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Pointwise Characterizations of Even Order Sobolev Spaces via Derivatives of Ball Averages

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Abstract. Let $\ell \in \mathbb{N}$ and $p \in (1, \infty]$. In this article, the authors establish several equivalent characterizations of Sobolev spaces $W^{2\ell+2,p}(\mathbb{R}^n)$ in terms of derivatives of ball averages. The novelty in the results of this article is that these equivalent characterizations reveal some new connections between the smoothness indices of Sobolev spaces and the derivatives on the radius of ball averages and also that, to obtain the corresponding results for higher order Sobolev spaces, the authors first establish the combinatorial equality: for any $\ell \in \mathbb{N}$ and $k \in \{0, \dots, \ell - 1\}$, $\sum_{j=0}^{2\ell}(-1)^j\binom{2\ell}{j}|\ell - j|^{2k} = 0$.

1 Introduction

It is well known that Sobolev spaces are very useful tools of analysis. The theory of Sobolev spaces on the Euclidean spaces $\mathbb{R}^n$, whose elements are differentiable functions, has been developed into a complete and mature theory in recent decades. Since the differential structures are not available on metric measure spaces, the problem of introducing Sobolev spaces on any metric measure space is one of the central topics in modern analysis. In 1996, Hajłasz [11] successfully introduced the concept of Hajłasz gradients which are used to characterize the first order Sobolev spaces in the setting of an arbitrary metric space equipped with a Borel measure. The main benefit of this characterization is that it does not need any derivatives and can be generalized to any metric measure space that might have no differential structure. This provides a new view and method for studying Sobolev spaces on any metric measure space. After the pioneering work of Hajłasz [11], Shanmugalingam [23] introduced another kind of the first order Sobolev spaces via upper and weak upper gradients. More progresses related to first order Sobolev spaces can be found in, for example, [12, 15, 16, 24].

Recently, Alabern et al. [1] established a new interesting characterization of Sobolev spaces on $\mathbb{R}^n$ via ball averages, which provides a way to introduce Sobolev spaces of any smoothness order on metric measure spaces. Inspired by [1], Sato [18–21] gave a weighted generalization of the main results in [1]. Certainly, ball averages, as a useful tool, can be used to characterize more function spaces. We refer the reader to articles [4, 8, 10, 13, 14, 22, 26–28] for some recent progress on the characterizations of Sobolev spaces in terms of ball averages. Some further characterizations of Besov

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and Triebel–Lizorkin spaces via ball averages were presented in a series of articles [3, 9, 25, 29]. Among these articles, Dai et al. [8] characterized Sobolev spaces via differences involving ball averages and studied Sobolev spaces of smoothness order 2ℓ ($\ell \in \mathbb{N}$) on spaces of homogeneous type in the sense of Coifman and Weiss in [5, 6]. In particular, when $p \in (1, \infty)$, it was proved in [8, Theorem 1.1] that a measurable function f belongs to Sobolev spaces $W^{2,p}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ and there exists a non-negative function $g \in L^p(\mathbb{R}^n)$ such that, for any $t \in (0, \infty)$ and almost every $x \in \mathbb{R}^n$,

$$\left|\frac{f(x) - B_t f(x)}{t^2}\right| \leq g(x); \tag{1.1}$$

moreover, $\|g\|_{L^p(\mathbb{R}^n)}$ is equivalent to $\|\Delta f\|_{L^p(\mathbb{R}^n)}$ with the positive equivalence constants independent of f and g, where $\Delta := \sum_{i=1}^n (\frac{\partial}{\partial x_i})^2$ denotes the Laplacian. Here and hereafter, we use the following notation: The *symbol* $L^p(\mathbb{R}^n)$, with any $p \in (1, \infty)$, denotes the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(\mathbb{R}^n)} := \left\{\int_{\mathbb{R}^n} |f(x)|^p \, dx\right\}^{1/p} < \infty,$$

and the *symbol* $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ denotes the set of all locally integrable functions on $\mathbb{R}^n$. For any $t \in (0, \infty)$, $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$B_t f(x) := \frac{1}{|B(x,t)|} \int_{B(x,t)} f(y) \, dy =: \fint_{B(x,t)} f(y) \, dy,$$

here and hereafter, $B(x,t) := \{y \in \mathbb{R}^n : |y - x| < t\}$.

