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A NOTE ON THE INTERSECTIONS OF THE BESICOVITCH SETS AND ERDŐS–RÉNYI SETS

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Abstract

For $x \in (0, 1]$ and a positive integer n, let $S_n(x)$ denote the summation of the first n digits in the dyadic expansion of x and let $r_n(x)$ denote the run-length function. In this paper, we obtain the Hausdorff dimensions of the following sets:

$$\Big\{x \in (0,1] : \liminf_{n\to\infty} \frac{S_n(x)}{n} = \alpha, \limsup_{n\to\infty} \frac{S_n(x)}{n} = \beta, \lim_{n\to\infty} \frac{r_n(x)}{\log_2 n} = \gamma\Big\},$$

where $0 \le \alpha \le \beta \le 1$, $0 \le \gamma \le +\infty$.

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1. Introduction

Suppose that $A, B \subset \mathbb{R}^n$ are two fractal sets; the intersection $A \cap B$ is often a fractal. Therefore, it is interesting to try to find the relationship between the Hausdorff dimension of this intersection and those of A and B. Unfortunately, we immediately find that we can say almost nothing in the general case although one often could hope that

$$\dim_H(A \cap B) = \max\{0, \dim_H A + \dim_H B - n\}. \tag{1.1}$$

Here and in the following, $\dim_H F$ denotes the Hausdorff dimension of the set F. For example, let C be the middle third Cantor set; Hawkes [12] proved that $\dim_H((C + t) \cap C) = \frac{1}{3}(\log 2/\log 3)$ for Lebesgue almost all $t \in [-1, 1]$. The result shows that the formula (1.1) does not hold even for 'many' simple fractal sets $C + t$ and C since the Hausdorff dimensions of the two sets are $\log 2/\log 3$. For more details about Hausdorff

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dimension and the results on the intersection of two fractals, we refer the reader to the famous book [8].

In this paper, we will consider the intersections of two typical fractal sets and our result shows that the formula (1.1) holds for these intersections. Let us firstly recall two kinds of fractal sets: Besicovitch sets and Erdős–Rényi sets.

Each $x \in (0,1]$ admits a nonterminating dyadic expansion:

$$x = \sum_{k=1}^{\infty} \frac{x_k}{2^k} = 0.x_1x_2\cdots,$$

where $x_k \in \{0,1\}$ for any $k \geq 1$. The infinite sequence $(x_1, x_2, x_3, \ldots)$ is called the dyadic digit sequence of x. For a positive integer n, let $S_n(x)$ be the summation of the first n digits of x. The classical Borel normal number theorem can be stated as follows:

$$\lim_{n\to\infty} \frac{S_n(x)}{n} = 1/2 \tag{1.2}$$

for Lebesgue almost all $x \in (0,1]$. In a fractal, it is natural to study the following level sets:

$$B(\alpha) = \Big\{x \in (0,1] : \lim_{n\to\infty} \frac{S_n(x)}{n} = \alpha\Big\}, \quad 0 \leq \alpha \leq 1.$$

Besicovitch [2] first studied the Hausdorff dimensions of this kind of sets and Eggleston [6] generalized Besicovitch's work to b-adic expansions, where $b \geq 2$ is an integer. More precisely, they established that

$$\dim_H B(\alpha) = \frac{H(\alpha)}{\log 2}.$$

Here $H(x)$ is the classical entropy function, which is defined as

$$H(x) = x\log x + (1-x)\log(1-x), \quad 0 \leq x \leq 1, \tag{1.3}$$

where we define $0\log 0 = 0$ by convention.

Nowadays, the sets $B(\alpha)$ are often called Besicovitch sets. It is worth mentioning that the above work of Besicovitch and Eggleston has been generalized in diverse directions; see [1, 9, 19, 21, 22] and references therein.

