ satisfying*

$$\begin{aligned} C_{ijklsr} &:= (-H^\mu h^\mu_{jl} + 2h^\mu_{jm} h^\mu_{ml})\delta_{ki}\delta_{sr} + 2h^\mu_{ik} h^\mu_{jl}\delta_{sr} + 2(p-2)h_{ik}h_{sr}\delta_{jl} \\ &\le b\delta_{ik}\delta_{jl}\delta_{sr} \end{aligned}$$

for some constant $b < 0$. Then any p-Yang–Mills field over M is unstable.

Remark When $p = 2$, our result is just Theorem C above (Xin [8]). For $M = \mathbb{S}^n$, we have $C_{ijklsr} = (2p - n)\delta_{ik}\delta_{jl}\delta_{sr}$, so *any p-Yang–Mills field over $\mathbb{S}^n$ with $n > 2p$ is unstable.*

Theorem 4.3 *Let M^n be a submanifold of $\mathbb{S}^{n+k}$ satisfying*

$$C_{ijklsr} := (-H^\mu h^\mu_{jl} + 2h^\mu_{jm}h^\mu_{ml})\delta_{ki}\delta_{sr} + 2h^\mu_{ik}h^\mu_{jl}\delta_{sr} + 2(p-2)h_{ik}h_{sr}\delta_{jl} < (n-2p)\delta_{ik}\delta_{jl}\delta_{sr}.$$

Then any p-Yang–Mills field over M is unstable.

2 Preliminaries

Denote by $d^\nabla : \Omega^p(\mathfrak{g}_E) \to \Omega^{p+1}(\mathfrak{g}_E)$ the exterior differential operator with respect to ∇, and by δ^∇ its adjoint operator. The Laplacian is defined by $\Delta^\nabla = d^\nabla\delta^\nabla + \delta^\nabla d^\nabla$. Set $D = \frac{d}{dt}\nabla^t|_{t=0}$, where $\nabla^t = \nabla + A^t$, $A^t \in \Omega^1(\mathfrak{g}_E)$ with $A^0 = 0$. The associated curvature R^{∇^t} of ∇^t is

$$R^{\nabla^t} = R^\nabla + d^\nabla A^t + \frac{1}{2}[A^t \wedge A^t].$$

Recall that for $\phi, \psi \in \mathfrak{g}_E$, $[\phi \wedge \psi]_{X,Y} := [\phi_X, \psi_Y] - [\phi_Y, \psi_X]$.

By direct computation, we have the following *first variational formula*:

$$\frac{d}{dt}YM_p(\nabla^t) = \int_M \|R^{\nabla^t}\|^{p-2}\Big\langle d^\nabla\Big(\frac{dA^t}{dt}\Big) + \Big[A^t \wedge \frac{dA^t}{dt}\Big], R^{\nabla^t}\Big\rangle. \tag{2.1}$$

It follows easily that

$$\frac{d}{dt}YM_p(\nabla^t)|_{t=0} = \int_M \langle \delta^\nabla(\|R^\nabla\|^{p-2}R^\nabla), D\rangle.$$

Consequently, the Euler–Lagrange equation of $YM_p(\,\cdot\,)$ is

$$\delta^\nabla(\|R^\nabla\|^{p-2}R^\nabla) = 0. \tag{2.2}$$

From

$$\frac{dR^{\nabla^t}}{dt} = d^\nabla\frac{dA^t}{dt} + \frac{1}{2}\frac{d}{dt}[A^t \wedge A^t]$$

and (2.1) we have

$$\frac{d}{dt}YM_p(\nabla^t) = \int_M \|R^{\nabla^t}\|^{p-2}\Big\langle \frac{dR^{\nabla^t}}{dt}, R^{\nabla^t}\Big\rangle.$$

Furthermore,

$$\frac{d^2}{dt^2}YM_p(\nabla^t) = (p-2)\int_M \|R^{\nabla^t}\|^{p-4}\Big\langle \frac{dR^{\nabla^t}}{dt}, R^{\nabla^t}\Big\rangle^2 + \int_M \|R^{\nabla^t}\|^{p-2}\Big\|\frac{dR^{\nabla^t}}{dt}\Big\|^2 + \int_M \Big\langle \frac{d^2R^{\nabla^t}}{dt^2}, R^{\nabla^t}\Big\rangle \|R^{\nabla^t}\|^{p-2}.$$

Hence, we have the following *second variational formula:*

$$
\begin{aligned}
\text{(2.3)} \qquad I_p(D) &:= \frac{d^2}{dt^2} YM_p(\nabla^t)|_{t=0} \\
&= (p-2)\int_M \|R^\nabla\|^{p-4}\langle d^\nabla D, R^\nabla\rangle^2 + \int_M \|R^\nabla\|^{p-2}\|d^\nabla D\|^2 \\
&\quad + \int_M \langle [D\wedge D], R^\nabla\rangle \|R^\nabla\|^{p-2}.
\end{aligned}
$$

Next, we derive a useful integral identity via the Weitzenböck formula. Let $\varphi \in \Omega^2(\mathfrak{g}_E)$, and let ω be a linear map-valued 2-form with $(\varphi\circ\omega)_{X,Y} := \frac{1}{2}\varphi_{e_j,\omega_{X,Y}e_j}$. Denote by R and Ric the Riemannian curvature tensor and Ricci curvature operator of M, respectively. Set

$$
\begin{aligned}
(\mathrm{Ric}\wedge I)_{X,Y} &:= \mathrm{Ric}(X)\wedge Y + X\wedge \mathrm{Ric}(Y), \\
\mathcal{R}^\nabla(\varphi)_{X,Y} &:= [R^\nabla_{e_j,X}, \varphi_{e_j,Y}] - [R^\nabla_{e_j,Y}, \varphi_{e_j,X}],
\end{aligned}
$$

where $(X\wedge Y)Z := \langle X, Z\rangle Y - \langle Y, Z\rangle X$.

