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STRUCTURE SPECIES AND CONSTRUCTIVE FUNCTORS

ANDREW A. BLANCHARD

Introduction. This communication is in essence a note tying together the structure theory of Bourbaki and the categorical structure theory exemplified in the works of Sonner, and Hedrlin, Pultr, and Trnkova, to name a few. The work of Bourbaki [**1**] appeared in 1957, and quietly succumbed to the explosion of category theory. Sonner [**6**] used the nomenclature of Bourbaki to classify certain types of functors, while Hedrlin *et al* use the functor approach, studying structures by way of the associated forgetful functors. More discussion of this approach appears in section 3.

The goal of this paper is to place the structure theory of Bourbaki on a categorical foundation; in so doing, the equivalence of the Bourbakian and the functor approach is established. This "equivalence" is realized in section 4 as the equivalence of two appropriate categories. It is a trivial matter to attach to a structure species a constructive functor, namely the associated forgetful functor (The term "constructive" is defined in section 3); this provides the bridge from Bourbaki to the functor approach, and was the basic justification for the latter point of view. What is at issue here is the reverse direction: does *any* constructive functor essentially arise from a structure species? This question is answered in the affirmative in section 4 (Theorem 4.1).

1. Categorical preliminaries. Let A be a category. For two objects a, b of A, we denote by $A(a, b)$ the set of morphisms of A with domain a and codomain b. We follow the usual notation for composition in A: the composite of x followed by y (when domain y = codomain x) is written $y \circ x$. We denote by Ob (A) (respectively Mor (A)) the objects (respectively morphisms) of A and by A^* the subcategory of A consisting of the isomorphisms (invertible morphisms) of A. Also, we denote by A^{op} the opposite category of A [**4**, p. 33].

A is said to be a *preorder* if $A(a, b)$ has at most one element for all objects a, b of A. We write $a < b$ if $A(a, b)$ is non-empty; $<$ is reflexive and transitive as a relation on Ob (A). If in addition $<$ is symmetric, A is said to be an *order*. Observe that $<$ is symmetric precisely when the only isomorphisms of the preorder A are the identity morphisms. Note also that a functor between preorders preserves the relation $<$.

If $\mathcal{U}$ is a universe, we say that A is $\mathcal{U}$-small if Ob (A) and Mor (A) are elements of $\mathcal{U}$. Similarly, a functor $F : A \to B$ is $\mathcal{U}$-small if A and B are; and, for completeness, a function $f : R \to T$ is $\mathcal{U}$-small if R and T are elements of $\mathcal{U}$.

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Throughout the remainder of this paper $\mathscr{U}$ shall denote a universe, $\mathscr{M}$ the category of $\mathscr{U}$-small functions, $\mathscr{F}$ the category of $\mathscr{U}$-small functors, and $\mathscr{O}$ the full subcategory of $\mathscr{F}$ generated by the $\mathscr{U}$-small orders. Note that $\mathscr{M}$, $\mathscr{F}$, and $\mathscr{O}$ are not $\mathscr{U}$-small categories.

Let T be an element of $\mathscr{U}$. To T we associate the category Dis (T), whose objects are the elements of T and whose only morphisms are identity morphisms, a so-called discrete category. This notion extends to a functor Dis : $\mathscr{M} \rightarrow \mathscr{O}$ (the category Dis (T) is actually an order) [**2**, p. 31]. We shall disregard in general the cumbersome Dis (T) and let T play a dual role, whose meaning shall be clear from context. If $(T, <)$ is a partial order, we define an order $T^{\perp}$ as follows: the objects of $T^{\perp}$ are elements of T; the morphisms of $T^{\perp}$ are the pairs (s, t) with $s < t$, and composition is defined by the formula

$$(s, t) \circ (r, s) = (r, t).$$

Here domain $(r, s) = r$, codomain $(r, s) = s$. It is an easy matter to show that $T^{\perp}$ is an order. Furthermore, the association is functorial: that is, there is a functor $(^{\perp})$ from the category $\mathscr{P}$ of partial orders (with non-decreasing functions as morphisms) to $\mathscr{O}$ [**2**, p. 31].

2. Structure Species and Σ-morphisms.

Definition. Let D be a subcategory of $\mathscr{F}$ and let $F : A \rightarrow D$ and $G : A \rightarrow D$ be functors. We say *F is a subfunctor of G* if for all objects a of A, $F(a)$ is a subcategory of $G(a)$, and if the inclusion is natural; that is, if $x : a \rightarrow b$ is a morphism of A, the diagram

$$\begin{array}{ccc} F(a) & \xrightarrow{\subset} & G(a) \\ {\scriptstyle F(x)}\downarrow & & \downarrow{\scriptstyle G(x)} \\ F(b) & \xrightarrow[\subset]{} & G(b) \end{array}$$

commutes in D.

