INVITED PAPER

COOPERATIVE GAME THEORY AND ITS INSURANCE APPLICATIONS

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ABSTRACT

This survey paper presents the basic concepts of cooperative game theory, at an elementary level. Five examples, including three insurance applications, are progressively developed throughout the paper. The characteristic function, the core, the stable sets, the Shapley value, the Nash and Kalai-Smorodinsky solutions are defined and computed for the different examples.

1. INTRODUCTION

Game theory is a collection of mathematical models to study situations of conflict and/or cooperation. It attempts to abstract out those elements that are common to many conflicting and/or cooperative encounters and to analyse these mathematically. Its goal is to explain, or to provide a normative guide for, rational behaviour of individuals confronted with strategic decisions or involved in social interaction. The theory is concerned with optimal strategic behaviour, equilibrium situations, stable outcomes, bargaining, coalition formation, equitable allocations, and similar concepts related to resolving group differences. The prevalence of competition in many human activities has made game theory a fundamental modeling approach in such diversified areas as economics, political science, operations research, and military planning.

In this survey paper, we will review the basic concepts of multiperson cooperative game theory, with insurance applications in mind. The reader is first invited to ponder the five following basic examples. Those examples will progressively be developed throughout the paper, to introduce and illustrate basic notions.

Example 1. United Nations Security Council

Fifteen nations belong to the United Nations Security Council: five permanent members (China, France, the United Kingdom, the Soviet Union, and the United States), and 10 nonpermanent members, on a rotating basis (in November 1990: Canada, Colombia, Cuba, Ethiopia, Finland, the Ivory Coast, Malaysia, Romania, Yemen, and Zaire). On substantive matters, including the investigation of a dispute and the application of sanctions, decisions require an affirmative vote from at least nine members, including *all* five permanent members. If one permanent member votes against, a resolution does not pass. This is the famous "veto right" of the "big five," used hundreds of times since 1945. This veto right obviously gives each permanent member a much larger power than the nonpermanent members. But how much larger? □

Example 2. Electoral representation in Nassau County [in LUCAS (1981)]

Nassau County, in the state of New York, has six municipalities, very unequal in population. The County Government is headed by a Board of six Supervisors, one from each municipality. In an effort to equalize citizen representation, Supervisors are given different numbers of votes. The following table shows the situation in 1964.

District	Population	%	No of Votes	%
Hempstead 1 } Hempstead 2 }	778,625	57.1	31	27.0
			31	27.0
Oyster Bay	285,545	22.4	28	24.3
North Hempstead	213,335	16.7	21	18.3
Long Beach	25,654	2.0	2	1.7
Glen Cove	22,752	1.8	2	1.7
	1,275,801		115	

A simple majority of 58 out of 115 is needed to pass a measure. Do the citizens of North Hempstead and Oyster Bay have the same political power in their Government? □

Example 3. Management of ASTIN money [LEMAIRE (1983)]

The Treasurer of ASTIN (player 1) wishes to invest the amount of 1,800,000 Belgian Francs on a short term (3 months) basis. In Belgium, the annual interest rate is a function of the sum invested.

Deposit	Annual Interest Rate
0-1,000,000	7.75%
1,000,000-3,000,000	10.25%
3,000,000-5,000,000	12%

The ASTIN Treasurer contacts the Treasurers of the International Actuarial Association (I.A.A. – player 2) and of the Brussels Association of Actuaries (A.A.Br. – player 3). I.A.A. agrees to deposit 900,000 francs in the common fund, A.A.Br. 300,000 francs. Hence the 3-million mark is reached and the interest rate will be 12%. How should the interests be split among the three associations? The common practice in such situations is to award each participant in the fund the same percentage (12%). Shouldn't ASTIN however be entitled to a higher rate, on the grounds that it can achieve a yield of 10.25% on its own, and the others only 7.75%? □

Example 4. Managing retention groups [Borch (1962)]

[For simplicity, several figures are rounded in this example]. Consider a group of $n_1 = 100$ individuals. Each of them is exposed to a possible loss of 1, with a probability $q_1 = 0.1$. Assume these persons decide to form a risk retention group, a small insurance company, to cover themselves against that risk. The premium charged will be such that the ruin probability of the group is less than 0.001. Assuming that the risks are independent, and using the normal approximation of the binomial distribution, the group must have total funds equal to

$$P_1 = n_1 q_1 + 3\sqrt{n_1 q_1 (1-q_1)} = 10+9 = 19.$$

Hence each person will pay, in addition to the net premium of 0.10, a safety loading of 0.09.

Another group consists of $n_2 = 100$ persons exposed to a loss of 1 with a probability $q_2 = 0.2$. If they form their own retention group under the same conditions, the total premium will be

$$P_2 = n_2 q_2 + 3\sqrt{n_2 q_2 (1-q_2)} = 20+12 = 32.$$

Assume now that the two groups decide to join and form one single company. In order to ensure that the ruin probability shall be less than 0.001, this new company must have funds amounting to

$$\begin{aligned} P_{12} &= n_1 q_1 + n_2 q_2 + 3\sqrt{n_1 q_1 (1-q_1) + n_2 q_2 (1-q_2)} \\ &= 10+20+15 \\ &= 45. \end{aligned}$$

Since $P_{12} = 45 < P_1 + P_2 = 51$, the merger results in a decrease of 6 of the total safety loading. How should those savings be divided between the two groups? A traditional actuarial approach would probably consist in dividing the safety loading in proportion to the net premiums. This leads to premiums of 15 and 30, respectively. The fairness of this rule is certainly open to question, since it awards group 1 most of the gain accruing from the formation of a single company. In any case the rule is completely arbitrary. □

Example 5. Risk exchange between two insurers

Insurance company C_1 owns a portfolio of risks, with a mean claim amount of 5 and a variance of 4. Company C_2's portfolio has a mean of 10 and a variance of 8. The two companies decide to explore the possibility to conclude a risk exchange agreement. Assume only linear risk exchanges are considered. Denote by x_1 and x_2 the claim amounts before the exchange, and by y_1 and y_2 the claim amounts after the exchange. Then the most general form of a linear risk exchange is

$$\begin{aligned} y_1 &= (1-\alpha)x_1 + \beta x_2 + K \\ y_2 &= \alpha x_1 + (1-\beta)x_2 - K \end{aligned} \qquad 0 \leq \alpha, \beta \leq 1$$

where K is a fixed (positive or negative) monetary amount. If $K = 5\alpha - 10\beta$, then $E(y_1) = E(x_1) = 5$ and $E(y_2) = E(x_2) = 10$. So the exchange does not modify expected claims, and we only need to analyse variances. Assuming independence,

$$\begin{aligned} \operatorname{Var}(y_1) &= 4(1-\alpha)^2 + 8\beta^2 \\ \operatorname{Var}(y_2) &= 4\alpha^2 + 8(1-\beta)^2 \end{aligned}$$

If, for instance, $\alpha = 0.2$ and $\beta = 0.3$, $\operatorname{Var}(y_1) = 3.28 < 4$ and $\operatorname{Var}(y_2) = 4.08 < 8$. Hence it is possible to improve the situation of both partners (if we assume, in this simple example, that companies evaluate their situation by means of the retained variance). Can we define "optimal" values of α and β? □

Those examples have several elements in common :

— Participants have some benefits to share (political power, savings, or money).
— This opportunity to divide benefits results from cooperation of all participants or a sub-group of participants.
— Individuals are free to engage in negotiations, bargaining, coalition formation.
— Participants have conflicting objectives; each wants to secure the largest part of the benefits for himself.