Lemma 2.1 *For any p-Yang–Mills field R^∇, we have*

$$
\begin{aligned}
\text{(2.4)} \qquad &\int_M \|R^\nabla\|^{p-2}\|\nabla R^\nabla\|^2 + (p-2)\int_M \|R^\nabla\|^{p-2}\|\nabla\|R^\nabla\|\|^2 \\
&\quad + \int_M \|R^\nabla\|^{p-2}\langle R^\nabla \circ (\mathrm{Ric}\wedge I + 2R), R^\nabla\rangle \\
&\quad + \int_M \|R^\nabla\|^{p-2}\langle \mathcal{R}(R^\nabla), R^\nabla\rangle = 0.
\end{aligned}
$$

Proof For any $\varphi \in \Omega^2(\mathfrak{g}_E)$, we have the following Weitzenböck formula [2]:

$$
\Delta^\nabla \varphi = \nabla^*\nabla\varphi + \varphi\circ(\mathrm{Ric}\wedge I + 2R) + \mathcal{R}^\nabla(\varphi).
$$

It follows that

$$
\frac{1}{2}\Delta\|\varphi\|^2 = \left\langle \Delta^\nabla\varphi, \varphi\right\rangle - \|\nabla\varphi\|^2 - \langle \varphi\circ(Ric\wedge I + 2R), \varphi\rangle - \left\langle \mathcal{R}^\nabla(\varphi), \varphi\right\rangle.
$$

Consequently,

$$
\begin{aligned}
\text{(2.5)} \qquad \frac{1}{p}\Delta\|\varphi\|^p &= \frac{1}{2}\|\varphi\|^{p-2}\Delta\|\varphi\|^2 - (p-2)\|\varphi\|^{p-2}\big\|\nabla\|\varphi\|\big\|^2 \\
&= \|\varphi\|^{p-2}[\langle \Delta^\nabla\varphi, \varphi\rangle - \|\nabla\varphi\|^2 - \langle\varphi\circ(\mathrm{Ric}\wedge I + 2R), \varphi\rangle \\
&\qquad - \langle \mathcal{R}^\nabla(\varphi), \varphi\rangle] - (p-2)\|\varphi\|^{p-2}\big\|\nabla\|\varphi\|\big\|^2 \\
&= \|\varphi\|^{p-2}\langle\Delta^\nabla\varphi, \varphi\rangle - \|\varphi\|^{p-2}\|\nabla\varphi\|^2 \\
&\qquad - \|\varphi\|^{p-2}\langle\varphi\circ(\mathrm{Ric}\wedge I + 2R), \varphi\rangle \\
&\qquad - \|\varphi\|^{p-2}\langle\mathcal{R}^\nabla(\varphi), \varphi\rangle - (p-2)\|\varphi\|^{p-2}\big\|\nabla\|\varphi\|\big\|^2.
\end{aligned}
$$

Now let $\varphi = R^\nabla$. Then by (2.2) we have $\delta^\nabla(\|R^\nabla\|^{p-2}R^\nabla) = 0$. Recall that R^∇ satisfies the Bianchi identity: $d^\nabla R^\nabla = 0$. From these we see that

$$\int_M \|R^\nabla\|^{p-2}\langle \Delta^\nabla R^\nabla, R^\nabla\rangle = \int_M \langle d^\nabla \delta^\nabla R^\nabla, \|R^\nabla\|^{p-2}R^\nabla\rangle = \int_M \langle \delta^\nabla R^\nabla, \delta^\nabla(\|R^\nabla\|^{p-2}R^\nabla)\rangle = 0. \tag{2.6}$$

Integrating (2.5) with $\varphi = R^\nabla$ and using (2.6), we obtain (2.4). ■

Let us choose orthonormal frames $\{X_a\}$ of $\mathfrak{g}_E$, and let

$$R^\nabla_{e_i,e_j} := f^a_{ij}X_a, \quad (\nabla_{e_k}R^\nabla)_{e_i,e_j} := f^a_{ijk}X_a.$$

Lemma 2.2

(i) *Let M^n be a submanifold of the Euclidean space $\mathbb{R}^{n+k}$. Then*

$$\langle R^\nabla \circ (\mathrm{Ric}\wedge I + 2R), R^\nabla\rangle = [-(H^\mu h^\mu_{jl} - h^\mu_{jm}h^\mu_{ml})\delta_{ki} + h^\mu_{ik}h^\mu_{jl}] f^a_{ji} f^a_{kl}.$$

(ii) *Let M^n be a submanifold of the sphere $\mathbb{S}^{n+k}$. Then*

$$\langle R^\nabla \circ (\mathrm{Ric}\wedge I + 2R), R^\nabla\rangle = [-(H^\mu h^\mu_{jl} - h^\mu_{jm}h^\mu_{ml})\delta_{ki} + h^\mu_{ik}h^\mu_{jl}] f^a_{ji} f^a_{kl} + 2(n-2)\|R^\nabla\|^2. \tag{2.7}$$