Let I be the identity on $L^p(\mathbb{R}^n)$. A natural question is: In (1.1), if we replace $\frac{I - B_t}{t^2}$ by $\frac{1}{t}\frac{\partial}{\partial t}(\frac{I - B_t}{t^2})$, whether or not such an inequality can still characterizes a Sobolev space? If yes, which Sobolev spaces can be characterized?

In this article, we answer these questions affirmatively. To be precise, we establish several new equivalent characterizations for Sobolev spaces $W^{4,p}(\mathbb{R}^n)$, with $p \in (1, \infty)$, in terms of $\frac{1}{t}\frac{\partial}{\partial t}(\frac{I - B_t}{t^2})$ (see Theorem 1.1). The corresponding results for Sobolev spaces $W^{2\ell+2,p}(\mathbb{R}^n)$, with $\ell \in \mathbb{N}$ and $p \in (1, \infty)$, are also obtained (see Theorem 1.4), via first establishing a combinatorial equality (see Theorem 1.3). The novelty of this approach exists in that these equivalent characterizations reveal some new connections between the smoothness indices of Sobolev spaces and the derivatives on the radius of ball averages.

To state the main results of this article, we begin with some basic notation. In what follows, we use $C^\infty(\mathbb{R}^n)$ to denote the set of all infinitely differentiable functions and $C_c^\infty(\mathbb{R}^n)$ the set of all $C^\infty(\mathbb{R}^n)$ functions with compact supports. Let $\mathcal{S}(\mathbb{R}^n)$ denote the space of all Schwartz functions, namely, the set of all functions φ in $C^\infty(\mathbb{R}^n)$ satisfying that, for any integer $\ell \in \mathbb{Z}_+$ and multi-index $\alpha \in (\mathbb{Z}_+)^n$,

$$\|\varphi\|_{\alpha,\ell} := \sup_{|\beta| \leq |\alpha|, \, x \in \mathbb{R}^n} (1 + |x|)^\ell \left|\partial^\beta \varphi(x)\right| < \infty.$$

Here and hereafter, for any $\beta := (\beta_1, \ldots, \beta_n) \in (\mathbb{Z}_+)^n$, we let $|\beta| := \beta_1 + \cdots + \beta_n$, $\partial^\beta := (\frac{\partial}{\partial x_1})^{\beta_1} \cdots (\frac{\partial}{\partial x_n})^{\beta_n}$ and $\beta! := \beta_1! \cdots \beta_n!$. These quasi-norms $\{\|\cdot\|_{\alpha,\ell}\}_{\alpha \in (\mathbb{Z}_+)^n, \ell \in \mathbb{Z}_+}$ also determine the topology of $\mathcal{S}(\mathbb{R}^n)$. We use $\mathcal{S}'(\mathbb{R}^n)$ to denote the dual space of $\mathcal{S}(\mathbb{R}^n)$, namely, the space of all *tempered distributions* on $\mathbb{R}^n$ equipped with the weak-$*$ topology. For any $\varphi \in \mathcal{S}'(\mathbb{R}^n)$, we use $\widehat{\varphi}$ to denote its *Fourier transform*, that is, for any $\xi \in \mathbb{R}^n$, $\widehat{\varphi}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} \varphi(x) e^{-ix\xi}\, dx$. Let $\alpha \in (0,\infty)$, $p \in (1,\infty)$, and $(-\Delta)^{\alpha/2}$ be the *fractional Laplacian* defined in terms of the distributional Fourier transform via $[(-\Delta)^{\alpha/2} f]^\wedge(\xi) := |\xi|^\alpha \widehat{f}(\xi)$ for any $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$. Recall that the *Sobolev space* $W^{\alpha,p}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{W^{\alpha,p}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \|(-\Delta)^{\alpha/2} f\|_{L^p(\mathbb{R}^n)} < \infty.$$

In what follows, $t \to 0^+$ means that $t \in (0,\infty)$ and $t \to 0$.