Definition 2.1. Let X be a $\mathscr{U}$-small category. A (covariant) *structure species* Σ *on* X is a pair (E, S) satisfying the following properties:

(1) $E : X \rightarrow \mathscr{O}$ and $S : X^* \rightarrow \mathscr{M}$ are functors;

(2) Dis $\circ\, S$ is a subfunctor of $E \circ J$, where $J : X^* \rightarrow X$ is the inclusion.

Remark. In order to preserve some of the traditional nomenclature of Bourbaki, we call E the *echelon* and S the *structure scheme* of Σ. As one expects, a *contravariant* structure species on X is by definition a covariant structure species on X^{op}. We shall interpret part 2 of the definition as follows: For each object a of X, $S(a)$ is a subset of Ob $(E(a))$; furthermore any isomorphism $x : a \rightarrow b$ of X induces a bijection from $S(a)$ to $S(b)$ which is the restriction of the bijective functor $E(x)$.

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Definition 2.2. Let $\Sigma = (E, S)$ be a structure species on X.

(a) We say that *(a, U) is a Σ-structure* or that *U is a Σ-structure on a* if $a \in \text{Ob}(X)$ and $U \in S(a)$.

(b) We say that *(x, U, V) is a Σ-morphism* if the following conditions are satisfied:

(i) x is a morphism of X, say $x : a \to b$;

(ii) U and V are Σ-structures on a and b respectively;

(iii) $E(x)(U) < V$, where $<$ is the relation defined on the order $E(b)$ as in section 1.

PROPOSITION 2.1. (a) *If U is a Σ-structure on a, then (Id_a, U, U) is a Σ-morphism.*

(b) *Let $x : a \to b$ and $y : b \to c$ be morphisms of X, and let U, V, W be Σ-structures on a, b, c, respectively. If (x, U, V) and (y, V, W) are Σ-morphisms, then $(y \circ x, U, W)$ is a Σ-morphism.*

Proof. Since $E(\text{Id}_a)(U) = \text{Id}_{E(a)}(U) = U < U$, part (a) is immediate. Part (b) follows from the fact that

$$E(y \circ x)(U) = E(y)(E(x)(U)) < E(y)(V) < W.$$

Proposition 2.1 allows us to construct a category X_Σ as follows: the objects of X_Σ are the Σ-structures (a, U), the morphisms of X_Σ are the Σ-morphisms (x, U, V), and composition is defined by the rule

$$(y, V, W) \circ (x, U, V) = (y \circ x, U, W).$$

We take pairs for the objects of X_Σ since then the equation domain $(y, \bar{V}, W)$ = codomain (x, U, V) implies $y \circ x$ is defined and $V = \bar{V}$. We also define a functor $F_\Sigma : X_\Sigma \to X$ as follows: F_Σ maps the object (a, U) of X_Σ into a, and $F_\Sigma(x, U, V) = x$. We summarize these results in the next proposition.

PROPOSITION 2.2. (a) *X_Σ is a $\mathscr{U}$-small category. Furthermore, an isomorphism of X_Σ is of the form $(x, U, S(x)(U))$ where $x \in X^*$ and $U \in S(\text{domain } x)$.*

(b) *$F_\Sigma : X_\Sigma \to X$ is a faithful functor.*

Proof. The fact that X_Σ is a category and F_Σ a functor is trivial. Furthermore,

$$\text{Ob}(X_\Sigma) = \bigcup \{\{a\} \times S(a) : a \in \text{Ob}(X)\} \in \mathscr{U};$$

$$\text{Mor}(X_\Sigma) \subset \bigcup \{X(a, b) \times S(a) \times S(b) : a, b \in \text{Ob}(X)\} \in \mathscr{U},$$

so $\text{Mor}(X_\Sigma) \in \mathscr{U}$. Thus X_Σ is $\mathscr{U}$-small.

If (x, U, V) is an isomorphism of X_Σ, then due to the nature of composition in X_Σ, $x \in X^*$ and the inverse of (x, U, V) is (x^{-1}, V, U). Then $E(x)(U) < V$ and $E(x^{-1})(V) < U$; thus we have

$$V = E(x)E(x^{-1})(V) < E(x)(U) < V,$$

so equality holds throughout. In particular $V = E(x)(U)$. Since $\text{Dis} \circ S$ is a subfunctor of $E \circ J$, $E(x)(U) = S(x)(U)$, and hence $(x, U, V) = (x, U, S(x)(U))$.