Cooperative game theory analyses those situations where participants' objectives are partially cooperative and partially conflicting. It is in the participants' interest to cooperate, in order to achieve the greatest possible total benefits. When it comes to sharing the benefits of cooperation, however, individuals have conflicting goals. Such situations are usually modeled as n-person cooperative games in characteristic function form, defined and illustrated in Section 2. Section 3 presents and discusses natural conditions, the individual and collective rationality conditions, that narrow the set of possible outcomes. Two concepts of solution are defined: the von Neumann-Morgenstern stable sets and the core. Section 4 is devoted to axiomatic approaches that aim at selecting a unique outcome. The main solution concept is here the Shapley value. Section 5 deals with two-person cooperative games without transferable utilities. The Nash and Kalai-Smorodinsky solution concepts are presented and applied to Example 5. A survey of some other solutions and concluding remarks are to be found in Sections 6 and 7.

2. CHARACTERISTIC FUNCTIONS

First, let us specify which situations will be considered in this paper, and some implicit assumptions.

— Participants are authorized to freely cooperate, negotiate, bargain, collude, make binding contracts with one another, form groups or subgroups, make threats, or even withdraw from the group.
— All participants are fully informed about the rules of the game, the payoffs under each possible situation, all strategies available, ...
— Participants are negotiating about sharing a given commodity (such as money or political power) which is fully transferable between players and evaluated in the same way by everyone. This excludes for instance games where participants evaluate their position by means of a concave utility function; risk aversion is not considered. (In other words, it is assumed that all individuals have linear utility functions). For this reason, the class of games defined here is called "Cooperative games with transferable utilities." This major assumption will be relaxed in Section 5.

Definition 1: An n-person game in characteristic function form Γ is a pair $[N, v]$, where $N = \{1, 2, \ldots, n\}$ is a set of n players. v is a real valued characteristic function on 2^N, the set of all subsets S of N. v assigns a real number $v(S)$ to each subset S of N, and $v(\Phi) = 0$.

Subsets S of N are called coalitions. The full set of players N is the grand coalition. Intuitively, $v(S)$ measures the worth or power that coalition S can achieve when its members act together. Since cooperation creates savings, it is assumed that v is superadditive, i.e., that

$$v(S \cup T) \geqslant v(S) + v(T) \text{ for all } T, S \subset N \text{ such that } S \cap T = \Phi.$$

Definition 2: Two n-person games Γ and Γ', of respective characteristic functions v and v', are said to be strategically equivalent if there exists numbers $k > 0$, $c_1, \ldots, c_n$ such that

$$v'(S) = kv(S) + \sum_{i \in S} c_i \quad \text{for all} \quad S \subseteq N.$$

The switch from v to v' only amounts to changing the monetary units and awarding a subsidy c_i to each player. Fundamentally, this operation doesn't change anything. Hence we only need to study one game in each class of strategically equivalent games. Therefore games are often normalized by assuming that the worth of each player is zero, and that the worth of the grand coalition is 1. [In the sequel expressions such as $v(\{1,3\})$ will be abbreviated as $v(13)$].

$$v(i) = 0 \qquad i = 1, \ldots, n \qquad v(N) = 1$$

Example 1. (UN Security Council). Since a motion either passes or doesn't, we can assign a worth of 1 to all winning coalitions, and 0 to all losing coalitions. The game can thus be described by the characteristic function

$v(S) = 1$ for all S containing all five permanent members and at least 4 nonpermanent members

$v(S) = 0$ for all other S. □

Games such that $v(S)$ can only be 0 or 1 are called simple games. One interesting class of simple games is the class of weighted majority games.

Definition 3: A weighted majority game

$$\Gamma = [M; w_1, \ldots, w_n],$$

where $w_1, \ldots, w_n$ are nonnegative real numbers and

$$M > \frac{1}{2} \sum_{i=1}^{n} w_i,$$

is the n-person cooperative game with characteristic function

$$v(S) = 1 \quad \text{if} \quad \sum_{i \in S} w_i \geq M$$

$$v(S) = 0 \quad \text{if} \quad \sum_{i \in S} w_i < M,$$

for all $S \subseteq N$. w_i is the power of player i (such as the number of shares held in a corporation). M is the required majority.

Example 1. It is easily verified that the UN Security Council's voting rule can be modelled as a weighted majority game. Each permanent member is awarded seven votes, each nonpermanent member one vote. The majority required to pass a motion is 39 votes. A motion can only pass if all five permanent members (35 votes) and at least four nonpermanent members (4 votes) are in favor. Without the adhesion of all permanent members, the majority of 39 votes cannot be reached.

$$\Gamma = [39; 7,7,7,7,7,1,1,1,1,1,1,1,1,1,1]$$

Does this mean that the power of each permanent member is seven times the power of nonpermanent members? □

Example 2. Nassau County's voting procedures form the weighted majority game [58; 31,31,28,21,2,2]. It clearly shows that numerical voting weights do not translate into political power. An inspection of all numerical possibilities reveals that the three least-populated municipalities have no voting power at all. Their combined total of 25 votes is never enough to tip the scales. To pass a motion simply requires the adhesion of two of the three largest districts. So the assigned voting weights might just as well be (31,31,28,0,0,0), or (1,1,1,0,0,0). We need a better tool than the number of votes to evaluate participants' strengths. □

Example 3. (ASTIN money). Straightforward calculations lead to the total interest each coalition can secure

$$\begin{aligned} v(1) &= 46{,}125 \\ v(2) &= 17{,}437.5 \\ v(3) &= 5{,}812.5 \\ v(12) &= 69{,}187.5 \\ v(13) &= 53{,}812.5 \\ v(23) &= 30{,}750 \\ v(123) &= 90{,}000 \end{aligned} \qquad \square$$