Proof (i) By using the Gauss equation, we can write the Riemannian curvature tensor and the Ricci curvature of M as

$$R_{ijkl} = h^\mu_{ik}h^\mu_{jl} - h^\mu_{il}h^\mu_{jk} \quad \text{and} \quad r_{jl} = H^\mu h^\mu_{jl} - h^\mu_{ji}h^\mu_{il},$$

respectively. Then

$$\begin{aligned}\langle R^\nabla \circ (Ric \wedge I + 2R), R^\nabla\rangle &= \frac{1}{2}[-2r_{lj}\langle R^\nabla_{e_j,e_k}, R^\nabla_{e_k,e_l}\rangle + R_{ijkl}\langle R^\nabla_{e_j,e_i}, R^\nabla_{e_k,e_l}\rangle]\\ &= \frac{1}{2}\Big[-2(H^\mu h^\mu_{jl} - h^\mu_{ji}h^\mu_{il})\langle R^\nabla_{e_j,e_k}, R^\nabla_{e_k,e_l}\rangle\\ &\qquad + (h^\mu_{ik}h^\mu_{jl} - h^\mu_{il}h^\mu_{jk})\langle R^\nabla_{e_j,e_i}, R^\nabla_{e_k,e_l}\rangle\Big]\\ &= -(H^\mu h^\mu_{jl} - h^\mu_{ji}h^\mu_{il}) f^a_{jk} f^a_{kl} + \frac{1}{2}(h^\mu_{ik}h^\mu_{jl} - h^\mu_{il}h^\mu_{jk}) f^a_{ji} f^a_{kl}\\ &= -(H^\mu h^\mu_{jl} - h^\mu_{ji}h^\mu_{il}) f^a_{jk} f^a_{kl} + h^\mu_{ik}h^\mu_{jl} f^a_{ji} f^a_{kl}\\ &= \left[-(H^\mu h^\mu_{jl} - h^\mu_{jm}h^\mu_{ml}\delta_{ki} + h^\mu_{ik}h^\mu_{lj}\right] f^a_{ji} f^a_{kl}.\end{aligned}$$

(ii) In this case, the Riemannian and Ricci curvature tensors can be written as

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h^\mu_{ik}h^\mu_{lj} - h^\mu_{il}h^\mu_{jk}) \quad \text{and} \quad r_{jl} = (n-1)\delta_{jl} + H^\mu h^\mu_{jl} - h^\mu_{ij}h^\mu_{il},$$

respectively, and (2.7) can be proved similarly. ∎

Later on, we will need the following.

Lemma 2.3 ([2]) *If* $\|R^\nabla\|^2 \le \frac{1}{2}\binom{n}{2}$, *then for* $n \ge 3$, *we have*

$$|\langle [R^\nabla_{e_k,e_i}, R^\nabla_{e_i,e_j}], R^\nabla_{e_j,e_k}\rangle| \le 2(n-2)||R^\nabla||^2.$$

Furthermore, when $n \ge 5$ *and* $R^\nabla \ne 0$, *the inequality is strict.*

Proof This is a corollary of [2, Proposition 5.6]. ∎

3 Gap Phenomena of p-Yang–Mills Fields

First, let M^n be a submanifold of $\mathbb{R}^{n+k}$. Suppose R^∇ is a p-Yang–Mills field over M. In this case, we have the following theorem on the gap phenomena of R^∇.

Theorem 3.1 *Suppose* M^n $(n \ge 5)$ *is a submanifold of* $\mathbb{R}^{n+k}$ *satisfying either*

$$(H^\mu h^\mu_{jl} - h^\mu_{jm}h^\mu_{ml})\delta_{ki} - h^\mu_{ik}h^\mu_{jl} \le (2-n)\delta_{jk}\delta_{il}$$

or

$$(H^\mu h^\mu_{jl} - h^\mu_{jm}h^\mu_{ml})\delta_{ki} - h^\mu_{ik}h^\mu_{jl} \le -(2-n)\delta_{ik}\delta_{jl}.$$

If a p*-Yang–Mills field* R^∇ *over* M *satisfies*

$$\|R^\nabla\|^2 \le \frac{1}{2}\binom{n}{2},$$

then $R^\nabla \equiv 0$.

Proof By Lemma 2.1,

$$\begin{aligned}
&\int_M \|R^\nabla\|^{p-2}\|\nabla R^\nabla\|^2 + (p-2)\int_M \|R^\nabla\|^{p-2}\big\|\nabla\|R^\nabla\|\big\|^2 \\
&\quad = -\int_M \|R^\nabla\|^{p-2}\langle R^\nabla \circ (Ric \wedge I + 2R), R^\nabla\rangle - \int_M \|R^\nabla\|^{p-2}\langle \mathfrak{R}^\nabla(R^\nabla), R^\nabla\rangle \\
&\quad := (\mathrm{I}) + (\mathrm{II}).
\end{aligned}$$

Using Lemma 2.2(i) and the assumptions on h^μ_{ij} of M, we have

$$(\mathrm{I}) \le 2(2-n)\int_M \|R^\nabla\|^p.$$

From Lemma 2.3, and noting that $\langle \mathcal{R}^\nabla(R^\nabla), R^\nabla \rangle = \langle [R^\nabla_{e_k,e_i}, R^\nabla_{e_i,e_j}], R^\nabla_{e_j,e_k} \rangle$, we see that if R^∇ is not identically zero, then

$$(\mathrm{II}) < 2(n-2) \int_M \|R^\nabla\|^p.$$

Combining these we deduce that

$$\int_M \|R^\nabla\|^{p-2} \|\nabla R^\nabla\|^2 + (p-2) \int_M \|R^\nabla\|^{p-2} \big\| \nabla \|R^\nabla\| \big\|^2 < 0,$$

which is a contradiction. Thus, $R^\nabla \equiv 0$. ■

In a similar way, we can prove the following.