Finally, for two objects (a, U) and (b, V) of X_Σ, the function $(x, U, V) \to x : X_\Sigma((a, U), (b, V)) \to X(a, b)$ is clearly injective, so F_Σ is faithful.

Remark. The characterization of isomorphisms of X_Σ above is a direct analogue of the definition of isomorphism in Bourbaki's treatise [**1**, p. 15]. We shall present examples and further comments in section 5.

Definition 2.3 Let $\Sigma = (E, S)$ and $\Sigma' = (E', S')$ be structure species on X. We say that $\Phi : \Sigma \to \Sigma'$ *is a homorphism of structure species* if the following conditions are satisfied:

(i) $\Phi : S \to S'$ is a natural transformation;

(ii) Whenever (x, U, V) is a Σ-morphism, where say $x : a \to b$, then $(x, \Phi_a(U), \Phi_b(V))$ is a Σ'-morphism.

If $\Phi : \Sigma \to \Sigma'$ and $\Psi : \Sigma' \to \Sigma''$ are homomorphisms of structure species, then $\Psi \cdot \Phi : \Sigma \to \Sigma''$ is a homomorphism of structure species, where $\Psi \cdot \Phi$ is the natural transformation defined by $(\Psi \cdot \Phi)_a = \Psi_a \circ \Phi_a$ for all objects a of X [**4**, p. 40]. We now can construct the *category of structure species on* X, denoted $\mathscr{S}_X$, whose objects are the structure species on X, and whose morphisms are the homomorphisms of structure species, with composition defined above.

3. The category of constructive functors on X. We now develop a category whose equivalence to $\mathscr{S}_X$ is the main thrust of this paper.

Definition 3.1 Let $F : Y \to X$ be a $\mathscr{U}$-small functor. We say that *F is a constructive functor on X* (or simply *F is constructive*) if F satisfies the following conditions:

(i) F is faithful;

(ii) For every object B of Y and for every isomorphism x of X with domain $F(B)$, there exists a unique isomorphism T of Y with domain B such that $F(T) = x$.

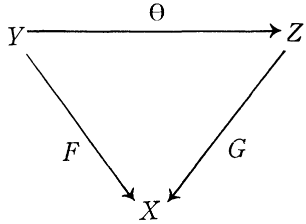
Remark. A constructive functor is the "structure species" of Sonner [**6**, p. 1333], so only the name is new. But the term "constructive" as used here bears no relation to the term as used by Hedrlin [**3**, p. 179]. Hedrlin's usage actually applies to functors which in the classical terminology are echelons $E : \mathscr{M} \to \mathscr{O}$ or $E : \mathscr{M}^{\mathrm{op}} \to \mathscr{O}$. The properties of constructive functors have been used and studied in some detail in [**5**], but this work is not germane to the discussion here. Let us agree that for an object a of X, an object A of Y such that $F(A) = a$ shall be called a *structure* on a; moreover, if $x : a \to b$ is a morphism of X, if A, B are structures on a and b respectively, and if $T : A \to B$ is a morphism of Y such that $F(T) = x$, x will be called an *F-morphism from A to B*. If F is faithful such a T is necessarily unique. Part (ii) of the definition now reads: "For every structure A on a, and for every isomorphism x in X with domain a (say $x : a \to b$), there exists (there can be constructed) a unique structure B on b rendering x an F-isomorphism from A to B."

Example. The so-called forgetful functors [**4**, p. 14] are constructive; e.g. the functor TOP $\to \mathscr{M}$ which assigns to each continuous function the underlying function (stripped of topology). ("Given a topological space X and a bijection $f : X \to Y$, there exists a unique topology on Y rendering f a homeomorphism".)

PROPOSITION 3.1. *If Σ is a structure species on X, then the functor $F_\Sigma : X_\Sigma \to X$ is constructive.*

Proof. F_Σ is faithful by virtue of Proposition 2.2. If (a, U) is a Σ-structure, and if x is an isomorphism of X with domain a, then $(x, U, S(x)(U))$ is the only isomorphism of X_Σ with domain (a, U) which maps into x via F_Σ, again by virtue of Proposition 2.2.

Definition 3.2 Let $F : Y \to X$ and $G : Z \to X$ be constructive functors on X. We say that $\Theta : F \to G$ is *a morphism of constructive functors* (on X) if $\Theta : Y \to Z$ is a functor such that $G \circ \Theta = F$.



Remark. It is easy (but irrelevant) to show that Θ is a constructive functor on Z.