Example 4. (Retention groups). This example differs from the others in the sense that figures here represent costs (to minimise) and not earnings (to maximise). Instead of a superadditive characteristic function $v(S)$, a cost function $c(S)$ is introduced. Scale economies make $c(S)$ a subadditive function

$$c(S \cup T) \leq c(S)+c(T) \text{ for all } S, T \subseteq N \text{ such that } S \cap T = \Phi$$

A "cost" game is equivalent to a "savings" game, of characteristic function

$$v(S) = \sum_{i \in S} c_i - c(S).$$

In the case of the example, $c(S)$ is the premium paid by each coalition

$$\begin{aligned} c(1) &= 19 \\ c(2) &= 32 \\ c(12) &= 45 \end{aligned} \qquad \square$$

3. von Neumann-Morgenstern stable sets and the core

Example 3. (ASTIN money). If they agree on a way to subdivide the profits of cooperation, the three Treasurers will have a total of 90,000 francs to share. Denote $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ the outcome (or payoff, or allocation): player i will receive the amount α_i. Obviously, the ASTIN Treasurer will only accept an allocation that awards him at least 46,125 francs, the amount he can secure by himself. This is the individual rationality condition. □

Definition 4: A payoff $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is individually rational if $\alpha_i \geq v(i)$ $i = 1, \ldots, n$.

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Definition 5: An imputation for a game $\Gamma = (N, v)$ is a payoff $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that

$$\alpha_i \geq v(i) \qquad i = 1, \ldots, n$$

$$\sum_{i=1}^{n} \alpha_i = v(N)$$

An imputation is an individually rational payoff that allocates the maximum amount. (This condition is also called "efficiency" or "Pareto-optimality").

Example 3. (ASTIN money). An imputation is any allocation such that

$$\begin{aligned} &\alpha_1+\alpha_2+\alpha_3 = 90{,}000 \\ &\alpha_1 \geq 46{,}125 \\ &\alpha_2 \geq 17{,}437.5 \\ &\alpha_3 \geq \ \ 5{,}812.5 \qquad \square \end{aligned}$$

Example 4. (Retention groups). In this cost example, an imputation is any set of premiums (α_1, α_2) such that

$$\begin{aligned} &\alpha_1+\alpha_2 = 45 \\ &\alpha_1 \leq 19 \\ &\alpha_2 \leq 32 \end{aligned}$$

Let us now add a third group of $n_3 = 120$ individual to this example, all subject to a loss of 1 with a probability $q_3 = 0.3$. A risk retention group with a ruin probability of .001 would require a total premium of

$$n_3 q_3 + 3\sqrt{n_3 q_3 (1-q_3)} = 36+15 = 51$$

If all three groups decide to merge to achieve a maximum reduction of the safety loading, the total premium will be

$$\begin{aligned} &n_1 q_1 + n_2 q_2 + n_3 q_3 + 3\sqrt{n_1 q_1 (1-q_1) + n_2 q_2 (1-q_2) + n_3 q_3 (1-q_3)} \\ &= 10+20+36+21 \\ &= 87 \end{aligned}$$

In this case an imputation is a payoff $(\alpha_1, \alpha_2, \alpha_3)$ such that

$$\begin{aligned} &\alpha_1+\alpha_2+\alpha_3 = 87 \\ &\alpha_1 \leq 19 \\ &\alpha_2 \leq 32 \\ &\alpha_3 \leq 51 \end{aligned}$$

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Are all those imputations acceptable to everybody? Consider the allocation (17, 31, 39). It is an imputation. It will however never be accepted by the first two groups. Indeed they are better off withdrawing from the grand coalition, forming coalition (12), and agreeing for instance on a payoff (15.5, 29.5). Player 3, the third group, cannot object to this secession since, left alone, he will be stuck to a premium of 51. He will be forced to make a concession during negotiations and accept a higher α_3. α_3 needs to be at least 42 to prevent players 1 and 2 to secede. This is the collective rationality condition: no coalition should have an incentive to quit the grand coalition. □

Definition 6: A payoff $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is collectively rational if

$$\sum_{i \in S} \alpha_i \geq v(S) \quad \text{for all} \quad S \subset N.$$

Definition 7: The core of the game is the set of all collectively rational payoffs.

The core of a game can be empty. When it is not, it usually consists of several, or an infinity, of points. It can also be defined using the notion of dominance.

Definition 8: Imputation $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ dominates imputation $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with respect to coalition S if

(i) $S \neq \Phi$

(ii) $\beta_i > \alpha_i$ for all $i \in S$

(iii) $v(S) \geq \sum_{i \in S} \beta_i$

So there exists a non-void set of players S, that all prefer β to α, and that has the power to enforce this allocation.

Definition 9: Imputation β dominates imputation α if there exists a coalition S such that β dominates α with respect to S.

Definition 7': The core is the set of all the undominated imputations.

Definitions 7 and 7' are equivalent.

Example 4. (Retention groups). The core is the set of all payoffs that allocate the total premium of 87, while satisfying the 3 individual and 3 collective rationality conditions.

$$\begin{aligned}
\alpha_1+\alpha_2+\alpha_3 &= 87\\
\alpha_1 &\leq 19\\
\alpha_2 &\leq 32\\
\alpha_3 &\leq 51\\
\alpha_1+\alpha_2 &\leq 45\\
\alpha_1 \quad +\alpha_3 &\leq 63.5\\
\alpha_2+\alpha_3 &\leq 75.3
\end{aligned}$$

So the core enables us to find upper and lower bounds for the premiums

$$\begin{aligned}
&\alpha_1+\alpha_2+\alpha_3 = 87\\
&11.7 \leq \alpha_1 \leq 19\\
&23.5 \leq \alpha_2 \leq 32\\
&42 \quad \leq \alpha_3 \leq 51
\end{aligned}$$

An allocation that violates any inequality leads to the secession of one or two groups. □

Example 3. (ASTIN money). The core consists of all payoffs such that

$$\begin{aligned}
&\alpha_1+\alpha_2+\alpha_3 = 90{,}000\\
&46{,}125 \quad \leq \alpha_1 \leq 59{,}250\\
&17{,}437.5 \leq \alpha_2 \leq 36{,}187.5\\
&5{,}812.5 \leq \alpha_3 \leq 20{,}812.5
\end{aligned}$$ □

Despite its intuitive appeal, the core was historically not the first concept that attempted to reduce the set of acceptable payoffs with rationality conditions. In their path-breaking work, VON NEUMANN and MORGENSTERN (1945) introduced the notion of stable sets.

