Poles and Polars of a Conic.

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(ABSTRACT.)

1. Being given a fixed line (the directrix) and two fixed points S, s (Fig. 15), then, if z and Z are two points on the directrix, the lines zp and ZP are said to correspond if zp be parallel to SZ and sz be parallel to ZP.

Theorem 1. If pairs of corresponding lines meet in p and P respectively, then sp is parallel to SP; and, if any line goes through p, the corresponding line goes through P.

Let pz, pz' correspond to PZ, PZ′, then sz and sz' are parallel to PZ, and PZ′. Hence

$$\frac{PZ}{ZZ'} = \frac{sz}{zz'}; \text{ and similarly } \frac{SZ}{ZZ'} = \frac{pz}{zz'}$$

$$\therefore \quad \frac{PZ}{ZS} = \frac{sz}{zp} \text{ and } P\hat{Z}S = \widehat{szp}.$$

Hence Δ PZS is similar to Δ szp and PS′ is parallel to sp; and clearly, if p is on zp, P is on the corresponding line ZP.

Theorem 2. If a pair of points p, q, correspond to a pair P, Q, and if pq and PQ meet the directrix in z and Z, then PQ is parallel to sz and pq is parallel to SZ.

Draw SZ parallel to zp, and through Z a line parallel to sz. Then P must lie on this line, Q therefore also lies on it, and therefore PQ is parallel to sz; and so on.

Theorem 3. If the point P moves in a conic with S as focus and the given line as directrix, p traces out a circle.

For, making PZ perpendicular to the directrix for convenience simply, we have

$$\frac{SP}{PZ} = e = \frac{sp}{sz} \quad \therefore \; sp = e.sz = \text{constant}.$$

The circle is an eccentric circle of the conic.

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2. Parallel lines on the one system correspond to lines meeting on the directrix in the other. Hence, propositions like the following can be proved.

If pairs of points be taken on three concurrent lines, the three points of intersection of lines joining pairs of corresponding points are collinear.

For the three concurrent lines can be transformed into three parallel lines and the pairs of points into pairs of points on parallel lines, a particular case in which the theorem is easily proved.

Tangents to a curve in one system correspond to tangents in the other, and chords of contact to chords of contact. Whence the usual theorems regarding tangents to a conic are easily proved.

The above transformation is easily seen to be a particular case of the projective transformation, its analytical representation being of the form

$$x = k/\mathrm{X},\ y = -l\mathrm{Y}/\mathrm{X}.$$

3. The following is an example of the application of the method in the proof of theorems regarding poles and polars (Fig. 16).

If QQ′, a chord of a conic with S as focus and the given line as directrix, passes through a fixed point O, PQ and PQ′, the tangents at Q and Q′ meet in P, which lies on a fixed straight line.

Let QQ′ meet the directrix in Z. Take any point s and draw sz parallel to QZ, zq parallel to SZ, sq parallel to SQ and sq' parallel SQ′. Then q and q' are on the eccentric circle of s. Draw so parallel to SO and let kp, the polar of o, cut the directrix in k. Finally draw sp parallel to SP.

Since sp bisects qsq', the tangents at q and q' meet in sp, and, since kp is the polar of o, the tangents meet on kp. Therefore pq and pq' are tangent to the eccentric circle, and they correspond to the lines PQ, PQ′. Hence p and k correspond to P and K and therefore KP is parallel to sk. Now K is a fixed point and KP is parallel to a fixed line. Hence the proposition is proved.

All the theorems regarding poles and polars to a conic may be proved in a similar manner.