

INJECTIVE HULLS AS COMPLETIONS

by PAUL D. BACSICH†

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A preliminary attempt is made to place the theory of completions of boolean algebras and of partially ordered sets in a wider context. The theory and construction of injective hulls in abelian categories is generalised and it is demonstrated that any variety with enough injectives admits injective hulls. Then the methods developed are applied to a non-algebraic bicategory, that of ordered sets.

1. Bicategories. We assume that the reader is familiar with the elementary aspects of the theory of bicategories as expounded in Isbell [12] and Semadeni [16].

Let K be a bicategory. We recall that an object A of K is said to be *injective* if whenever $e: B \rightarrow C$ is an injection and $f \in \text{Hom}(B, A)$, then there is $h \in \text{Hom}(C, A)$ such that $he = f$. An injection $e: A \rightarrow A'$ is called *proper* if it is not an equivalence, and *dense* if, whenever $f \in \text{Hom}(A', B)$ is such that $fe: A \rightarrow B$ is an injection, then f is an injection. A' is called an *extension of A by e* if $e: A \rightarrow A'$ is an injection.

In this section we state a number of simple lemmas which are necessary for the proof of the main theorem in §2. Proofs for the first two are supplied so as to establish the techniques to use in the nonabelian case, and using these the other lemmas should be obvious. (A treatment of the abelian case can be found in [14], Section III.2.)

LEMMA 1.1. *The composition of dense injections is a dense injection.*

Proof. Let $e: A \rightarrow B, f: B \rightarrow C$ be dense injections. Then $fe: A \rightarrow C$ is an injection. Let $g \in \text{Hom}(C, D)$ be such that gfe is an injection. Then gf is an injection as e is dense, and so g is an injection as f is dense.

LEMMA 1.2. *Let $e: A \rightarrow B, e': A \rightarrow C$ be dense injections, $f \in \text{Hom}(B, C)$ be such that $fe = e'$. Then f is a dense injection.*

Proof. f is an injection, as e is dense. Now let $g \in \text{Hom}(C, D)$ be such that gf is an injection. Then $gfe = ge'$ is an injection, and so g is an injection as e' is dense.

DEFINITION 1.3.

(1) A pair (e, A') is an *injective extension* of A if A' is injective and $e: A \rightarrow A'$ is an injection.

(2) An extension (e, A') of A is called *dense* if e is dense.

(3) (e, A') is an *injective hull* of A if (e, A') is a dense injective extension of A .

(4) A dense extension (e, A') of A is *maximal* if A' has no proper dense extension.

LEMMA 1.4. *Let $u: A \rightarrow B$ be a dense injection, (e, D) an injective extension of A . Then there is an injection $e': B \rightarrow D$ such that $e'u = e$.*

By noting that a retraction is a surjection, we can establish

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LEMMA 1.5. *If A is injective, then A has no proper dense extensions.*

LEMMA 1.6. *Any two injective hulls of A are isomorphic.*

We say that K has *enough injectives* if every object of K has an injective extension. An injection $e: A \rightarrow D$ of K is said to be *reducible* if there is a morphism $f: D \rightarrow B$ such that $fe: A \rightarrow B$ is a dense injection.

THEOREM 1.7. *Suppose that K has enough injectives and that every injection of K is reducible. Let A be an object of K . If A has no proper dense extensions, then A is injective.*

Proof. Let $e: A \rightarrow D$ be an injective extension of A . Since e is reducible, let $f: D \rightarrow B$ be a morphism such that $fe: A \rightarrow B$ is a dense injection. Hence fe is an equivalence. Thus A is a retract of D by $(fe)^{-1}f$, and so A is injective.

THEOREM 1.8. *Suppose that K has enough injectives. Then the following are equivalent:*

- (1) *Every injection of K is reducible and every object of K has a maximal dense extension.*
- (2) *Every object of K has an injective hull.*

2. Varieties. We recall that a *variety* is an equational class of algebras. It is known that any variety forms a bicategory in a natural way, with morphisms interpreted as homomorphisms, injections as *embeddings* (i.e. 1–1 homomorphisms), and surjections as surjective homomorphisms. We say that A is a *dense subalgebra* of B if A is a subalgebra of B and the inclusion map $A \rightarrow B$ is dense.

The following well-known property of varieties will be used implicitly several times: *If $e: A \rightarrow B$ is an injection, then there is an algebra A' and an isomorphism $f: A' \rightarrow B$ such that A is a subalgebra of A' and $e = f|_A$.*

We first find a more algebraic characterisation of dense subalgebras.