If $\Theta : F \to G$ and $\Theta' : G \to H$ are morphisms of constructive functors on X, then $\Theta' \circ \Theta : F \to H$ is a morphism of constructive functors, because $H \circ \Theta' \circ \Theta = G \circ \Theta = F$. Hence we construct the category of constructive functors on X, denoted $\mathscr{C}_X$, as follows: the objects of $\mathscr{C}_X$ are the constructive functors on X, and the morphisms of $\mathscr{C}_X$ are the morphisms of constructive functors on X with composition as indicated. Due to the nature of this composition and to the characterization of isomorphism of categories [**4**, p. 14] we have immediately the following result:

PROPOSITION 3.2. *A morphism $\Theta : F \to G$ of $\mathscr{C}_X$ is an isomorphism of $\mathscr{C}_X$ if and only if Θ is bijective both on objects and morphisms.*

Remark. The category $\mathscr{C}_{\mathscr{M}}$ is the framework for much of the discussion of Hedrlin, Pultr, and Trnkova (e.g. [**3**; **5**]). More precisely, given two forgetful functors $F : K \to \mathscr{M}$ and $F' : K' \to \mathscr{M}$ a *realization of K in K'* is a full injective functor $\Phi : K \to K'$ such that $F' \circ \Phi = F$, i.e. a morphism from F to F' in

$\mathscr{C}_{\mathscr{M}}$ with certain extra properties. Other terms of their design, e.g. pseudo-realization [**5**], strong embedding [**5**], and representation [**3**], are done in generalizations of $\mathscr{C}_{\mathrm{M}}$. As these are specifics of the functor approach, we shall not discuss details here.

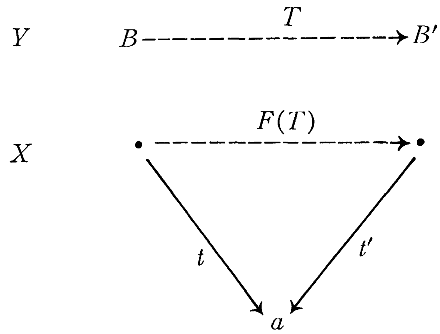
4. The equivalence of $\mathscr{S}_X$ and $\mathscr{C}_X$.

THEOREM 1. *Let $F : Y \to X$ be a constructive functor. Then there exists a structure species Σ on X and an isomorphism of categories $\Theta : Y \to X_\Sigma$ such that $F_\Sigma \circ \Theta = F$.*

Remark. This theorem asserts the existence of a structure species Σ such that F_Σ and F are isomorphic objects of $\mathscr{C}_X$.

Proof. We divide the proof into several steps.

(A) *Construction of the echelon E.* Let a be an object of X. Denote by L_a the set of the pairs $(B, t) \in \mathrm{Ob}\ (Y) \times \mathrm{Mor}\ (X)$ such that $F(B) =$ domain t and codomain $t = a$. For (B, t) and (B', t') in L_a, write $(B, t) \prec (B', t')$ if there exists $T : B \to B'$ in Y such that $t = t' \circ F(T)$.



Now $\prec$ is a reflexive and transitive relation in L_a; denote by Q_a the quotient set of L_a determined by the equivalence relation "$R \in L_a$ and $S \in L_a$ and $R \prec S$ and $S \prec R$". Following usual conventions we denote elements of Q_a by $[R]$ for $R \in L_a$. The relation defined on Q_a by $[R] \prec [S]$ if $R \prec S$ for some representative R of $[R]$, S of $[S]$ is an order relation (partial order) in Q_a. Denote by $E(a)$ the order $(Q_a)^{\perp}$. (See section 1.)

If $x : a \to b$ is a morphism of X, there is an induced function $L_x : (B, t) \to (B, x \circ t) : L_a \to L_b$; moreover, if $(B, t) \prec (B', t')$ in L_a, then $(B, x \circ t) \prec (B', x \circ t')$ in L_b, and so L_x induces a non-decreasing function $Q_x : Q_a \to Q_b$ by passage to quotients. Hence we get a functor $E(x) : E(a) \to E(b)$ defined by $E(x) = (Q_x)^{\perp}$. But if $x' \circ x$ is defined in X, then $L_{x' \circ x} = L_{x'} \circ L_x$, and so by passage to quotients and beyond, $E(x' \circ x) = E(x') \circ E(x)$. Thus we have defined a functor $E : X \to \mathscr{O}$. (Observe that if a is an object of X, we have $L_{\mathrm{Id}_a} = \mathrm{Id}_{L_a}$, and so $E(\mathrm{Id}_a) = \mathrm{Id}_{E(a)}$). Observe also that these constructions remain in the universe $\mathscr{U}$.