Definition 10: A von Neumann-Morgenstern stable set of a game $\Gamma = (N, v)$ is a set L of imputations that satisfy the two following conditions

(i) (External stability) To each imputation $\alpha \notin L$ corresponds an imputation $\beta \in L$ that dominates α.

(ii) (Internal stability) No imputation of L dominates another imputation of L.

Stable sets are however usually very difficult to compute.

The main drawback of the core and the stable sets seems to be that, in most cases, they contain an infinity of allocations. For instance, the core and the stable set of all 2-person games simply consist of all imputations. It would be preferable to be able to single out a unique, "fair" payoff for each game. This is what the Shapley value achieves.

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4. THE SHAPLEY VALUE

Example 3. (ASTIN money). Assume the ASTIN Treasurer decides to initiate the coalition formation process. Playing alone, he would make $v(1) = 46{,}125$. If player 2 decides to join, coalition (12) will make $v(12) = 69{,}187.5$. Assume player 1 agrees to award player 2 the entire benefits of cooperation; player 2 receives his entire *admission value* $v(12) - v(1) = 23{,}062.5$. Player 3 joins in a second stage, and increases the total gain to 90,000. If he is allowed to keep his entire admission value $v(123) - v(12) = 20{,}812.5$, we obtain the payoff

$$[46{,}125; \quad 23{,}062.5; \quad 20{,}812.5]$$

This allocation of course depends on the order of formation of the grand coalition. If player 1 joins first, then player 3, and finally player 2, and if everyone keeps his entire admission value, the following payoff results

$$[46{,}125; \quad 36{,}187.5; \quad 7{,}687.5]$$

The four other player permutations [(213), (231), (312), (321)] lead to the respective payoffs

$$[51{,}750; \quad 17{,}437.5; \quad 20{,}812.5]$$

$$[59{,}250; \quad 17{,}437.5; \quad 13{,}312.5]$$

$$[48{,}000; \quad 36{,}187.5; \quad 5{,}812.5]$$

$$[59{,}250; \quad 24{,}937.5; \quad 5{,}812.5]$$

Assume we now decide to take the average of those six payoffs, to obtain the final allocation

$$[51{,}750; \quad 25{,}875 \; ; \quad 12{,}375 \;]$$

We have in fact computed the Shapley value of the game, the expected admission value when all player permutations are equiprobable. □

The Shapley value is the only outcome that satisfies the following set of three axioms [SHAPLEY, 1953)].

Axiom 1 (Symmetry). For all permutations Π of players such that $v[\Pi(S)] = v(S)$ for all S, $\alpha_{\Pi(i)} = \alpha_i$.

A symmetric problem has a symmetric solution. If there are two players that cannot be distinguished by the characteristic function, that contribute the same amount to each coalition, they should be awarded the same payoff. This axiom is sometimes also called anonymity; it implies that the selected allocation only depends on the characteristic function, and not, for instance, on the numbering of the players.

Axiom 2 (Dummy players). If, for a player i, $v(S) = v(S \backslash i) + v(i)$ for each coalition to which he can belong, then $\alpha_i = v(i)$.

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A dummy player does not contribute any scale economy to any coalition. The worth of any coalition only increases by $v(i)$ when he joins. Such an inessential player cannot claim to receive any share of the benefits of cooperation.

Axiom 3 (Additivity). Let $\Gamma = (N, v)$ and $\Gamma' = (N, v')$ be two games, and $\alpha(v)$ and $\alpha'(v)$ their respective payoffs. Then $\alpha(v+v') = \alpha(v)+\alpha(v')$ for all players.

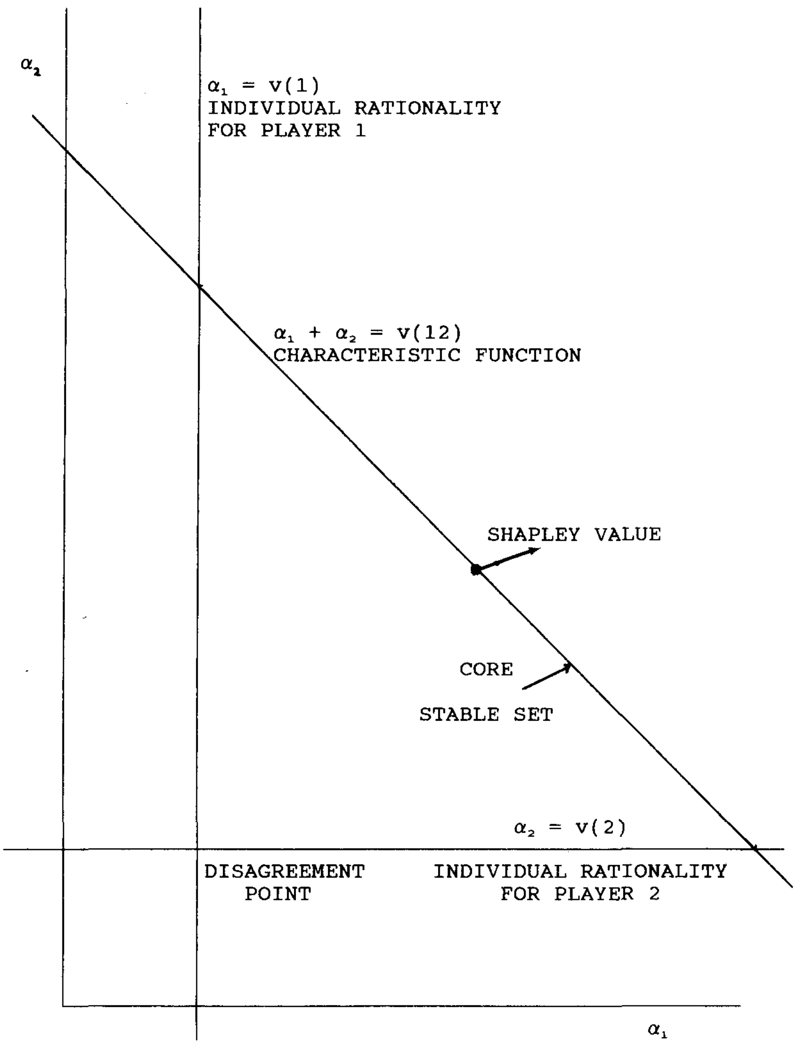

FIGURE 1. Two-person cooperative game with transferable utilities.

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Payoffs resulting from two distinct games should be added. While the first two axioms seem quite justified, the latter has been criticized. It rules out all interactions between the two games, for instance.