Recently, another kind of fractal sets arising from the digit sequence of $x \in (0,1]$, Erdős–Rényi sets, has been widely studied by many authors. For each $n \geq 1$ and $x \in (0,1]$, the run-length function $r_n(x)$ is defined as the length of the longest run of 1's in $(x_1, x_2, \ldots, x_n)$, that is,

$$r_n(x) = \max\{\ell : x_{i+1} = \cdots = x_{i+\ell} = 1 \text{ for some } 0 \leq i \leq n-\ell\}.$$

Erdős and Rényi [7] (see also [23]) proved the following asymptotic behavior of r_n: for Lebesgue almost all $x \in (0,1]$,

$$\lim_{n\to\infty} \frac{r_n(x)}{\log_2 n} = 1. \tag{1.4}$$

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This is the well-known Erdős–Rényi limit theorem. There are many generalizations of this theorem; see [5, 18, 25] and references therein. Again, it is interesting to study the following level sets:

$$E(\gamma) = \Big\{x \in (0,1] : \lim_{n\to\infty} \frac{r_n(x)}{\log_2 n} = \gamma\Big\}, \quad 0 \le \gamma \le +\infty.$$

Following Chen and Wen [4], we call $E(\gamma)$ Erdős–Rényi sets. Moreover, Chen and Wen [4] proved that

$$\dim_H E(\gamma) = 1, \quad 0 \le \gamma \le +\infty. \tag{1.5}$$

Clearly, the two quantities $S_n(x)$ and $r_n(x)$ reflect different properties of the dyadic digit sequence of the point x. It is interesting to consider those points with some properties associated with them simultaneously. For $0 \le \alpha \le 1$, $0 \le \beta \le +\infty$, Chen and Wen [4] first studied the Hausdorff dimensions of the following interesting intersections:

$$\underline{S}(\alpha,\beta) = \Big\{x \in (0,1] : \liminf_{n\to\infty} \frac{S_n(x)}{n} \ge \alpha,\ \lim_{n\to\infty} \frac{r_n(x)}{\log_2 n} = \beta\Big\}.$$

They obtained that

$$\dim_H \underline{S}(\alpha,\beta) = \sup_{\alpha\le t\le 1} \frac{H(t)}{\log 2}.$$

Later, Zhang and Peng [26] proved that

$$\dim_H E(\alpha,\beta) = \frac{H(\alpha)}{\log 2},$$

where

$$E(\alpha,\beta) = \Big\{x \in (0,1] : \lim_{n\to\infty} \frac{S_n(x)}{n} = \alpha,\ \lim_{n\to\infty} \frac{r_n(x)}{\log_2 n} = \beta\Big\}.$$

Borel's normal number theorem tells us that the rate of growth of $S_n(x)$ is $1/2$ for almost all $x \in (0,1]$, and the Erdős–Rényi limit theorem tells us that the rate of growth of $r_n(x)$ is $\log_2 n$ for almost all $x \in (0,1]$. However, the limits in (1.2) and (1.4) may not exist. Therefore, it is natural to study the exceptional sets in the above Borel normal number theorem and Erdős–Rényi limit theorem. In particular, Carbone *et al.* [3] proved that

$$\dim_H D_{\alpha,\beta} = \min\Big(\frac{H(\alpha)}{\log 2}, \frac{H(\beta)}{\log 2}\Big), \tag{1.6}$$

where

$$D_{\alpha,\beta} := \Big\{x \in (0,1] : \liminf_{n\to\infty} \frac{S_n(x)}{n} = \alpha, \limsup_{n\to\infty} \frac{S_n(x)}{n} = \beta\Big\}, \quad 0 \le \alpha \le \beta \le 1.$$

Let us remark that the above result has been generalized in diverse directions; see [13, 17, 21, 22] and references therein. Of course, there are also many works on the exceptional sets arising from the Erdős–Rényi limit theorem or its generalizations; see [10, 14, 15, 20, 24, 27] and references therein.

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In [16], we consider the intersections of the classical Besicovitch set and some kind of 'worst' exceptional sets in the Erdős–Rényi limit theorem, that is,

$$E^{\varphi}_{\alpha,\max} = \Big\{x \in (0, 1] : \lim_{n\to\infty} \frac{S_n(x)}{n} = \alpha, \liminf_{n\to\infty} \frac{r_n(x)}{\varphi(n)} = 0, \limsup_{n\to\infty} \frac{r_n(x)}{\varphi(n)} = +\infty\Big\}.$$

Here $\varphi : \mathbb{N} \to (0, +\infty)$ is a function satisfying $\lim_{n\to+\infty} \varphi(n) = +\infty$.