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(B) *Construction of the structure scheme S.* We first construct a functor $\bar{S} : X^* \to \mathscr{M}$ from which we derive S. Let $a \in \mathrm{Ob}\,(X)$; denote by $\bar{S}(a)$ the set of objects A of Y such that $F(A) = a$. If $x : a \to b$ is an isomorphism of X, then there is an induced bijection $\bar{S}(x) : \bar{S}(a) \to \bar{S}(b)$, due the fact that F is constructive. (Simply associate to each $A \in \bar{S}(a)$ the codomain of T, where T is the unique isomorphism of Y with domain A such that $F(T) = x$.) If x' and x are composable isomorphisms, then $\bar{S}(x' \circ x) = \bar{S}(x') \circ \bar{S}(x)$, which follows from the uniqueness of the isomorphisms T. Also, $\bar{S}(\mathrm{Id}_a) = \mathrm{Id}_{\bar{S}(a)}$, since Id_A is a lift of the isomorphism Id_a with domain A, and the constructiveness of F guarantees its uniqueness. Thus we have constructed a functor $\bar{S} : X^* \to \mathscr{M}$.

For $a \in \mathrm{Ob}\,(X)$, define a function ϕ_a by

$$\phi_a : A \to [(A, \mathrm{Id}_a)] : \bar{S}(a) \to Q_a.$$

We claim that ϕ_a is injective. Indeed, if $[(A, \mathrm{Id}_a)] = [(A', \mathrm{Id}_a)]$ then $(A, \mathrm{Id}_a) \prec (A', \mathrm{Id}_a)$ and $(A', \mathrm{Id}_a) \prec (A, \mathrm{Id}_a)$ in L_a. Hence there exists $T : A \to A'$ and $S : A' \to A$ such that $\mathrm{Id}_a = \mathrm{Id}_a \circ F(T)$ and $\mathrm{Id}_a = \mathrm{Id}_a \circ F(S)$. But then $F(T \circ S) = \mathrm{Id}_a$ and $F(S \circ T) = \mathrm{Id}_a$, and so $T \circ S = \mathrm{Id}_{A'}$ and $S \circ T = \mathrm{Id}_A$ since F is faithful. Thus T is an isomorphism of Y with domain A such that $F(T) = \mathrm{Id}_a$. But Id_A has the same property, so $T = \mathrm{Id}_A$ since F is constructive. Hence $A' = \text{codomain } T = \text{codomain } \mathrm{Id}_A = A$, and ϕ_a is injective.

Now if $x : a \to b$ is an isomorphism of X, the square

$$\begin{array}{ccc} \bar{S}(a) & \xrightarrow{\phi_a} & Q_a \\ {\scriptstyle \bar{S}(x)}\downarrow & & \downarrow{\scriptstyle Q_x} \\ \bar{S}(b) & \xrightarrow{\phi_b} & Q_b \end{array}$$

commutes. Indeed, if $A \in \bar{S}(a)$, then

$$Q_x(\phi_a(A)) = Q_x([A, \mathrm{Id}_a)]) = [(A, x)],$$

while $\phi_b(\bar{S}(x)(A)) = [(B, \mathrm{Id}_b)]$, where B is the codomain of the unique isomorphism T such that domain $T = A$ and $F(T) = x$. But $(A, x) \prec (B, \mathrm{Id}_b)$ because $x = \mathrm{Id}_b \circ F(T)$, and $(B, \mathrm{Id}_b) \prec (A, x)$ because $\mathrm{Id}_b = x \circ F(T^{-1})$; hence $[(A, x)] = [(B, \mathrm{Id}_b)]$. In conclusion, the bijection $\bar{S}(x)$ induces a bijection from $\phi_a(\bar{S}(a))$ to $\phi_b(\bar{S}(b))$, which we denote $S(x)$; we denote $\phi_a(\bar{S}(a))$ by $S(a)$. Moreover, the inclusion of $S(a)$ into $E(a)$ is natural, as the above square shows. In other words, the functor S defined above is a subfunctor of $E \circ J$, i.e., $\Sigma = (E, S)$ is a structure species on X.

Before we continue the proof, let us gather our thoughts. A Σ-structure is of the form (a, U) where $a \in \mathrm{Ob}\,(X)$ and $U \in S(a)$; in our case a Σ-structure is of the form $(a, [(A, \mathrm{Id}_a)])$ where $a \in \mathrm{Ob}\,(X)$, $A \in \mathrm{Ob}\,(Y)$ and $F(A) = a$. Conversely, if $A \in \mathrm{Ob}\,(Y)$, then $(F(A), [(A, \mathrm{Id}_{F(A)})])$ is a Σ-structure; let us

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denote $[(A, \mathrm{Id}_{F(A)})]$ by $\Theta(A)$. Observe now that the function

$$A \to (F(A), \Theta(A)) : \mathrm{Ob}\ (Y) \to \mathrm{Ob}\ (X_\Sigma)$$

is bijective.