Theorem 3.2 *Let M^n $(n \geq 5)$ be a submanifold of $\mathbb{S}^{n+k}$ satisfying either*

$$(H^\mu h^\mu_{jl} - h^\mu_{jm} h^\mu_{ml})\delta_{ki} - h^\mu_{ik} h^\mu_{jl} \leq b\delta_{jk}\delta_{il}$$

or

$$(H^\mu h^\mu_{jl} - h^\mu_{jm} h^\mu_{ml})\delta_{ki} - h^\mu_{ik} h^\mu_{jl} \leq -b\delta_{ik}\delta_{jl}$$

for some $b \leq 0$. If a p-Yang–Mills field R^∇ over M satisfies

$$\|R^\nabla\|^2 \leq \frac{1}{2}\binom{n}{2},$$

then $R^\nabla \equiv 0$.

We remark that if we let $M^n = S^n \subset \mathbb{R}^{n+1}$ in Theorem 3.1, then it is easy to see that

$$(H^\mu h^\mu_{jl} - h^\mu_{jm} h^\mu_{ml})\delta_{ki} - h^\mu_{ik} h^\mu_{jl} = (n-2)\delta_{jl}\delta_{ki}.$$

Therefore, Theorem 3.1 generalizes the theorem of Bourguignon and Lawson mentioned above (Theorem B). More generally, for convex hypersurfaces M^n of $\mathbb{R}^{n+1}$, if we write $h^{n+1}_{ij} := h_{ij} = \lambda_i \delta_{ij}$ where λ_i is the i-th principal curvature of M, $i = 1, 2, \ldots, n$, $H := \lambda_1 + \lambda_2 + \cdots + \lambda_n$, then

$$(H^\mu h^\mu_{jl} - h^\mu_{jm} h^\mu_{ml})\delta_{ki} - h^\mu_{ik} h^\mu_{jl} = (H\lambda_j - \lambda_j \lambda_l - \lambda_i \lambda_j)\delta_{jl}\delta_{ki}.$$

We thus obtain the following.

Corollary 3.3 *Suppose M^n $(n \geq 5)$ is a convex hypersurface of $\mathbb{R}^{n+1}$ satisfying*

$$\lambda_j(H - \lambda_i - \lambda_j) \leq n - 2, \quad i, j = 1, 2, \ldots, n,$$

where λ_i is the i-th principal curvature and H is the mean curvature of M. Then any p-Yang–Mills field R^∇ over M with $\|R^\nabla\|^2 \leq \frac{1}{2}\binom{n}{2}$ must identically vanish.

Similarly, we also have the following.

Corollary 3.4 *Suppose M^n ($n \geq 5$) is a convex hypersurface of $\mathbb{S}^{n+1}$ satisfying*

$$\lambda_j(H - \lambda_i - \lambda_j) \leq 0, \quad i, j = 1, 2, \ldots, n,$$

where λ_i is the i-th principal curvature and H is the mean curvature of M. Then any p-Yang–Mills field R^∇ over M with $\|R^\nabla\|^2 \leq \frac{1}{2}\binom{n}{2}$ must identically vanish.

4 Instability of p-Yang–Mills Fields

In this section, we will prove some results on instability of p-Yang–Mills fields R^∇ over submanifolds M^n of $\mathbb{R}^{n+k}$ and $\mathbb{S}^{n+k}$.

Theorem 4.1 *Let M^n be a submanifold of $\mathbb{R}^{n+k}$ satisfying*

$$C_{ijklsr} := (-H^\mu h^\mu_{jl} + 2h^\mu_{jm}h^\mu_{ml})\delta_{ki}\delta_{sr} + 2h^\mu_{ik}h^\mu_{jl}\delta_{sr} + 2(p-2)h_{ik}h_{sr}\delta_{jl} \leq b\delta_{ik}\delta_{jl}\delta_{sr}$$

for some constant $b < 0$. Then any p-Yang–Mills field over M is unstable.

Proof We first note that for tangent vectors V, X to M, let $D = i_V R^\nabla$. Then $D_X = (i_V R^\nabla)_X = R^\nabla_{V,X}$, and

$$\begin{aligned}(d^\nabla D)_{e_i,e_j} &= (\nabla_{e_i} D)_{e_j} - (\nabla_{e_j} D)_{e_i}\\ &= (\nabla_{e_i} R^\nabla)_{V,e_j} - (\nabla_{e_j} R^\nabla)_{V,e_i} + R^\nabla_{\nabla_{e_i}V,e_j} - R^\nabla_{\nabla_{e_j}V,e_i}.\end{aligned}$$

Now take the standard orthonormal basis $\{E_A \mid A = 1, 2, \ldots, n+k\}$ of $\mathbb{R}^{n+k}$, and choose $V_A := v^i_A e_i$ to be the tangent part of E_A. Here the indices A, B, C run from 1 to $n+k$. We note that

$$v^B_A v^C_A = \delta_{BC}, \quad \nabla_{e_i} V_A = v^\mu_A h^\mu_{ij} e_j. \tag{4.1}$$