In this paper, we naturally consider the Hausdorff dimensions of the intersections of exceptional sets in Borel's normal number theorem and Erdős–Rényi sets. Our result complements the ones in [4, 16] and [26]. More precisely, for $0 \le \alpha \le \beta \le 1$, $0 \le \gamma \le +\infty$, define

$$\begin{aligned} C^{\gamma}_{\alpha,\beta} &:= D_{\alpha,\beta} \cap E(\gamma) \\ &= \Big\{x \in (0, 1] : \liminf_{n\to\infty} \frac{S_n(x)}{n} = \alpha, \limsup_{n\to\infty} \frac{S_n(x)}{n} = \beta, \lim_{n\to\infty} \frac{r_n(x)}{\log_2 n} = \gamma\Big\}. \end{aligned} \tag{1.7}$$

Now we can state our main result.

THEOREM 1.1. *For* $0 \le \alpha \le \beta \le 1, 0 \le \gamma \le +\infty$,

$$\dim_H C^{\gamma}_{\alpha,\beta} = \min\Big(\frac{H(\alpha)}{\log 2}, \frac{H(\beta)}{\log 2}\Big).$$

The following result follows immediately from Theorem 1.1.

COROLLARY 1.2. *For any* $0 \le \gamma \le +\infty$,

$$\dim_H C^{\gamma} = 1,$$

where

$$C^{\gamma} := \Big\{x \in (0, 1] : \liminf_{n\to\infty} \frac{S_n(x)}{n} < \limsup_{n\to\infty} \frac{S_n(x)}{n}, \lim_{n\to\infty} \frac{r_n(x)}{\log_2 n} = \gamma\Big\}.$$

We would like to emphasize again that it is very difficult to determine whether the formula (1.1) holds even if one of the original sets has full Hausdorff dimension. However, combining (1.5) and (1.6), our results show that the formula (1.1) does hold for $C^{\gamma}_{\alpha,\beta}$ and C^{γ}.

2. Preliminaries

In this section, we present some notation and classical tools which we need in the next section.

For $n \in \mathbb{N}$, let

$$\{0, 1\}^n = \{(\omega_1, \ldots, \omega_n) : \omega_i \in \{0, 1\}, i = 1, \ldots, n\}$$

and

$$\{0, 1\}^* = \bigcup_{n\in\mathbb{N}} \{0, 1\}^n.$$

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For each $\omega = (\omega_1, \ldots, \omega_n) \in \{0, 1\}^n$, the length of the word ω is n and we call ω an n-word. That is, $\{0, 1\}^n$ and $\{0, 1\}^*$ denote the families of words with length of n and all finite words, respectively.

For two words $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \{0, 1\}^n$ and $\tau = (\tau_1, \tau_2, \ldots, \tau_m) \in \{0, 1\}^m$, we denote their concatenation by $\omega\tau = (\omega_1, \ldots, \omega_n, \tau_1, \ldots, \tau_m)$, which is a word of length $n + m$. Moreover, for $W_1, \ldots, W_n \subset \{0, 1\}^*$, we write

$$W_1 \cdots W_n = \{(u_1, \ldots, u_n) : u_i \in W_i, 1 \le i \le n\}.$$

In particular, a^m means $\underbrace{a \cdots a}_{m \text{ times}}$, where $a = 0$ or 1. For convenience, Zhang and Peng [26] introduced the notion of an (N, M)-word. For an integer N satisfying $0 \le N < M$, the M-word

$$(x_1, x_2, \ldots, x_{M-1}, 0) \in \{0, 1\}^M \quad \text{with} \sum_{i=1}^{M-1} x_i = N$$

is called an (N, M)-word. The family of all (N, M)-words is denoted by $W_M(N)$.

By Stirling's formula and simple calculation, we can get the following useful estimate, which is closely related to the entropy function.