If $x : a \to b$ is a morphism of X, $\Theta(A)$ a Σ-structure on a, $\Theta(B)$ a Σ-structure on b, then the following statements are successively equivalent:

$(x, \Theta(A), \Theta(B))$ is a Σ-morphism;

$(x, [(A, \mathrm{Id}_a)], [(B, \mathrm{Id}_b)]$ is a Σ-morphism; (definition of Θ)

$E(x)([(A, \mathrm{Id}_a)]) < [(B, \mathrm{Id}_b)]$; (definition of a Σ-morphism)

$[(A, x)]) < [(B, \mathrm{Id}_b)]$ in $E(b)$; (action of $E(x)$)

$(A, x) \prec (B, \mathrm{Id}_b)$ in L_b; (definition of $<$ in $E(b) = (Q_b)^{\perp}$)

There exists $T : A \to B$ such that $x = \mathrm{Id}_b \circ F(T)$; (Definition of $\prec$)

x is an F-morphism from A to B. (see section 3).

Thus $\mathrm{Ob}\ (X_\Sigma)$ is the collection $(F(A), \Theta(A))$ for $A \in \mathrm{Ob}\ (Y)$, and $\mathrm{Mor}\ (X_\Sigma)$ is in essence the collection of the F-morphisms.

(C) *Construction of the isomorphism of categories* $\Theta : Y \to X_\Sigma$.

Θ is defined on objects by the function

$$A \to (F(A), \Theta(A)); \mathrm{Ob}\ (Y) \to \mathrm{Ob}\ (X_\Sigma),$$

and Θ is defined on morphisms by the function

$$T \to (F(T), \Theta(A), \Theta(B)) : \mathrm{Mor}\ (Y) \to \mathrm{Mor}\ (X_\Sigma),$$

where A = domain T and B = codomain T. Now Θ is functorial, for if $A \in \mathrm{Ob}\ (Y)$,

$$\begin{aligned}\Theta(\mathrm{Id}_A) &= (F(\mathrm{Id}_A), \Theta(A), \Theta(A)) = (\mathrm{Id}_{F(A)}, \Theta(A), \Theta(A)) \\ &= \mathrm{Id}_{(F(A),\Theta(A))} = \mathrm{Id}_{\Theta(A)}.\end{aligned}$$

If $T : A \to B$ and $T' : B \to C$, we have

$$\begin{aligned}\Theta(T' \circ T) &= (F(T' \circ T), \Theta(A), \Theta(C)) \\ &= (F(T'), \Theta(B), \Theta(C)) \circ (F(T), \Theta(A), \Theta(B)) \\ &= \Theta(T') \circ \Theta(T)\end{aligned}$$

As observed earlier, the functor Θ is bijective on objects; on morphisms, the inverse of Θ is given by the function

$$(x, \Theta(A), \Theta(B)) \to T : \mathrm{Mor}\ (X_\Sigma) \to \mathrm{Mor}\ (Y),$$

where $T : A \to B$ is such that $F(T) = x$. The existence of such a T is guaranteed by the characterization of Σ-morphism, and its uniqueness follows since F is faithful.

Finally, $F_\Sigma \circ \Theta = F$, because if $T : A \to B$ is in Y,

$$F_\Sigma \circ \Theta(T) = F_\Sigma(F(T), \Theta(A), \Theta(B)) = F(T).$$

This concludes the proof of Theorem 1.

Now suppose $\Phi : \Sigma \to \Sigma'$ is a homomorphism of structure species on X. Then there is a functor $K(\Phi)$ from X_Σ to $X_{\Sigma'}$ which sends each Σ-structure (a, U) into the Σ'-structure $(a, \Phi_a(U))$, and each Σ-morphism (x, U, V) into the Σ'-morphism $(x, \Phi_a(U), \Phi_b(V))$, where $a =$ domain x and $b =$ codomain x. Moreover,

$$K(\Phi) : F_\Sigma \to F_{\Sigma'}$$

is a morphism of constructive functors, because

$$F_{\Sigma'} \circ K(\Phi)(x, U, V) = F_{\Sigma'}(x, \Phi_a(U), \Phi_b(V)) = x = F_\Sigma(x, U, V).$$

Thus we have a correspondence K from $\mathscr{S}_X$ to $\mathscr{C}_X$ which sends each structure species Σ on X into F_Σ, and each homomorphism of structure species $\Phi : \Sigma \to \Sigma'$ into the morphism of constructive functors $K(\Phi) : F_\Sigma \to F_{\Sigma'}$.