Shapley has shown that one and only one allocation satisfies the three axioms

$$\alpha_i = (1/n!) \sum_S (s-1)!\,(n-s)!\,[v(S)-v(S\setminus i)] \qquad i = 1, \ldots, n$$

where s is the number of members of a coalition S.

The Shapley value can be *interpreted* as the mathematical expectation of the admission value, when all orders of formation of the grand coalition are equiprobable. In computing the value, one can assume, for convenience, that all players enter the grand coalition one by one, each of them receiving the entire benefits he brings to the coalition formed just before him. All orders of formation of N are considered and intervene with the same weight $1/n!$ in the computation. The combinatorial coefficient results from the fact that there are $(s-1)!\,(n-s)!$ ways for a player to be the last to enter coalition S: the $s-1$ other players of S and the $n-s$ players of $N\setminus S$ can be permuted without affecting i's position.

In a two-player game, the Shapley value is

$$\alpha_1 = (1/2)\,[v(12)+v(1)-v(2)]$$

$$\alpha_2 = (1/2)\,[v(12)+v(2)-v(1)]$$

It is the middle of the segment $\alpha_1+\alpha_2 = v(12)$, $\alpha_1 \geq v(1)$, $\alpha_2 \geq v(2)$. This is illustrated in Figure 1.

Example 1. (UN Security Council). In a weighted majority game, the admission value of a player is either 0 or 1. One simply has to compute the probability that a player clinches victory for a motion. In the UN Security Council game, the power of a nonpermanent member i is the probability that he enters ninth in any coalition that already includes the five permanent members. It is

$$\alpha_i = \binom{8}{3} \underbrace{(5/15)\ (4/14)\ (3/13)\ (2/12)\ (1/11)}_{\text{all five permanent before } i}\ \underbrace{(9/10)\ (8/9)\ (7/8)}_{\text{3 of the nonpermanent before } i}\ \underset{\substack{\downarrow \\ i \text{ then enters}}}{(1/7)}$$

$$= 0.1865\,\%$$

By symmetry, the power for each permanent member is

$$\alpha_i = 19.62\,\%$$

So permanent nations are 100 times more powerful than nonpermanent nations. [Note: in practice a permanent member may abstain without impairing the validity of an affirmative vote. While this rule complicates the analysis of the game, it only changes the second decimal of the Shapley value]. □

Example 2. (Nassau County). The Shapley value of the districts is (1/3, 1/3, 1/3, 0, 0, 0). This analysis led the County authorities to change the voting rules by increasing the required majority from 58 to 63. There are now no more dummy players, and the new power indices are [0.283, 0.283, 0.217, 0.117, 0.050, 0.050]. This is certainly much closer to the original intention. □

Example 4. (Retention groups). In the two-company version of this game, the Shapley value is [16,29]. In the three-company version, the value is [14.5, 26.9, 45.6]. The traditional pro rata approach leads to [13.2, 26.4, 47.4]. It does not take into account the savings each member brings to the grand coalition, or its threat possibilities. It is unfair to the third group, because it fails to give proper credit to the important reduction (10) of the total safety loading it brings to the grand coalition. □

The Shapley value may lie outside the core. In the important subclass of convex games, however, it will always be in the core.

Definition 11. A game is convex if, for all $S \subseteq T \subseteq N$, for all $i \notin T$,

$$v(T \cup i) - v(T) \geq v(S \cup i) - v(S).$$

A game is convex when it produces large economies of scale; a "snow-balling" effect makes it increasingly interesting to enter a coalition as its number of members increases. In particular, it is always preferable to be the last to enter the grand coalition N. The core of convex games is always non-void. Furthermore, it coincides with the unique von Neumann-Morgenstern stable set. It is a compact convex polyhedron, of dimension at most $n-1$. The Shapley value lies in the center of the core, in the sense that it is the center of gravity of the core's external points.

5. TWO-PERSON GAMES WITHOUT TRANSFERABLE UTILITIES

Example 5. (Risk exchange). As shown in the presentation of the example, selecting $\alpha = 0.2$ and $\beta = 0.3$ results in a decrease of Var (y_1) of 0.72, and a decrease of Var (y_2) of 3.92. This risk exchange treaty is represented as point 1 in Figure 2.

In this figure the axes measure the respective variance reductions, p_1 and p_2. Point 2 corresponds to $\alpha = \beta = 0.4$. It dominates point 1, since it leads to a greater variance reduction for both companies. Point 3 is $\alpha = 0.53$, $\beta = 0.47$; it dominates points 1 and 2. It can be shown that no point can dominate point 3, and that all treaties such that $\alpha + \beta = 1$ neither dominate nor are dominated by point 3. For instance, point 4 ($\alpha = 0.7$, $\beta = 0.3$) will be preferred to point 3 by