Lemma 2.1. *For any natural numbers n and k with $0 \le k \le n$, the following estimate holds:*

$$\log \binom{n}{k} = nH\Big(\frac{k}{n}\Big) + O(\log n) \quad \text{as } n \to \infty,$$

where $H(\cdot)$ is the entropy function defined as in (1.3) and the notation $f(n) = O(g(n))$ means that $f(n)/g(n)$ is bounded as $n \to \infty$.

Moran sets play an important role in fractal geometry due to their controlled constructions and nice dimensional results. To get the lower estimate for Hausdorff dimension of a set, a powerful method is to construct a Moran-type subset in it and then use the known dimension result on the Moran-type set to get the lower bound. We next present the dimension result on homogeneous Moran sets established in [11], which has become a classical tool to estimate the lower bound of the Hausdorff dimension of a fractal set.

Let us firstly recall the definition of a homogeneous Moran set. Let $\{m_i\}_{i\ge1}$ be a sequence of positive integers and $\{c_i\}_{i\ge1}$ be a sequence of positive numbers satisfying $m_i \ge 2, 0 < c_i < 1, m_1c_1 \le \delta$ and $m_ic_i \le 1$ for any $i \ge 2$, where δ is some positive number. Define

$$D = \bigcup_{i\ge0} D_i,$$

where

$$D_0 = \emptyset, \quad D_i = \{(\omega_1, \ldots, \omega_i) : 1 \le \omega_j \le m_j, 1 \le j \le i\}.$$

For $\omega = (\omega_1, \ldots, \omega_m) \in D_m$ and $\tau = (\tau_1, \ldots, \tau_n) \in D_n$, we again use $\omega\tau$ to denote the concatenation of the two words.

Suppose that $J \subset \mathbb{R}$ is a closed interval of length δ. Consider the collection of closed subintervals $\mathcal{F} = \{J_\sigma : \sigma \in D\}$ of J satisfying:

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(i) $J_\emptyset = J$;

(ii) for any $k \geq 1$ and $\sigma \in D_{k-1}, J_{\sigma*1}, J_{\sigma*2}, \ldots, J_{\sigma*m_k}$ are subintervals of J_σ and $\mathrm{int}(J_{\sigma*i}) \cap \mathrm{int}(J_{\sigma*j}) = \emptyset$ $(i \neq j)$, where $\mathrm{int}(A)$ denotes the interior of A;

(iii) for any $k \geq 1$ and any $\sigma \in D_{k-1}, 1 \leq j \leq m_k$,

$$c_k = \frac{|J_{\sigma*j}|}{|J_\sigma|},$$

where $|A|$ denotes the diameter of A.

Then we call

$$E = \bigcap_{k\geq 1} \bigcup_{\sigma\in D_k} J_\sigma$$

a homogeneous Moran set determined by $\mathcal{F}$.

Lemma 2.2. *Let E be the Moran set defined as above. Then*

$$\dim_H E \geq \liminf_{j\to\infty} \frac{\log(m_1 \cdots m_j)}{-\log(c_1 \cdots c_{j+1} m_{j+1})}.$$

Finally, we end this section with an easy inequality which will be frequently used later.

Lemma 2.3. *Let $\{a_n\}_{n\geq 1}, \{b_n\}_{n\geq 1}, \{c_n\}_{n\geq 1}, \{d_n\}_{n\geq 1}$ be four sequences of positive numbers and suppose that*

$$\lim_{n\to\infty} \frac{a_n}{c_n} = A, \quad \lim_{n\to\infty} \frac{b_n}{d_n} = B.$$

Then

$$\min(A, B) \leq \liminf_{n\to\infty} \frac{a_n + b_n}{c_n + d_n} \leq \max(A, B).$$

The above inequalities also holds if we replace limit inferior with limit superior.

3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1.

By (1.6), (1.7) and the monotonicity of Hausdorff dimension, we only need to show that

$$\dim_H C^\gamma_{\alpha,\beta} \geq \min\Big(\frac{H(\alpha)}{\log 2}, \frac{H(\beta)}{\log 2}\Big).$$

To do this, for fixed $0 \leq \alpha \leq \beta \leq 1$, $0 \leq \gamma \leq +\infty$, we next construct a homogeneous Moran set E such that

$$E \subset C^\gamma_{\alpha,\beta} \tag{3.1}$$

and

$$\dim_H E \geq \min\Big(\frac{H(\alpha)}{\log 2}, \frac{H(\beta)}{\log 2}\Big). \tag{3.2}$$

We divide the proof into three cases according to the value of γ.