Then for $D_A := i_{V_A} R^\nabla$, $A = 1, 2, \ldots, n+k$, it follows from (2.3) that

$$\begin{aligned}\sum_A I_p(D_A) &= (p-2)\sum_A \int_M \|R^\nabla\|^{p-4}\langle R^\nabla, d^\nabla D_A\rangle^2\\ &\quad + \sum_A \int_M \|R^\nabla\|^{p-2}\|d^\nabla D_A\|^2 + \sum_A \int_M \langle R^\nabla, [D_A \wedge D_A]\rangle \|R^\nabla\|^{p-2}.\end{aligned} \tag{4.2}$$

Since for $i = 1, 2, \ldots, n$ and $A = 1, 2, \ldots, n+k$,

$$\begin{aligned}(d^\nabla D_A)_{e_i,e_j} &= (\nabla_{e_i} R^\nabla)_{V_A,e_j} - (\nabla_{e_j} R^\nabla)_{V_A,e_i} + R^\nabla_{\nabla_{e_i}V_A,e_j} - R^\nabla_{\nabla_{e_j}V_A,e_i}\\ &= v^l_A(\nabla_{e_i} R^\nabla)_{e_l,e_j} - v^l_A(\nabla_{e_j} R^\nabla)_{e_l,e_i} + v^\mu_A h^\mu_{il} R^\nabla_{e_l,e_j} - v^\mu_A h^\mu_{jl} R^\nabla_{e_l,e_i},\end{aligned} \tag{4.3}$$

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we have

$$\begin{aligned}\langle R^{\nabla}, d^{\nabla} D_A\rangle &= \frac{1}{2}\langle R^{\nabla}_{e_i,e_j}, (d^{\nabla} D_A)_{e_i,e_j}\rangle\\
&= \frac{1}{2}\nu^l_A\langle R^{\nabla}_{e_i,e_j}, (\nabla_{e_i} R^{\nabla})_{e_l,e_j}\rangle - \frac{1}{2}\nu^l_A\langle R^{\nabla}_{e_i,e_j}, (\nabla_{e_j} R^{\nabla})_{e_l,e_i}\rangle\\
&\quad + \frac{1}{2}\nu^{\mu}_A h^{\mu}_{il}\langle R^{\nabla}_{e_i,e_j}, R^{\nabla}_{e_l,e_j}\rangle - \frac{1}{2}\nu^{\mu}_A h^{\mu}_{jl}\langle R^{\nabla}_{e_i,e_j}, R^{\nabla}_{e_l,e_i}\rangle\\
&= \nu^l_A\langle R^{\nabla}_{e_i,e_j}, (\nabla_{e_i} R^{\nabla})_{e_l,e_j}\rangle + \nu^{\mu}_A h^{\mu}_{il}\langle R^{\nabla}_{e_i,e_j}, R^{\nabla}_{e_l,e_j}\rangle,\end{aligned}$$

from which with (4.1) we have

$$\begin{aligned}\sum_A \langle R^{\nabla}, d^{\nabla} D_A\rangle^2 = \sum_l &\langle R^{\nabla}_{e_i,e_j}, (\nabla_{e_i} R^{\nabla})_{e_l,e_j}\rangle^2\\
&+ h^{\mu}_{il} h^{\mu}_{tm}\langle R^{\nabla}_{e_i,e_j}, R^{\nabla}_{e_l,e_j}\rangle\langle R^{\nabla}_{e_t,e_s}, R^{\nabla}_{e_m,e_s}\rangle.\end{aligned} \tag{4.4}$$

Using the second Bianchi identity, we have

$$\begin{aligned}\langle R^{\nabla}_{e_i,e_j}, (\nabla_{e_i} R^{\nabla})_{e_l,e_j}\rangle &= -\langle R^{\nabla}_{e_i,e_j}, (\nabla_{e_l} R^{\nabla})_{e_j,e_i}\rangle - \langle R^{\nabla}_{e_i,e_j}, (\nabla_{e_j} R^{\nabla})_{e_i,e_l}\rangle\\
&= \langle R^{\nabla}_{e_i,e_j}, (\nabla_{e_l} R^{\nabla})_{e_i,e_j}\rangle - \langle R^{\nabla}_{e_j,e_i}, (\nabla_{e_j} R^{\nabla})_{e_l,e_i}\rangle,\end{aligned}$$

which implies

$$\sum_{ij}\langle R^{\nabla}_{e_i,e_j}, (\nabla_{e_i} R^{\nabla})_{e_l,e_j}\rangle = \frac{1}{2}\sum_{ij}\langle R^{\nabla}_{e_i,e_j}, (\nabla_{e_l} R^{\nabla})_{e_i,e_j}\rangle = \langle R^{\nabla}, \nabla_{e_l} R^{\nabla}\rangle.$$

Putting this into (4.4) then yields

$$\begin{aligned}\sum_A \langle R^{\nabla}, d^{\nabla} D_A\rangle^2 &= \sum_l \langle R^{\nabla}, \nabla_{e_l} R^{\nabla}\rangle^2 + h^{\mu}_{il} h^{\mu}_{tm}\langle R^{\nabla}_{e_i,e_j}, R^{\nabla}_{e_l,e_j}\rangle\langle R^{\nabla}_{e_t,e_s}, R^{\nabla}_{e_m,e_s}\rangle\\
&= \|R^{\nabla}\|^2 \|\nabla \| R^{\nabla}\| \|^2 + h^{\mu}_{il} h^{\mu}_{tm}\langle R^{\nabla}_{e_i,e_j}, R^{\nabla}_{e_l,e_j}\rangle\langle R^{\nabla}_{e_t,e_s}, R^{\nabla}_{e_m,e_s}\rangle.\end{aligned}$$