LEMMA 2.1. *Let K be a variety. Suppose that A is a subalgebra of B . Then A is a dense subalgebra of B if and only if, for any congruence J on B with $J \cap A^2 = \Delta_A$, we have $J = \Delta_B$ (where Δ_A is the graph of equality on A).*

Proof. Let $e: A \rightarrow B$ be the inclusion map. Let $f \in \text{Hom}(B, D)$, J be the kernel of f . Then $J \cap A^2$ is the kernel of fe . The lemma is now obtained by using the fact that a morphism $f': C \rightarrow C'$ is an injection if and only if the kernel of f' is Δ_C .

COROLLARY 2.2. *Suppose that K is one of the following varieties: abelian groups, boolean algebras, commutative rings. Let A be a subalgebra of B . Then A is a dense subalgebra of B if and only if, whenever I is an ideal of B such that $I \cap A = 0$, then $I = 0$.*

As a further refinement we obtain the following result of [3].

COROLLARY 2.3. *Let K be the variety of boolean algebras, A be a subalgebra of B . Then A is a dense subalgebra of B if and only if A is dense in B in the usual boolean-algebraic sense that whenever $b \in B$ is nonzero, there is $a \in A$ such that $0 < a \leq b$.*

The next result is Proposition VII.2.5 of Cohn [7], but we shall give a proof here as we wish to generalise the method later.

THEOREM 2.4. *Let K be any variety. Then each injection of K is reducible.*

Proof. Let A be a subalgebra of D . Consider the set C of congruences J on D such that $J \cap A^2 = \Delta_A$, ordered by inclusion. Then the union of any chain in C belongs to C . Hence C has a maximal member J , by Zorn's Lemma. Let D/J be the quotient algebra of D by J , $f: D \rightarrow D/J$ the canonical surjection. Let $e: A \rightarrow D$ be the inclusion map. Now fe has kernel $J \cap A^2 = \Delta_A$, so that fe is an injection. Let J' be a nontrivial congruence on D/J , and let J_1 be the congruence on D given by $dJ_1 d'$ if and only if $f(d)J'f(d')$. Then J is a proper subset of J_1 , and so there are distinct points $a, b \in A$ with $aJ_1 b$. For these points, $f(a)J'f(b)$ and $f(a) \neq f(b)$. Hence fe is a dense injection.

The next few results will establish that any variety with enough injectives admits injective hulls.

LEMMA 2.5. *Let A be a subalgebra of an algebra D . Let G be the collection of all subalgebras C of D such that A is a dense subalgebra of C , ordered by inclusion. Then G has a maximal member.*

Proof. G is nonvoid as $A \in G$. Let $(C_i: i \in T)$ be a chain in G . Then $C = \bigcup_{i \in T} C_i$ is a subalgebra of D , and also $C^2 = \bigcup_{i \in T} C_i^2$, since we have a chain. Now let J be a congruence on C such that $J \cap A^2 = \Delta_A$. Then $J = \bigcup_{i \in T} (J \cap C_i^2)$, and $J \cap C_i^2$ is a congruence on C_i with $(J \cap C_i^2) \cap A^2 = \Delta_A$, so that $J \cap C_i^2$ is the diagonal on C_i . Hence $J = \Delta_C$. This shows that A is a dense subalgebra of C . Thus the union of a chain in G belongs to G , and so, by Zorn's Lemma, G has a maximal member.

LEMMA 2.6. *Let K be a variety with enough injectives, A an algebra of K . Then A has a maximal dense extension.*

Proof. Let D be an injective algebra of K such that A is a subalgebra of D . Define G as in Lemma 2.5 and let B be a maximal member of G . We show that B has no proper dense extension. Let (e, B') be a dense extension of B . Then there is an injection $f: B' \rightarrow D$ with $fe(b) = b$ for $b \in B$, by Lemma 1.4. Let B'' be the image of B' under f . Then B is a subalgebra of B'' and B'' is a subalgebra of D . Now B is a dense subalgebra of B'' (as B'' is isomorphic to B') and so A is a dense subalgebra of B'' , by Lemma 1.1. Thus $B = B''$, and so f is an isomorphism of B' onto B . Hence $e: B \rightarrow B'$ is an isomorphism, and so B is a maximal dense extension of A .

THEOREM 2.7. *Let K be a variety with enough injectives. Then every algebra of K has an injective hull.*

Proof. Use Lemma 2.6 and Theorem 2.4, and then Theorem 1.8.