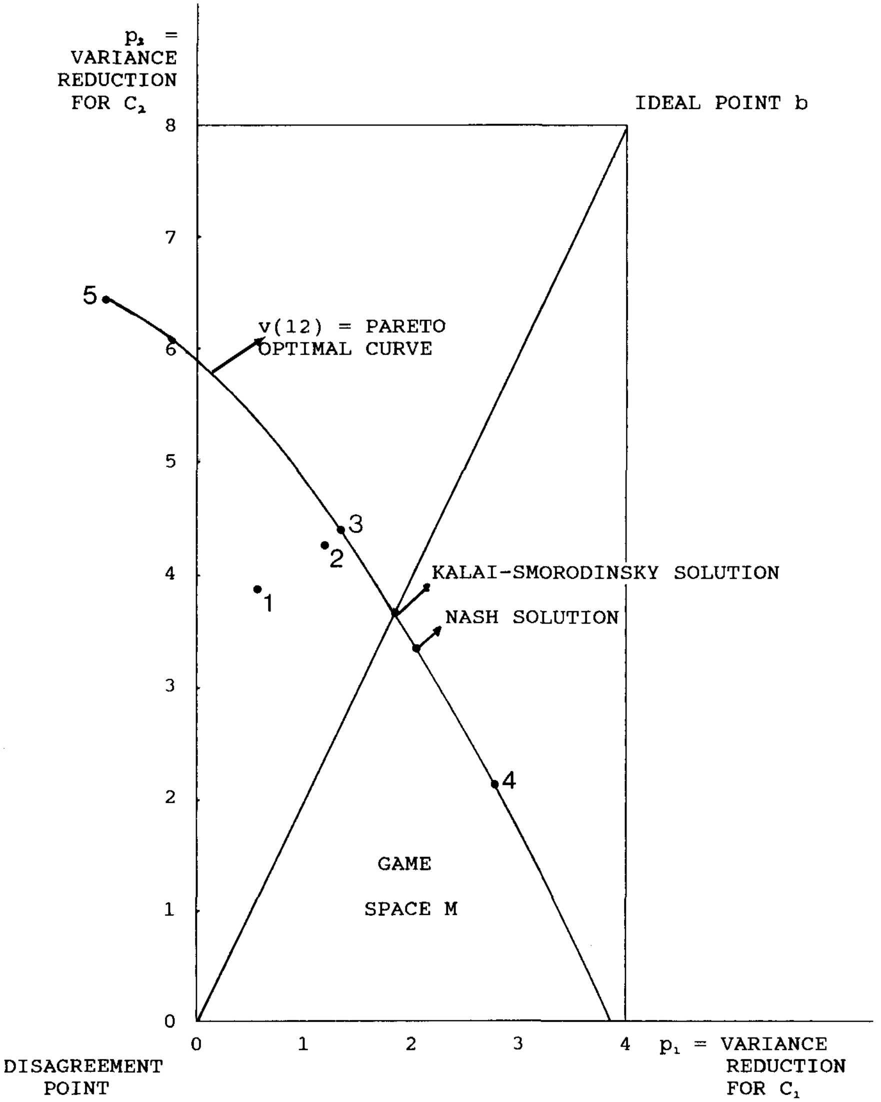

FIGURE 2. Two-person cooperative game without transferable utilities.

C_1. However C_2 will prefer point 3 to point 4. Hence neither point dominates the other. The set of all treaties such that $\alpha+\beta = 1$ forms curve $v(12)$, the Pareto-optimal surface. Points to the north-east of $v(12)$ cannot be attained. All points to the south-west of $v(12)$ correspond to a given selection of α and β. The convex set of all attainable points, including the boundary $v(12)$, is called the game space M. That space is limited by the Pareto-optimal curve and the two axes. The axes represent the two individual rationality conditions: no company will accept a treaty that results in a variance increase. For instance point 5 ($\alpha = 0.35$, $\beta = 0.65$) will not be accepted by C_1. While each point in the game space is attainable, it is in both companies' interest to cooperate to reach the Pareto-optimal curve. Any point that does not lie on the north-east boundary is dominated by a Pareto-optimal point. Once the curve is reached, however, the players' interests become conflicting. C_1 will negotiate to reach a point as far east as possible, while C_2 will attempt to move the final treaty north. If the players cannot reach an agreement, no risk exchange will take place. The disagreement point results in no variance reduction.

Hence all the elements of a two-player game are present in this simplified risk exchange example. In fact, Figure 2 closely resembles Figure 1, with an important difference: the Pareto-optimal set of treaties $v(12)$ is a curve in Figure 2, while the characteristic function $v(12)$ in Figure 1 is a straight line. This is due to the non-transferability of utilities in the risk exchange example. The players are "trading" variances, but an increase of 1 of Var (y_1) results in a decrease of Var (y_2) that is not equal to 1. Example 5 is a two-person cooperative game without transferable utility. □

Definition 12. A two-person cooperative game without transferable utilities is a couple (M, d), where $d = (d_1, d_2)$ is the disagreement point (the initial utilities of the players). M, the game space, is a convex compact set in the two-dimensional space E^2 of the players' utilities; it represents all the payoffs that can be achieved.

Such a game is often called a two-person bargaining game. Let B be the set of all pairs (M, d). Since no player will accept a final payoff that does not satisfy the individual rationality condition, M can be limited to the set of points (p_1, p_2) such that $p_1 \geq d_1$ and $p_2 \geq d_2$. Our goal is to select a unique payoff in M.

Definition 13. A solution (or a value) is a rule that associates to each bargaining game a payoff in M. It is thus a mapping $f: B \to E^2$ such that $f(M, d)$ is a point $p = (p_1, p_2)$ of M for all $(M, d) \in \mathrm{B}$; $f_1(M, d) = p_1$ and $f_2(M, d) = p_2$.

The first solution concept for bargaining games was developed in 1950 by Nash. The Nash solution satisfies the four following axioms.

Axiom 1. Independence of linear transformations

The solution cannot be affected by linear transformations performed on the players' utilities. For all (M, d) and all real numbers $a_i > 0$ and b_i, let (M', d') be the game defined by $d_i' = a_i d_i + b_i$ $(i = 1, 2)$ and $M' = \{q \in E^2 | \exists p \in M$ such that $q_i = a_i p_i + b_i\}$. Then $f_i(M', d') = a_i f_i(M, d) + b_i \quad i = 1, 2$.

This axiom is hard to argue with. It only reflects the information contained in utility functions. Since utilities are only defined up to linear transformations, it should be the same for solutions.

Axiom 2. Symmetry

All symmetric games have a symmetric solution. A game is symmetric if $d_1 = d_2$ and $(p_1, p_2) \in M \Rightarrow (p_2, p_1) \in M$. The axiom requires that, in this case, $f_1(M, d) = f_2(M, d)$.

Like axiom 1, axiom 2 requires that the solution only depends on the information contained in the model. A permutation of the two players should not modify the solution, if they cannot be differentiated by the rules of the game. Two players with the same utility function and the same initial wealth should receive the same payoff if the game space is symmetric.

Axiom 3. Pareto-optimality

The solution should be on the Pareto-optimal curve. For all $(M, d) \in B$, if p and $q \in M$ are such that $q_i > p_i$ $(i = 1, 2)$, then p cannot be the solution: $f(M, d) \neq p$.

Axiom 4. Independence of irrelevant alternatives

The solution does not change if we remove from the game space any point other than the disagreement point and the solution itself. Let (M, d) and (M', d) be two games such that M' contains M and $f(M', d)$ is an element of M. Then $f(M, d) = f(M', d)$.

This axiom formalizes the negotiation procedure. It requires that the solution, which by axiom 3 must lie on the upper boundary of the game space, depends on the shape of this boundary only in its neighbourhood, and not on distant points. It expresses the fact that, during negotiations, the set of the alternatives likely to be selected is progressively reduced. At the end, the solution only competes with very close points, and not with proposals already eliminated during the first phases of the discussion. Nash's axioms thus model a bargaining procedure that proceeds by narrowing down the set of acceptable points. Each player makes concessions until the final point is selected.

NASH (1950) has shown that one and only one point satisfies the four axioms. It is the point that maximizes the product of the two players' utility gains. Nash's solution is the function f, defined by $f(M, d) = p$, such that $p \geq d$ and $(p_1 - d_1)(p_2 - d_2) \geq (q_1 - d_1)(q_2 - d_2)$, for all $q \neq p \in M$.