Hence

$$\begin{aligned}(p-2)\sum_A \int_M \|R^{\nabla}\|^{p-4}\langle R^{\nabla}, d^{\nabla} D_A\rangle^2 &= (p-2)\int_M \|R^{\nabla}\|^{p-2}\|\nabla \|R^{\nabla}\|\|^2\\
&\quad + (p-2)\int_M \|R^{\nabla}\|^{p-4} h^{\mu}_{il} h^{\mu}_{tm}\langle R^{\nabla}_{e_i,e_j}, R^{\nabla}_{e_l,e_j}\rangle\langle R^{\nabla}_{e_t,e_s}, R^{\nabla}_{e_m,e_s}\rangle.\end{aligned} \tag{4.5}$$

The second term on the right-hand side can be written as

$$(p-2)\int_M \|R^{\nabla}\|^{p-4} h^{\mu}_{il} h^{\mu}_{tm} f^a_{ij} f^a_{lj} f^b_{ts} f^b_{ms} = (p-2)\int_M \|R^{\nabla}\|^{p-4} h^{\mu}_{ik} h^{\mu}_{sr}\delta_{jl}\delta_{qt} f^a_{ij} f^a_{kl} f^b_{st} f^b_{rq}.$$

Inserting this into (4.5) yields:

$$(4.6)\quad (p-2)\sum_A \int_M \|R^\nabla\|^{p-4}\langle R^\nabla, d^\nabla D_A\rangle^2 = (p-2)\int_M \|R^\nabla\|^{p-2}\|\nabla\|R^\nabla\|\|^2 + \int_M \|R^\nabla\|^{p-4}[(p-2)h^\mu_{ik}h^\mu_{sr}\delta_{jl}\delta_{qt}]f^a_{ij}f^a_{kl}f^b_{st}f^b_{rq}.$$

Now we compute the second term on the right-hand side of (4.2). By (4.3),

$$\sum_A \|d^\nabla D_A\|^2 = \frac{1}{2}\sum_A \langle (d^\nabla D_A)_{e_i,e_j}, (d^\nabla D_A)_{e_i,e_j}\rangle = f^a_{ijk}f^a_{ijk} - f^a_{kji}f^a_{kij} + h^\mu_{ik}h^\mu_{il}f^a_{kj}f^a_{lj} - h^\mu_{ik}h^\mu_{jl}f^a_{kj}f^a_{li}.$$

Since from the Bianchi identity we have $f^a_{kji}f^a_{kij} = \frac{1}{2}f^a_{ijk}f^a_{ijk} = \|\nabla R^\nabla\|^2$, therefore

$$\sum_A \|d^\nabla D_A\|^2 = \|\nabla R^\nabla\|^2 + (h^\mu_{ik}h^\mu_{il}f^a_{kj}f^a_{lj} - h^\mu_{ik}h^\mu_{jl}f^a_{kj}f^a_{li}).$$

Consequently,

$$(4.7)\quad \sum_A \int_M \|R^\nabla\|^{p-2}\|d^\nabla D_A\|^2 = \int_M \|R^\nabla\|^{p-2}\|\nabla R^\nabla\|^2 + \int_M \|R^\nabla\|^{p-2}(h^\mu_{ik}h^\mu_{il}f^a_{kj}f^a_{lj} - h^\mu_{ik}h^\mu_{jl}f^a_{kj}f^a_{li}).$$

As for the third term on the right-hand side of (4.2), we first note that

$$\begin{aligned}\langle R^\nabla, [D_A\wedge D_A]\rangle &= \frac{1}{2}\langle R^\nabla_{e_j,e_k}, [D_A\wedge D_A]_{e_j,e_k}\rangle \\ &= \langle R^\nabla_{e_j,e_k}, [D_{A,e_j}, D_{A,e_k}]\rangle = -\langle R^\nabla_{e_j,e_k}, [D_{A,e_k}, D_{A,e_j}]\rangle \\ &= -\langle R^\nabla_{e_j,e_k}, [R^\nabla_{V_A,e_k}, R^\nabla_{V_A,e_j}]\rangle = -v^i_A v^l_A\langle R^\nabla_{e_j,e_k}, [R^\nabla_{e_i,e_k}, R^\nabla_{e_l,e_j}]\rangle \\ &= -\langle R^\nabla_{e_j,e_k}, [R^\nabla_{e_i,e_k}, R^\nabla_{e_i,e_j}]\rangle = \langle \mathcal{R}^\nabla(R^\nabla), R^\nabla\rangle.\end{aligned}$$

Hence,

$$(4.8)\qquad \sum_A \int_M \langle R^\nabla, [D_A\wedge D_A]\rangle\|R^\nabla\|^{p-2} = \int_M \langle \mathcal{R}^\nabla(R^\nabla), R^\nabla\rangle\|R^\nabla\|^{p-2}.$$