PROPOSITION 4.1. *$K : \mathscr{S}_X \to \mathscr{C}_X$ is a functor.*

Proof. Clearly $K(\mathrm{Id}_\Sigma) = \mathrm{Id}_{F_\Sigma}$; if $\Phi : \Sigma \to \Sigma'$ and $\Psi : \Sigma' \to \Sigma''$ are composable homomorphisms of structure species, then

$$\begin{aligned} K(\Psi \cdot \Phi)(x, U, V) &= (x, (\Psi \cdot \Phi)_a(U), (\Psi \cdot \Phi)_b(V)) \\ &= (x, \Psi_a(\Phi_a(U)), \Psi_b(\Phi_b(V))) \\ &= K(\Psi)(x, \Phi_a(U), \Phi_b(V)) \\ &= K(\Psi)(K(\Phi)(x, U, V)), \end{aligned}$$

so K is a functor.

THEOREM 2. *$K : \mathscr{S}_X \to \mathscr{C}_X$ is an equivalence of categories.*

Proof. We shall prove that K is faithful and full, and that every object of $\mathscr{C}_X$ is isomorphic to the image under K of an object of $\mathscr{S}_X$[**4**, p. 91]. This last assertion is the content of Theorem 1, so we now prove K is faithful and full.

K is faithful. Suppose that $\Phi : \Sigma \to \Sigma'$ and $\Psi : \Sigma \to \Sigma'$ are homomorphisms of structure species with $K(\Phi) = K(\Psi)$. Let $a \in \mathrm{Ob}\ (X)$; then for any Σ-structure U on a, (Id_a, U, U) is a Σ-morphism, and

$$\begin{aligned} (\mathrm{Id}_a, \Phi_a(U), \Phi_a(U)) &= K(\Phi)(\mathrm{Id}_a, U, U) \\ &= K(\Psi)(\mathrm{Id}_a, U, U) \\ &= (\mathrm{Id}_a, \Psi_a(U), \Psi_a(U)); \end{aligned}$$

hence $\Phi_a = \Psi_a$, and this is true for all $a \in \mathrm{Ob}\ (X)$. In other words, $\Phi = \Psi$.

K is full. Suppose $\Sigma = (E, S)$ and $\Sigma' = (E', S')$ are structure species on X, and $\Theta : F_\Sigma \to F_{\Sigma'}$ is a morphism of constructive functors; we must construct a homomorphism of structure species $\Phi : \Sigma \to \Sigma'$ such that $K(\Phi) = \Theta$.

Let $a \in \mathrm{Ob}\ (X)$; for each $U \in S(a)$, we have

$$\Theta(\mathrm{Id}_a, U, U) = (\mathrm{Id}_a, W, W)$$

for some $W \in S'(a)$, as Θ maps identities into identities. Denote this W by $\Phi_a(U)$; this defines a function $\Phi_a : S(a) \to S'(a)$ for each $a \in \mathrm{Ob}\ (X)$. Notice that if $x : a \to b$ is a morphism of X and if (x, U, V) is a Σ-morphism, then

$$(x, \Phi_a(U), \Phi_b(V)) = \Theta(x, U, V)$$

and so is necessarily a Σ'-morphism. (The last equality above follows since $F_{\Sigma'} \circ \Theta = F_\Sigma$.)

Finally, if $x : a \to b$ is an isomorphism of X and if U is a Σ-structure on X, then $(x, U, S(x)(U))$ is an isomorphism of X_Σ, so that

$$(x, \Phi_a(U), \Phi_b(S(x)(U))) = \Theta(x, U, S(x)(U))$$

is an isomorphism of $X_{\Sigma'}$. Due to the characterization of isomorphism of $X_{\Sigma'}$ (Proposition 2.2 (a)), we necessarily have $\Phi_b(S(x)(U)) = S'(x)(\Phi_a(U))$. In other words, $\Phi_b \circ S(x) = S'(x) \circ \Phi_a$; i.e. we have shown that

$$\Phi = (\Phi_a)_{a \in \mathrm{Ob}(X)} : \Sigma \to \Sigma' \text{ is a homomorphism}$$

of structure species. By construction, $K(\Phi) = \Theta$, and so K is full.