Example 5. (Risk exchange). In this example, the players' objective is to reduce the variance of their claims. Hence $d = (0, 0)$: if the companies cannot agree on a risk exchange treaty, they will keep their original portfolio, with no improvement. The players' variance reductions are

$$p_1 = 4-4(1-\alpha)^2-8\beta^2$$
$$p_2 = 8-4\alpha^2-8(1-\beta)^2$$

Maximising the product $p_1 p_2$, under the condition $\alpha+\beta = 1$, leads to the Nash solution

$$\alpha = 0.613$$
$$\beta = 0.387$$
$$p_1 = 2.203$$
$$p_2 = 3.491 \qquad \square$$

Nash's axiom 4 has been criticised by KALAI and SMORODINSKY (1975), who proved that Nash's solution does not satisfy a monotonicity condition. Consider the two games represented in Figure 3. The space of game 1 is the four-sided figure whose vertices are at *d*, *A*, *B*, *D*. The Nash solution is *B*. The space of game 2 is the figure whose vertices are at *d*, *A*, *C*, *D*. From the second player's point of view, game 2 seems more attractive, since he stands to gain more if the first player's payoff is between *E* and *D*. So one would expect the second player's payoff to be larger in game 2. This is not the case, since the Nash solution of game 2 is *C*.

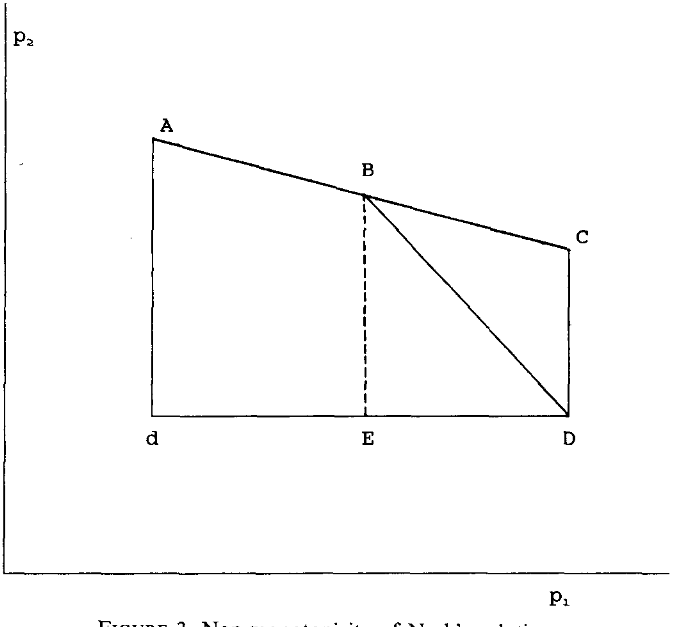

FIGURE 3. Non-monotonicity of Nash's solution.

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Axiom 5. Monotonicity. Let $b(M) = (b_1, b_2)$ the "ideal" point formed by the maximum possible payoffs (see Figure 2): $b_i = \max\{p_i | (p_1, p_2) \in M\}$ $(i = 1, 2)$. If (M, d) and (M', d) are two games such that M contains M' and $b(M) = b(M')$, then $f(M, d) \geq f(M', d)$.

KALAI and SMORODINSKY have shown that one and only one point satisfies axioms 1, 2, 3, and 5. It is situated at the intersection of the Pareto-optimal curve and the straight line linking the disagreement point and the ideal point.

Example 5. It is easily verified that the equation of the Pareto-optimal curve is $\sqrt{8-p_1}+\sqrt{4-p_2} = 12$. Since the ideal point is (4,8), the line joining d and b has equation $p_2 = 2p_1$. Kalai-Smorodinsky's solution point, at the intersection, is

$$\begin{aligned} \alpha &= 0.5858 \\ \beta &= 0.4142 \\ p_1 &= 1.9413 \\ p_2 &= 3.8821 \end{aligned}$$

It is slightly more favourable to player 2 than Nash's solution. □

6. OTHER SOLUTION CONCEPTS – OVERVIEW OF LITERATURE

Stable sets and the core are the most important solution concepts of game theory that attempt to reduce the number of acceptable allocations by introducing intuitive conditions. Both notions however can be criticized.

Stable sets are difficult to compute. Some games have no stable sets. Some others have several. Moreover, the dominance relation is neither antisymmetric nor transitive. It is for instance possible that an imputation β dominates an imputation α with respect to one coalition, while α dominates β with respect to another coalition. Therefore an imputation inside a stable set may be dominated by an imputation outside.

The concept of core is appealing, because it satisfies very intuitive rationality conditions. However, there exists vast classes of games that have an empty core: the rationality conditions are conflicting. Moreover, several examples have been built for which the core provides a counter-intuitive payoff, as shown in Example 6.

Example 6. A pair of shoes

Player 1 owns a left shoe. Players 2 and 3 each own a right shoe. A pair can be sold for \$ 100. How much should 1 receive if the pair is sold? Surprisingly, the core totally fails to catch the threat possibilities of coalition (23) and selects the paradoxical allotment (100, 0, 0). Any payoff that awards a positive amount to 2 or 3 is dominated; for instance (99, 1, 0) is dominated by (99.5, 0, 0.5).

Moreover, the paradox remains if we assume that there are 999 left shoes and 1000 right shoes. The game is now nearly symmetrical, but the owners of right shoes still receive nothing. The Shapley value is $(66\frac{2}{3}, 16\frac{2}{3}, 16\frac{2}{3})$, definitely a much better representation of the power of each player than the core. □

Many researchers feel that the core is too static a concept, that it does not take into account the real dynamics of the bargaining process. In addition, laboratory experiments consistently produce payoffs that lie outside the core. This led AUMANN and MASCHLER (1964) to define the bargaining set. This set explicitly recognizes the fact that a negotiation process is a multi-criteria situation. Players definitely attempt to maximise their payoff, but also try to enter into a "safe" or "stable" coalition. Very often, it is observed that players willingly give up some of their profits to join a coalition that they think has fewer chances to fall apart. This behaviour is modelled through a dynamic process of "threats" and "counter-threats." A payoff is then considered stable if all objections against it can be answered by counter-objections.