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Case 1. $0 < \gamma < +\infty$.

Let $N_n = 2^{n^2}, n \geq 1$. Then it is not difficult to check that there exists some positive integer n_0 such that

$$\gamma \log_2(nN_n) \geq n \tag{3.3}$$

for $n \geq n_0$ and

$$\lim_{n\to\infty} \frac{N_1 + 2N_2 + \cdots + nN_n}{(n+1)N_{n+1}} = 0, \quad \lim_{n\to\infty} \frac{\log_2(nN_n)}{\log_2((n+1)N_{n+1})} = 1. \tag{3.4}$$

Recall that, for integers N, M with $0 \leq N < M$, $W_M(N)$ denotes the family of all (N, M)-words. For $k \geq 1$, define

$$\mathcal{W}_{2k-1} = \{\omega_1\omega_2 \cdots \omega_{N_{2k-1}-1} u_{N_{2k-1}} : u_{N_{2k-1}} = \omega_{N_{2k-1}} 1^{[\gamma \log_2((2k-1)N_{2k-1})]} 0, \\ \omega_i \in W_{2k-1}([\alpha(2k-1)]), 1 \leq i \leq N_{2k-1}\}$$

and

$$\mathcal{W}_{2k} = \{\omega_1\omega_2 \cdots \omega_{N_{2k}-1} u_{N_{2k}} : u_{N_{2k}} = \omega_{N_{2k}} 1^{[\gamma \log_2(2kN_{2k})]} 0, \\ \omega_i \in W_{2k}([\beta(2k)]), 1 \leq i \leq N_{2k}\}.$$

Here and in the following, the notation $[x]$ denotes the integer part of x.

Finally, define

$$E = \{0.v_1v_2 \cdots \in (0, 1] : v_i \in \mathcal{W}_i, \forall i \geq 1\}.$$

We next show that E is the desired homogeneous Moran set.

Write

$$A_k = \sum_{i=1}^{2k-1} (iN_i + [\gamma \log_2(iN_i)] + 1)$$

and

$$B_k = \sum_{i=1}^{2k} (iN_i + [\gamma \log_2(iN_i)] + 1).$$

That is, A_k and B_k are the lengths of the words in $\mathcal{W}_1 \cdots \mathcal{W}_{2k-1}$ and $\mathcal{W}_1 \cdots \mathcal{W}_{2k}$, respectively.

For any $n \geq N_1$, there exists some positive k such that

$$A_k \leq n < A_k + 2kN_{2k} + [\gamma \log_2(2kN_{2k})] + 1$$

or

$$B_k \leq n < B_k + (2k+1)N_{2k+1} + [\gamma \log_2((2k+1)N_{2k+1})] + 1.$$

We only consider the case $A_k \leq n < A_k + 2kN_{2k} + [\gamma \log_2(2kN_{2k})] + 1$ since the second case can be treated in the same way.

First, we show that

$$\lim_{n\to\infty} \frac{r_n(x)}{\log_2 n} = \gamma \tag{3.5}$$

for any $x \in E$.

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By the construction of E and (3.3),

$$r_n(x) = [\gamma \log_2((2k-1)N_{2k-1})] \quad \text{or} \quad [\gamma \log_2(2kN_{2k})].$$

Therefore,

$$\frac{[\gamma \log_2((2k-1)N_{2k-1})]}{\log_2(A_k + 2kN_{2k} + [\gamma \log_2(2kN_{2k})] + 1)} \le \frac{r_n(x)}{\log_2 n} \le \frac{[\gamma \log_2(2kN_{2k})]}{\log_2 A_k}.$$

By (3.4), we can check that both the first and the third terms tend to γ as $k \to \infty$. That is, (3.5) holds.