Substituting (4.6), (4.7) and (4.8) into (4.2) yields

$$\sum_A I_p(D_A) = (p-2)\int_M \|R^\nabla\|^{p-2}\big\|\nabla\|R^\nabla\|\big\|^2 + \int_M \|R^\nabla\|^{p-2}\|\nabla R^\nabla\|^2$$
$$+ \int_M \|R^\nabla\|^{p-2}(h^\mu_{ik}h^\mu_{il}f^a_{kj}f^a_{lj} - h^\mu_{ik}h^\mu_{jl}f^a_{kj}f^a_{li})$$
$$+ \int_M \|R^\nabla\|^{p-4}[(p-2)h^\mu_{ik}h^\mu_{sr}\delta_{jl}\delta_{qt}]f^a_{ij}f^a_{kl}f^b_{st}f^b_{rq}$$
$$+ \int_M \langle \mathcal{R}^\nabla(R^\nabla), R^\nabla\rangle \|R^\nabla\|^{p-2}.$$

By Lemma 2.1, we obtain that

$$\sum_A I(D_A) = -\int_M \|R^\nabla\|^{p-2}\langle R^\nabla \circ (Ric\wedge I + 2R), R^\nabla\rangle$$
$$+ \int_M \|R^\nabla\|^{p-4}[(p-2)h^\mu_{ik}h^\mu_{sr}\delta_{jl}\delta_{qt}]f^a_{ij}f^a_{kl}f^b_{st}f^b_{rq}$$
$$+ \int_M \|R^\nabla\|^{p-2}(h^\mu_{ik}h^\mu_{il}f^a_{kj}f^a_{lj} - h^\mu_{ik}h^\mu_{jl}f^a_{kj}f^a_{li}).$$

Using Lemma 2.2(i), we then have

$$\sum_A I(D_A) = \int_M \|R^\nabla\|^{p-2}[-(H^\mu h^\mu_{jl} - h^\mu_{jm}h^\mu_{ml})\delta_{ki} + h^\mu_{ik}h^\mu_{jl}]f^a_{ij}f^a_{kl}$$
$$+ \int_M \|R^\nabla\|^{p-4}[(p-2)h^\mu_{ik}h^\mu_{sr}\delta_{jl}\delta_{qt}]f^a_{ij}f^a_{kl}f^b_{st}f^b_{rq}$$
$$+ \int_M \|R^\nabla\|^{p-2}(h^\mu_{ik}h^\mu_{il}f^a_{kj}f^a_{lj} - h^\mu_{ik}h^\mu_{jl}f^a_{kj}f^a_{li})$$
$$= \int_M \|R^\nabla\|^{p-2}[(-H^\mu h^\mu_{jl} + 2h^\mu_{jm}h^\mu_{ml})\delta_{ki} + 2h^\mu_{ik}h^\mu_{jl}]f^a_{ij}f^a_{kl}$$
$$+ \int_M \|R^\nabla\|^{p-4}[(p-2)h^\mu_{ik}h^\mu_{sr}\delta_{jl}\delta_{qt}]f^a_{ij}f^a_{kl}f^b_{st}f^b_{rq}$$
$$= \frac{1}{2}\int_M \|R^\nabla\|^{p-4}[(-H^\mu h^\mu_{jl} + 2h^\mu_{jm}h^\mu_{ml})\delta_{ki} + 2h^\mu_{ik}h^\mu_{jl}]f^a_{ij}f^a_{kl}f^b_{st}f^b_{rq}$$
$$+ \int_M \|R^\nabla\|^{p-4}[(p-2)h^\mu_{ik}h^\mu_{sr}\delta_{jl}\delta_{qt}]f^a_{ij}f^a_{kl}f^b_{st}f^b_{rq}$$
$$= \frac{1}{2}\int_M \|R^\nabla\|^{p-4}[(-H^\mu h^\mu_{jl} + 2h^\mu_{jm}h^\mu_{ml})\delta_{ki}\delta_{sr}\delta_{tq} + 2h^\mu_{ik}h^\mu_{jl}\delta_{sr}\delta_{tq}$$
$$+ 2(p-2)h^\mu_{ik}h^\mu_{sr}\delta_{jl}\delta_{qt}]f^a_{ij}f^a_{kl}f^b_{st}f^b_{rq}$$

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$$= \frac{1}{2}\int_M \|R^\nabla\|^{p-4}[(-H^\mu h^\mu_{jl} + 2h^\mu_{jm}h^\mu_{ml})\delta_{ki}\delta_{sr} + 2h^\mu_{ik}h^\mu_{jl}\delta_{sr} + 2(p-2)h^\mu_{ik}h^\mu_{sr}\delta_{jl}] f^a_{ij} f^a_{kl} f^b_{st} f^b_{rt}.$$

Let $C_{ijklsr} := (-H^\mu h^\mu_{jl} + 2h^\mu_{jm}h^\mu_{ml})\delta_{ki}\delta_{sr} + 2h^\mu_{ik}h^\mu_{jl}\delta_{sr} + 2(p-2)h^\mu_{ik}h^\mu_{sr}\delta_{jl}$. Then

$$\sum_A I(D_A) = \frac{1}{2}\int_M \|R^\nabla\|^{p-4} C_{ijklsr} f^a_{ij} f^a_{kl} f^b_{st} f^b_{rt}. \tag{4.9}$$

By the assumption on C_{ijklsr}, we obtain that

$$\begin{aligned}\sum_A I(D_A) &\le \frac{1}{2} b \int_M \|R^\nabla\|^{p-4}\delta_{ik}\delta_{jl}\delta_{sr} f^a_{ij} f^a_{kl} f^b_{st} f^b_{rt}\\ &= \frac{b}{2}\int_M \|R^\nabla\|^{p-4} f^a_{ij} f^a_{ij} f^b_{st} f^b_{st}\\ &= 2b\int_M \|R^\nabla\|^p < 0.\end{aligned}$$

Therefore, R^∇ is unstable. This completes the proof. ■

Corollary 4.2 *Let M^n be a convex hypersurface of $\mathbb{R}^{n+1}$ with principal curvature $\lambda_1, \lambda_2, \dots, \lambda_n$ and mean curvature $H = \sum_i \lambda_i$ satisfying*

$$H\lambda_j > 2\lambda_i\lambda_j + 2\lambda_j^2 + (2p-4)\lambda_i\lambda_k, \quad \forall i, j, k = 1, 2, \dots, n,$$

then any p-Yang–Mills field R^∇ over M is unstable. In particular, any p-Yang–Mills field over $\mathbb{S}^n$ $(n > 2p)$ is unstable.