Example 7. Consider the three-person game

$$\begin{aligned} v(1) &= v(2) = v(3) = 0 \\ v(12) &= v(13) = 100 \\ v(23) &= 50 \end{aligned}$$

The core of this game is empty. For instance, the players will not agree on an allocation like [75, 25, 0], because it is dominated by [76, 0, 24]. Bargaining set theory, on the other hand, claims that such a payoff is stable. If player 1 threatens 2 of a payoff [76, 0, 24], this objection can be met with the counter-objection [0, 25, 25]. Player 2 shows that, without the help of player 1, he can protect his payoff of 25, while player 3 receives more in the counter-objection than in the objection. Similarly, objection [0, 27, 23] of player 2 against [75, 25, 0] can be counter-objected by [75, 0, 25]. So, if a proposal [75, 25, 0] arises during the bargaining process, it is probable that it will be selected as final payoff. Any objection, by either player 1 or player 2, can be countered by the other. On the other hand, a proposal like [80, 20, 0] is unstable. Player 2 can object that he and player 3 will get more in [0, 21, 29]. Player 1 has no counter-objection, because he cannot keep his 80 while offering player 3 at least 29.

Thus, in addition to all undominated payoffs (the core), the bargaining set also contains all payoffs against which there exists objections, providing they can be met by counter-objections. The bargaining set for this example consists of the four points

$$\begin{array}{l} [\ 0,\ \ 0,\ \ 0] \\ [75,\ \ 0,\ 25] \\ [75,\ 25,\ \ 0] \\ [\ 0,\ 25,\ 25] \end{array} \qquad \square$$

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The bargaining set is never empty. It always contains the core. For more details, consult OWEN (1968, 1982) or AUMANN and MASCHLER (1964).

In 1965, DAVIS and MASCHLER defined the kernel of a game, a subset of the bargaining set. In 1969, SCHMEIDLER introduced the nucleolus, a unique payoff, included in the kernel. It is defined as the allocation that minimises successively the largest coalitional excesses

$$e(\alpha, S) = v(S) - \sum_{i \in S} \alpha_i$$

The excess is the difference between a payoff a coalition can achieve and the proposed allocation. Hence it measures the amount ("the size of the complaint") by which coalition S as a group falls short of its potential $v(S)$ in allocation α. If the excess is positive, the payoff is outside the core (and so the nucleolus exists even when the core is empty). If the excess is negative, the proposed allocation is acceptable, but the coalition nevertheless has interest in obtaining the smallest possible $e(\alpha, S)$. The nucleolus is the imputation that minimises (lexicographically) the maximal excess. Since it is as far away as possible of the rationality conditions, it lies in the middle of the core. It is computed by solving a finite sequence of linear programs. Variants of the nucleolus, like the proportional and the disruptive nucleolus, are surveyed among others in LEMAIRE (1983). The proportional nucleolus, for instance, results when the excesses are defined as

$$e(\alpha, S) = \left[v(S) - \sum_{i \in S} \alpha_i\right] \Big/ v(S)$$

Since it consists of a single point, the nucleolus (also called the lexicographic center) provides an alternative to the Shapley value. The Shapley value has been subjected to some criticisms, mainly focussing on the additivity axiom and the fact that people joining a coalition receive their full admission value.

Example 3. (ASTIN money). The Shapley value, computed in Section 4, is

[51,750; 25,875; 12,375]

It awards an interest of 11.5% to ASTIN and I.A.A., and 16.5% to A.A.Br. This allocation is much too generous towards A.A.Br.'s Treasurer, who takes a great advantage from the fact that he is essential to reach the 3-million mark. His admission value is extremely high (in proportion to the funds supplied) when he comes in last. The nucleolus is

[52,687.5; 24,937.5; 12,375]

or, in percentages

[11.71; 11.08; 16.5]

It recognises the better bargaining position of ASTIN versus I.A.A., but still favours A.A.Br. Both the Shapley value and the nucleolus, defined in an additive way, fail in this multiplicative problem. The proportional nucleolus suggests

$$[54{,}000; \quad 27{,}000; \quad 9{,}000]$$

or, in percentages,

$$[12; \quad 12; \quad 12],$$

thereby justifying common practice. □

Only the case of the two-person games without transferable utilities has been reviewed in Section 5. A book by ROTH (1980) is devoted entirely to this case. It provides a thorough analysis of Nash's and Kalai-Smorodinsky's solutions. The generalisation of those models to the n-person case has proved to be very difficult. In the two-person case, the disagreement point is well defined: if the players don't agree, they are left alone. In the n-person case, if a general agreement in the grand coalition cannot be reached, sub-coalitions may form. Also, some players may wish to explore other avenues, like possible business partners outside the closed circle of the n players. This is an objection against modeling market situations as non-transferable n-person games. Such games ignore external opportunities, such as competitive outside elements. See SHAPLEY (1964) and LEMAIRE (1974, 1979) for definitions of values in the n-player case.

Though somewhat dated by now, the book by LUCE and RAIFFA (1957) is still an excellent introduction to game theory and utility theory. It provides an insightful critical analysis of the most important concepts. An excellent book that surveys recent developments is OWEN (1968, 1982, especially the second edition). A booklet edited by LUCAS (1981) provides an interesting, simple, abundantly illustrated analysis of the basics of cooperative and non-cooperative game theory. Finally, the proceedings of a conference on applied game theory [BRAMS, SCHOTTER, SCHWODIAUER (1979)] provide a fascinating overview (from a strategic analysis of the Bible to the mating of crabs) of applications of the theory.

7. CONCLUSIONS

Game theory solutions have been effectively implemented in numerous situations. A few of those applications are

- — allocating taxes among the divisions of McDonnell-Douglas Corporation
- — subdividing renting costs of WATS telephone lines at Cornell University
- — allocating tree logs after transportation between the Finnish pulp and paper companies
- — sharing maintenance costs of the Houston medical library
- — financing large water resource development projects in Tennessee
- — sharing construction costs of multipurpose reservoirs in the United States

— subdividing costs of building an 80-kilometer water supply tunnel in Sweden
— setting landing fees at Birmingham Airport
— allotting water among agricultural communities in Japan
— subsidising public transportation in Bogota

Cooperative game theory deals with competition, cooperation, conflicts, negotiations, coalition formation, allocation of profits. Consequently one would expect numerous applications of the theory in insurance, where competitive and conflicting situations abound. It has definitely not been the case. The first article mentioning game theory in the *ASTIN Bulletin* was authored by BORCH (1960a). In subsequent papers, BORCH (1960b, 1963) progressively developed his celebrated risk exchange model, which in fact is an *n*-person cooperative game without transferable utilities. This model has further been developed by in the 1970s by Lemaire and several of his students [BATON and LEMAIRE (1981a, 1981b), BRIEGLEB and LEMAIRE (1982), LEMAIRE (1977, 1979)]. The ASTIN Bulletin has yet to find a third author attracted by game theory! It is hoped that this survey paper will contribute to disseminate some knowledge about the situations game theory models, so that the risk exchange model will not stand for a long time as its lone actuarial application.

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