Next we show that

$$\liminf_{n\to\infty} \frac{S_n(x)}{n} = \alpha, \quad \limsup_{n\to\infty} \frac{S_n(x)}{n} = \beta \tag{3.6}$$

for any $x \in E$.

On one hand, write

$$C_k = \sum_{i=1}^{k-1} ([\alpha(2i-1)]N_{2i-1} + [\gamma \log_2((2i-1)N_{2i-1})] + [\beta(2i)]N_{2i} + [\gamma \log_2((2i)N_{2i})]).$$

It is easy to check that

$$S_{A_k}(x) = C_k + [\alpha(2k-1)]N_{2k-1} + [\gamma \log_2((2k-1)N_{2k-1})]$$

and

$$\begin{aligned} S_{B_k}(x) = C_k &+ [\alpha(2k-1)]N_{2k-1} + [\gamma \log_2((2k-1)N_{2k-1})] \\ &+ [\beta(2k)]N_{2k} + [\gamma \log_2((2k)N_{2k})]. \end{aligned}$$

It follows from (3.4) that

$$\frac{S_{A_k}(x)}{A_k} = \frac{C_k + [\alpha(2k-1)]N_{2k-1} + [\gamma \log_2((2k-1)N_{2k-1})]}{B_{k-1} + (2k-1)N_{2k-1} + [\gamma \log_2((2k-1)N_{2k-1})] + 1}$$

tends to α as $k \to \infty$. That is,

$$\lim_{k\to\infty} \frac{S_{A_k}(x)}{A_k} = \alpha. \tag{3.7}$$

In a similar way,

$$\lim_{k\to\infty} \frac{S_{B_k}(x)}{B_k} = \beta. \tag{3.8}$$

On the other hand, recall that

$$A_k \le n < A_k + 2kN_{2k} + [\gamma \log_2(2kN_{2k})] + 1.$$

We consider the following three cases.

Assume that there exists some positive integer $1 \le p < N_{2k}$ such that

$$A_k + p(2k) \le n < A_k + (p+1)(2k).$$

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Then

$$\frac{S_{A_k}(x) + p[\beta(2k)]}{A_k + (p+1)[2k]} \leq \frac{S_n(x)}{n} \leq \frac{S_{A_k}(x) + (p+1)[\beta(2k)]}{A_k + p[2k]}.$$

It follows from (3.7) and Lemma 2.3 that

$$\alpha \leq \liminf_{n\to\infty} \frac{S_n(x)}{n} \leq \limsup_{n\to\infty} \frac{S_n(x)}{n} \leq \beta.$$

Assume that

$$A_k + 2kN_{2k} < n \leq A_k + 2kN_{2k} + [\gamma \log_2(2kN_{2k})].$$

Then

$$\frac{S_{B_k}(x) - [\gamma \log_2(2kN_{2k})]}{B_k - 1} \leq \frac{S_n(x)}{n} \leq \frac{S_{B_k}(x)}{B_k - [\gamma \log_2(2kN_{2k})] - 1}.$$

It follows from (3.4) and (3.8) that

$$\lim_{n\to\infty} \frac{S_n(x)}{n} = \beta.$$

Assume that

$$A_k + 2kN_{2k} + [\gamma \log_2(2kN_{2k})] \leq n < A_k + 2kN_{2k} + [\gamma \log_2(2kN_{2k})] + 1.$$

Then

$$\frac{S_{B_k}(x)}{B_k} \leq \frac{S_n(x)}{n} \leq \frac{S_{B_k}(x)}{B_k - 1}.$$

It follows from (3.8) that

$$\lim_{n\to\infty} \frac{S_n(x)}{n} = \beta.$$

Combining (3.7), (3.8) and the above estimates, we claim that (3.6) holds and therefore (3.1) is proved.

We next prove (3.2).