We observe that, if K has an injective separator D (in the sense of 5.1 of [16]), then K has enough injectives. For let A be an algebra of K , $I = \text{Hom}(A, D)$, and $e: A \rightarrow D^I$ be given by $e(a)(m) = m(a)$ for $m \in I$, $a \in A$. Then e is an injection, while D^I is injective. Hence the following varieties admit injective hulls (the algebras in parentheses are injective separators):

- (1) Abelian groups (the circle group)—by §9.4 of [16],

- (2) Boolean algebras (the 2-element algebra), by Column 8 of the Table in [16],
- (3) Distributive lattices with 0 and 1 (the 2-element lattice)—by Lemma 3.1 of [2],
- (4) Semilattices with 0 and 1 (the 2-element semilattice), by the remark after Corollary 4.7,
- (5) p -rings with 1 (the finite field with p elements), for prime p —by §5 of [13].

A general treatment of injective hulls in abelian categories will be found in Freyd [9] (note that only (1) above is abelian).

3. Boolean algebras. Let us consider the variety of boolean algebras. By replacing congruences by *ideals* and making the obvious adjustments in Theorem 2.4 and Lemmas 2.5 and 2.6, we obtain a very simple proof of the result: Every boolean algebra has an injective hull. It now follows that every complete boolean algebra is injective, by using the following two lemmas, the first due to Halmos [11] and the second to Sikorski [18, Theorem 23.1].

LEMMA 3.1. *Every injective boolean algebra is complete.*

Proof. Every such is a retract of some 2^I and so is complete.

LEMMA 3.2. *Let B be a complete boolean algebra, A a dense subalgebra of B . Then A is a regular subalgebra of B , and, for $b \in B$, $b = \sup_B \{a \in A : a \leq b\}$.*

Thus, if A is itself complete, $A = B$. By applying Theorem 2.7, the next theorem is obvious.

THEOREM 3.3. *Every complete boolean algebra is injective.*

This result was first proved by Sikorski in [17], using step-by-step extension and Zorn's Lemma. Gleason [10] has given a proof using the Stone duality, and Scott [15] has proved it using Boolean-valued models of set theory. The above technique of deriving Theorem 3.3 is due, in dual form, to Gleason (see Lemma 2.3 and Theorem 2.5 of [10]). It first appeared explicitly in [8].

Lemma 3.2 (together with Corollary 2.3) also shows that there is a general-algebraic proof of the existence of the *completion* of a boolean algebra A . For any injective hull of A clearly has properties (i), (iii), and (iv) of Theorem 35.2 in [18].

4. Partially ordered sets. Let K be the category of (partially) ordered sets and order-preserving maps. This can be made into a bicategory by defining $e: A \rightarrow B$ to be an injection precisely if e is an order-isomorphism of A onto a subset of B . By adapting the abstract construction of §2, we shall show that K admits injective hulls, and then these will be characterised as Dedekind–MacNeille completions.

Our first task is to show that the algebraic notions of congruence and kernel have suitable analogues in K . We recall that a *preorder* is a reflexive transitive relation.

LEMMA 4.1. *Let $f: (A, \leq) \rightarrow (B, \leq')$ be a morphism of K . Define a relation \leq_f on A as follows: $x \leq_f y$ if and only if $f(x) \leq' f(y)$. Then \leq_f is a preorder on A extending \leq , and f is an injection if and only if $\leq_f = \leq$ (as subsets of A^2). We call \leq_f the kernel of f .*

Proof. If $x \leq y$, then $f(x) \leq' f(y)$ and so $x \leq_f y$. Hence $\leq \subseteq \leq_f$. Clearly \leq_f is a

preorder. The proof is completed by recalling that f is an injection if and only if, for all $x, y \in A$, $x \leq y$ if and only if $f(x) \leq' f(y)$.

LEMMA 4.2. *Let \leq be a preorder on A . Define a relation E on A by xEy if and only if $x \leq y$ and $y \leq x$. Then E is an equivalence and A/E admits a natural partial order \leq^0 such that $x \leq y$ if and only if $\bar{x} \leq^0 \bar{y}$, where \bar{x} is the E -equivalence class of x , for $x \in A$. Further, if \leq_1 is a partial order on A with $\leq_1 \subseteq \leq$, then the canonical map $f: (A, \leq_1) \rightarrow (A/E, \leq^0)$ is order-preserving and $\leq_f = \leq$.*

LEMMA 4.3. *Let (B, \leq) be an ordered set, A a subset of B , \leq_A the restriction of \leq to A . Then the inclusion map $e: (A, \leq_A) \rightarrow (B, \leq)$ is dense if and only if, for every preorder R on B extending \leq , if $R \cap A^2 = \leq_A$, then $R = \leq$.*

Proof. Let $f: (B, \leq) \rightarrow (C, \leq')$ be order-preserving. Now it is sufficient to remark that f is an injection if and only if $\leq_{f \circ e} = \leq_A$, and that $\leq_{f \circ e} = \leq_f \cap A^2$.