5. Comments and examples. There is some difference between this exposition and the treatise of Bourbaki [**1**] other than the obvious categorical reformulation. The morphisms of a Bourbakian structure species (called σ-morphisms) are not bound quite so tightly to the structures as they are here. This apparent divergence is justified by two facts: the morphisms of the classical structures are heavily linked to their structures, and secondly, Theorem 1 provides a bridge. (It can be shown the set of the σ-morphisms of a Bourbakian structure species forms a category and gives rise to a constructive functor [**6**, p. 1333].) As an example of the classical case, consider the structure of groups: a group is a pair (A, G) such that $G \subset A \times A \times A$ is the graph of a group law on A. A group homomorphism f: $(A, G) \to (A', G')$ is a function $f : A \to A'$ such that $(f \times f \times f)(G) \subset G'$. The property of group homomorphism is thus reduced to a problem of order, and yet is strongly and irrevocably tied to the group structure. Similar examples (continuous mappings, open mappings, pointed mappings, monoid homomorphisms, etc.) show that whether a mapping is a morphism is a question of order.

This question of order was observed by Hedrlin *et al* [**3**] when considering relational systems, and more specifically the realization of general relational systems in simple relational systems; the morphisms of relational systems are precisely those which respect the relations, i.e. those which carry relations into subsets of the corresponding relations. This type of work may best be classified under what Bourbaki termed "deduction of structures". For details see [**1**, p. 17; **3**, p. 181].

The basic building blocks for the classical structure are two echelons $\mathscr{P}^+ : \mathscr{M} \to \mathscr{O}$ and $\mathscr{P}^- : \mathscr{M}^{\mathrm{op}} \to \mathscr{O}$ which we now describe. Recall the set-valued functors P^+ and P^- where for $f : A \to B$ in $\mathscr{M}$

$$P^+(f) : X \to f(X) : P(A) \to P(B)$$

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and

$$P^-(f) : Y \to f^{-1}(Y) : P(B) \to P(A)$$

(here $P(A)$ denotes the power set (set of subsets) of A). P^+ is covariant and P^- is contravariant. Observe that $P^+(f)$ and $P^-(f)$ are in actuality non-decreasing functions when we consider the ordered set $(P(A), \subset)$, so we define the functors $\mathscr{P}^+$ and $\mathscr{P}^-$ by $\mathscr{P}^+(f) = (P^+(f))^\perp$ and $\mathscr{P}^-(f) = (P^-(f))^\perp$.

Example. Continuous mappings on topological spaces. Denote by E the echelon $\mathscr{P}^+ \circ P^- : \mathscr{M}^{\text{op}} \to \mathscr{M} \to \mathscr{O}$. For $A \in \text{Ob}(\mathscr{M})$, denote by $T(A)$ the set of topologies on A; then if $f : A \to B$ is a bijection, $P^+(P^-(f))$ induces a bijection $T(f) : T(B) \to T(A)$. (This is the vestigial remains of the "transportability" criterion of Bourbaki.) $T : (\mathscr{M}^{\text{op}})^* \to \mathscr{M}$ is then a functor, and $\Sigma = (E, T)$ is a contravariant structure species on $\mathscr{M}$.

If $f : A \to B$ is a function, U a topology on A, and V a topology on B, then the following statements are successively equivalent: (f^{op}, V, U) is a Σ-morphism (f^{op} is f as an element of $\mathscr{M}^{\text{op}}$); $E(f)(V) < U$ in $E(A) = \mathscr{P}^+(P^-(A)) = (P(P(A)))^\perp$; $\mathscr{P}^+(P^-(f))(V) < U$ in $E(A)$; $(\mathscr{P}^+(P^-(f))(V), U)$ is a morphism of $(P(P(A)))^\perp$; $P^+(P^-(f)(V)) \subset U$ (definition of $(^\perp)$); for all $Y \in V$, $P^-(f)(Y) \in U$; for all $Y \in V$, $f^{-1}(Y) \in U$; $f : (A, U) \to (B, V)$ is a continuous mapping. It is worthwhile to note that starting with the echelon $\mathscr{P}^+ \circ P^+ : \mathscr{M} \to \mathscr{O}$, we can construct in an analogous fashion the structure species of open mappings on topological spaces.

References

1. N. Bourbaki, *Elements de mathematique: theorie des ensembles*, Chapter 4 (Hermann, Paris, 1957).
2. P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory* (Springer-Verlag, New York, 1967).
3. Z. Hedrlin, A. Pultr, and V. Trnkova, *Concerning a categorical approach to topological and algebraic theories*, General Topology and its Relations to Modern Analysis and Algebra, II, Proc. Second Prague Topological Symposium, 1966 (Academia, Prague, 1967).
4. S. MacLane, *Categories for the working mathematician* (Springer-Verlag, New York, 1971).
5. A. Pultr, *On full embeddings of concrete categories with respect to forgetful functors*, Comment. Math. Univ. Carolinae *9* (1968), 281–305.
6. J. Sonner, *Lifting inductive and projective limits*, Can. J. Math. *19* (1967), 1329–1339.

United States Naval Academy,
Annapolis, Maryland