It is not difficult to check that E can be regarded as a homogeneous Moran set and the associated parameters $\{m_i\}_{i\geq 1}$ and $\{c_i\}_{i\geq 1}$ are defined as follows:

$$m_i = \begin{cases} \binom{2k-1}{[\alpha(2k-1)]}, & \sum_{j=0}^{2k-2} N_j < i \leq \sum_{j=0}^{2k-1} N_j; \\ \binom{2k}{[\beta(2k)]}, & \sum_{j=0}^{2k-1} N_j < i \leq \sum_{j=0}^{2k} N_j, \end{cases} \tag{3.9}$$

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$$c_i = \begin{cases} 2^{-(2k-1)}, & \sum_{j=0}^{2k-2} N_j < i < \sum_{j=0}^{2k-1} N_j; \\ 2^{-((2k-1)+[\gamma \log_2((2k-1)N_{2k-1})]+1)}, & i = \sum_{j=0}^{2k-1} N_j; \\ 2^{-2k}, & \sum_{j=0}^{2k-1} N_j < i < \sum_{j=0}^{2k} N_j; \\ 2^{-(2k+[\gamma \log_2(2kN_{2k})]+1)}, & i = \sum_{j=0}^{2k} N_j. \end{cases} \tag{3.10}$$

Here $k \geq 1$ and we set $N_0 = 0$ for convenience.

Next we will use Lemma 2.2 to estimate the lower Hausdorff dimension of E. For any positive integer $j > N_1$, there exists a positive integer k such that either $\sum_{i=0}^{2k-1} N_i \leq j < \sum_{i=0}^{2k} N_i$ or $\sum_{i=0}^{2k} N_i \leq j < \sum_{i=0}^{2k+1} N_i$. Again, we only consider the case that $\sum_{i=0}^{2k-1} N_i \leq j < \sum_{i=0}^{2k} N_i$.

We can write $j = \sum_{i=1}^{2k-1} N_i + q$ with $0 \leq q < N_{2k}$. We consider the following two cases.

Subcase 1. $j + 1 < \sum_{i=1}^{2k} N_i$. By (3.9) and Lemma 2.1,

$$\begin{aligned} \log(m_1 \cdots m_j) &= \sum_{i=1}^{k} N_{2i-1} \log\binom{2i-1}{[\alpha(2i-1)]} + \sum_{i=1}^{k-1} N_{2i} \log\binom{2i}{[\beta(2i)]} + q \log\binom{2k}{[\beta(2k)]} \\ &= \sum_{i=1}^{k} N_{2i-1}\Big((2i-1)H\Big(\frac{[\alpha(2i-1)]}{2i-1}\Big) + O(\log(2i-1))\Big) \\ &\quad + \sum_{i=1}^{k-1} N_{2i}\Big((2i)H\Big(\frac{[\beta(2i)]}{2i}\Big) + O(\log(2i))\Big) \\ &\quad + q\Big((2k)H\Big(\frac{[\beta(2k)]}{2k}\Big) + O(\log(2k))\Big). \end{aligned}$$

On the other hand, by (3.10) and the choice of $\{N_n\}_{n\geq 1}$,

$$\begin{aligned} -\log(c_1 \cdots c_{j+1} m_{j+1}) &\leq -\log(c_1 \cdots c_{j+1}) \\ &= \Big(\sum_{i=1}^{2k-2} iN_i + (2k-1)N_{2k-1} + q(2k)\Big) \log 2 \\ &\quad + O((2k-1)^2 + \log(2k-1)). \end{aligned}$$

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By (3.4),

$$\lim_{k\to\infty}\frac{\sum_{i=1}^{k}(2i-1)N_{2i-1}H\Big(\frac{[\alpha(2i-1)]}{2i-1}\Big)+\sum_{i=1}^{k-1}(2i)N_{2i}H\Big(\frac{[\beta(2i)]}{2i}\Big)}{\sum_{i=1}^{2k-2}iN_i+(2k-1)N_{2k-1}+O((2k-1)^2+\log(2k-1))}=H(\alpha).$$

Observe that

$$\lim_{k\to\infty}\frac{q(2k)H\Big(\frac{[\beta(2k)]}{2k}\Big)}{q(2k)}=H(\beta).$$

By Lemma 2.3 and the above estimates,

$$\liminf_{j\to\infty}\frac{\log(m_1\cdots m_j)}{-\log(c_1\cdots c_{j+1}m_{j+1})}\geq\min\Big(\frac{H(\alpha)}{\log 2},\frac{H(\beta)}{\log 2}\Big).$$