THEOREM 4.4. *Every injection of K is reducible.*

Proof. Let e be an injection of K . We may assume that e is an inclusion map $A \rightarrow B$. Let R be a preorder on B maximal such that $\leq \subseteq R$ and $R \cap A^2 = \leq_A$ (by using a proof similar to that of Theorem 2.4). Let $f: B \rightarrow C$ be a morphism with kernel R . Then $f \circ e$ is a dense injection (proof as in Theorem 2.4).

The next two lemmas will demonstrate that K has enough injectives. Lemma 4.5 is a result of [3].

LEMMA 4.5. *Every complete lattice is injective in K .*

Proof. Let L be a complete lattice, B an ordered set, A a subset of B , $f \in \text{Hom}(A, L)$. Define $g: B \rightarrow L$ by $g(x) = \sup \{f(a) : a \in A \text{ and } a \leq x\}$. Then g extends f and is order-preserving.

LEMMA 4.6. *Every ordered set P is order-embeddable in a complete atomic boolean algebra.*

Proof. This is classical: we can embed P in the power set of P .

COROLLARY 4.7. *Every injective in K is a complete lattice.*

Proof. Use Lemma 4.6 and an argument similar to that for Lemma 3.1.

Essentially the same methods will show that in the variety of (meet) semilattices with 0 and 1, the injectives are precisely the complete Brouwerian lattices (by noting that the embedding of Lemma 4.6 preserves meets and that a meet-retract of a boolean algebra is pseudo-complemented—this has been observed independently in [5]). Thus the 2-element semilattice is an injective separator.

THEOREM 4.8. *K admits injective hulls.*

Proof. Let A be an object of K . Let A be a subset of D , where D is injective (by Lemmas 4.6 and 4.5). By carrying out the proof of Lemma 2.5 using preorders in place of congruences, and using an argument similar to that in Lemma 2.6, we can show that A has a maximal dense extension. But this is injective by Theorem 1.8.

We now show that injective hulls have a concrete realisation as (Dedekind–MacNeille) completions. The standard construction of the completion can be found in Birkhoff [4].

THEOREM 4.9. *Let A be an ordered set. Then any injective hull of A is a completion.*

Proof. Let $L \supseteq A$ be an injective hull of A . Then L is a complete lattice by Corollary 4.7. Also, there is a preorder R on L extending \leq_L , given by xRy if and only if $I_x \subseteq I_y$, where $I_x = \{a \in A : a \leq x\}$. Now $R = \leq_L$ as $R \cap A^2 = \leq_A$. Thus, for $x \in L$, $x = \sup I_x$, and the dual of this also holds.

Note that as we wish to *construct* the completion, we cannot use Proposition 4.2 of [3], which assumes its existence.

By dualising the definitions of §1, we arrive at the notions of *dense surjection* and *projective cover*. We now show that K admits projective covers. An object A of K is said to be *free* if A is totally unordered. The reader can easily check the following facts:

- (1) Every free object is projective.
- (2) Every object is a quotient of a free object.
- (3) Every projective object is free.
- (4) A natural projective cover of (A, \leq) is $(u, (A, \leq_1))$, where \leq_1 is equality on A and u is the identity map.

5. Concluding remarks. The methods developed here can also be applied to give a short abstract proof of Cohen's result (see [6]) that the bicategory K of normed linear spaces and linear contractions admits injective hulls. For K has enough injectives (by Column 11 of the Table in [16]), and by replacing congruences by *seminorms* the obvious analogues of Theorem 2.4 and Lemmas 2.5 and 2.6 can easily be obtained.

It has been brought to the notice of the author that there is a general construction of injective hulls in Daigneault [8]. However, this would not apply to partially ordered sets or normed linear spaces, and for varieties, it leads to a more complicated proof (as the conditions C3 and C6 of [8] are nontrivial to verify in general).

An abstract approach to injective hull constructions and further investigation of completions (e.g. in metric spaces) can be found in [1].

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MATHEMATICAL INSTITUTE
OXFORD UNIVERSITY