Proof Direct calculations show that for submanifold M^n in $\mathbb{R}^{n+1}$, the following holds:

$$C_{ijklsr} = [2\lambda_i\lambda_j + 2\lambda_j\lambda_l - H\lambda_j + (2p-4)\lambda_i\lambda_s]\delta_{ik}\delta_{jl}\delta_{sr}.$$

In particular, for $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, $C_{ijklsr} = (2p-n)\delta_{ik}\delta_{jl}\delta_{sr}$. The conclusions then follow from these and Theorem 4.1. ■

This result generalizes [8, Theorem 3] and [3, Theorem 5.3]. Now let us consider the case that M^n is a submanifold of the sphere $\mathbb{S}^{n+k}$. We note that the second formula in (4.1) becomes

$$\nabla_{e_i} V_A = (v^\mu_A h^\mu_{ij} + v^{n+k+1}_A \delta_{ij}) e_j. \tag{4.10}$$

Here h^μ_{ij} is a component of the second fundamental form of M in $\mathbb{S}^{n+k}$.

Theorem 4.3 *Let M^n be a submanifold of $\mathbb{S}^{n+k}$ satisfying*

$$C_{ijklsr} := (-H^\mu h^\mu_{jl} + 2h^\mu_{jm}h^\mu_{ml})\delta_{ki}\delta_{sr} + 2h^\mu_{ik}h^\mu_{jl}\delta_{sr} + 2(p-2)h_{ik}h_{sr}\delta_{jl} < (n-2p)\delta_{ik}\delta_{jl}\delta_{sr}.$$

Then any p-Yang–Mills field over M is unstable.

Proof Comparing to the proof of Theorem 4.1 and using (4.10), it follows that (4.6) becomes:

$$(p-2)\sum_A \int_M \|R^\nabla\|^{p-4}\langle R^\nabla, d^\nabla D_A\rangle^2 = (p-2)\int_M \|R^\nabla\|^{p-2}\|\nabla\|R^\nabla\|\|^2 + \int_M \|R^\nabla\|^{p-4}[(p-2)h^\mu_{ik}h^\mu_{sr}\delta_{jl}\delta_{qt}] f^a_{ij} f^a_{kl} f^b_{ts} f^b_{rq} + 4(p-2)\int_M \|R^\nabla\|^p. \tag{4.11}$$

Also, corresponding to (4.7) we have

$$\sum_A \int_M \|R^\nabla\|^{p-2}\|d^\nabla D_A\|^2 = \int_M \|R^\nabla\|^{p-2}\|\nabla R^\nabla\|^2 + \int_M \|R^\nabla\|^{p-2}(h^\mu_{ik}h^\mu_{il} f^a_{kj} f^a_{lj} - h^\mu_{ik}h^\mu_{jl} f^a_{kj} f^a_{li}) + 4\int_M \|R^\nabla\|^p. \tag{4.12}$$

We note that (4.8) remains unchanged, that is, we still have

$$\sum_A \int_M \langle R^\nabla, [D_A \wedge D_A]\rangle \|R^\nabla\|^{p-2} = -\int_M \langle \mathcal{R}^\nabla(R^\nabla), R^\nabla\rangle \|R^\nabla\|^{p-2}. \tag{4.13}$$

Putting (4.11), (4.12) and (4.13) into (4.2) gives

$$\begin{aligned}\sum_A I(D_A) = &-\int_M \|R^\nabla\|^{p-2}\langle R^\nabla \circ (Ric \wedge I + 2R), R^\nabla\rangle \\ &+ \int_M \|R^\nabla\|^{p-4}[(p-2)h^\mu_{ik}h^\mu_{sr}\delta_{jl}\delta_{qt}] f^a_{ij} f^a_{kl} f^b_{st} f^b_{rq} \\ &+ \int_M \|R^\nabla\|^{p-2}(h^\mu_{ik}h^\mu_{il} f^a_{kj} f^a_{lj} - h^\mu_{ik}h^\mu_{jl} f^a_{kj} f^a_{li}) \\ &+ (4p-4)\int_M \|R^\nabla\|^p.\end{aligned}$$

Similar to deriving (4.9), except that here we use Lemma 2.2(ii) instead of Lemma 2.2(i), we have

$$\sum_A I(D_A) = \frac{1}{2}\int_M \|R^\nabla\|^{p-4} C_{ijklsr} f^a_{ij} f^a_{kl} f^b_{st} f^b_{rt} + (4p-2n)\int_M \|R^\nabla\|^p.$$

Since $C_{ijklsr} < (n-2p)\delta_{ik}\delta_{jl}\delta_{sr}$, it follows that

$$\sum_A I(D_A) < (2n-4p)\int_M \|R^\nabla\|^p + (4p-2n)\int_M \|R^\nabla\|^p = 0,$$

which means that R^∇ is unstable. ■

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