Subcase 2. $j+1=\sum_{i=1}^{2k}N_i$. By simple calculation, similar to the above case,

$$\begin{aligned}\log(m_1\cdots m_j)&=\sum_{i=1}^{k}N_{2i-1}\log\Big(\frac{2i-1}{[\alpha(2i-1)]}\Big)\\&\quad+\sum_{i=1}^{k-1}N_{2i}\log\Big(\frac{2i}{[\beta(2i)]}\Big)+(N_{2k}-1)\log\Big(\frac{2i}{[\beta(2i)]}\Big)\\&=\sum_{i=1}^{k}N_{2i-1}\Big((2i-1)H\Big(\frac{[\alpha(2i-1)]}{2i-1}\Big)+O(\log(2i-1))\Big)\\&\quad+\sum_{i=1}^{k-1}N_{2i}\Big((2i)H\Big(\frac{[\beta(2i)]}{2i}\Big)+O(\log(2i))\Big)\\&\quad+(N_{2k}-1)\Big((2k)H\Big(\frac{[\beta(2k)]}{2k}\Big)+O(\log(2k))\Big).\end{aligned}$$

On the other hand,

$$\begin{aligned}-\log(c_1\cdots c_{j+1}m_{j+1})&\leq-\log(c_1\cdots c_{j+1})\\&=\Big(\sum_{i=1}^{2k-1}iN_i+(2k)(N_{2k}-1)\Big)\log 2+O((2k)^2+\log(2k)).\end{aligned}$$

Again,

$$\liminf_{j\to\infty}\frac{\log(m_1\cdots m_j)}{-\log(c_1\cdots c_{j+1}m_{j+1})}\geq\frac{H(\beta)}{\log 2}\geq\min\Big(\frac{H(\alpha)}{\log 2},\frac{H(\beta)}{\log 2}\Big).$$

Therefore, (3.2) holds.

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Case 2. $\gamma = 0$.

In this and the next case, we only give the key constructions of the desired homogeneous Moran set since the proofs are similar to that in Case 1.

Let $N_n = 2^{n^2}, n \geq 1$ and, for $k \geq 1$, let

$$\mathcal{W}_{2k-1} = \{\omega_1\omega_2 \cdots \omega_{N_{2k-1}-1}u_{N_{2k-1}} : u_{N_{2k-1}} = \omega_{N_{2k-1}}1^{2k-1}0, \\ \omega_i \in W_{2k-1}([\alpha(2k-1)]), 1 \leq i \leq N_{2k-1}\}$$

and

$$\mathcal{W}_{2k} = \{\omega_1\omega_2 \cdots \omega_{N_{2k}-1}u_{N_{2k}} : u_{N_{2k}} = \omega_{N_{2k}}1^{2k}0, \omega_i \in W_{2k}([\beta(2k)]), 1 \leq i \leq N_{2k}\}.$$

Define

$$E = \{0.v_1v_2 \cdots \in (0, 1] : v_i \in \mathcal{W}_i, \forall i \geq 1\}.$$

Case 3. $\gamma = +\infty$.

Let $N_n = 2^{n^2}, n \geq 1$ and, for $k \geq 1$, let

$$\mathcal{W}_{2k-1} = \{\omega_1\omega_2 \cdots \omega_{N_{2k-1}-1}u_{N_{2k-1}} : u_{N_{2k-1}} = \omega_{N_{2k-1}}1^{(2k-1)^3}0, \\ \omega_i \in W_{2k-1}([\alpha(2k-1)]), 1 \leq i \leq N_{2k-1}\}$$

and

$$\mathcal{W}_{2k} = \{\omega_1\omega_2 \cdots \omega_{N_{2k}-1}u_{N_{2k}} : u_{N_{2k}} = \omega_{N_{2k}}1^{(2k)^3}0,\ \omega_i \in W_{2k}([\beta(2k)]), 1 \leq i \leq N_{2k}\}.$$

Define

$$E = \{0.v_1v_2 \cdots \in (0, 1] : v_i \in \mathcal{W}_i, \forall i \geq 1\}.$$

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