Theorem 1.1 *Let $p \in (1,\infty)$. Then the following statements are mutually equivalent:*

(i) $f \in W^{4,p}(\mathbb{R}^n)$;

(ii) $f \in L^p(\mathbb{R}^n)$ *and there exist a set $E \subset (0,\infty)$ of Lebesgue measure* 0 *and a function $g_1 \in L^p(\mathbb{R}^n)$ such that*

$$\lim_{t \in (0,\infty)\setminus E,\, t \to 0^+} \frac{1}{t} \frac{\partial}{\partial t}\left(\frac{f - B_t f}{t^2}\right) = g_1 \quad \text{in } \mathcal{S}'(\mathbb{R}^n);$$

(iii) $f \in L^p(\mathbb{R}^n)$ *and there exists a non-negative function $g_2 \in L^p(\mathbb{R}^n)$ such that, for almost every $t \in (0,\infty)$ and $x \in \mathbb{R}^n$,*

$$\left|\frac{\partial}{\partial t}\left(\frac{f(x) - B_t f(x)}{t^2}\right)\right| \le t g_2(x);$$

(iv) $f \in L^p(\mathbb{R}^n)$ *and there exist a non-negative function $g_3 \in L^p(\mathbb{R}^n)$ and positive constants C_1 and C_2 (depending only on n) such that, for almost every $t \in (0,\infty)$ and $x \in \mathbb{R}^n$,*

$$\fint_{B(x,t)} \left|\frac{\partial}{\partial t}\left(\frac{f(y) - B_{C_1 t} f(y)}{t^2}\right)\right| dy \le t \fint_{B(x,C_2 t)} g_3(y)\, dy;$$

(v) $f \in L^p(\mathbb{R}^n)$ *and there exist a non-negative function $g_4 \in L^p(\mathbb{R}^n)$ and a positive constant C_3 (depending only on n) such that, for almost every $t \in (0,\infty)$ and $x \in \mathbb{R}^n$,*

$$\left|\fint_{B(x,t)} \frac{\partial}{\partial t}\left(\frac{f(y) - B_{C_3 t} f(y)}{t^2}\right) dy\right| \le t g_4(x). \tag{1.2}$$

Moreover, if $f \in W^{4,p}(\mathbb{R}^n)$, then, for any $i \in \{1,2,3,4\}$, the function g_i, in the above statements, can be chosen such that $\|g_i\|_{L^p(\mathbb{R}^n)}$ is equivalent to $\|\Delta^2 f\|_{L^p(\mathbb{R}^n)}$ with the positive equivalence constants depending only on n and p.

Let $q \in (1,\infty)$, $c \in (0,\infty)$ and $K \in (0,\infty]$. For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the *sharp maximal function* $f^{K,*}_{c,q}$ of f is defined by setting, for any $x \in \mathbb{R}^n$,

$$f^{K,*}_{c,q}(x) := \operatorname*{ess\,sup}_{t \in (0,K)} \frac{1}{t}\left[\fint_{B(x,t)} \left|\frac{\partial}{\partial t}\left(\frac{f(y) - B_{ct} f(y)}{t^2}\right)\right|^q dy\right]^{\frac{1}{q}}.$$

Applying Theorem 1.1, we also obtain the following equivalent characterization of $W^{4,p}(\mathbb{R}^n)$ via the above sharp maximal function.

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Corollary 1.2 *Let $p \in (1,\infty)$, $q \in (1,p)$, $c \in (0,\infty)$ and $K \in (0,\infty]$. Then $f \in W^{4,p}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ and $f^{K,*}_{c,q} \in L^p(\mathbb{R}^n)$. Moreover, if $f \in W^{4,p}(\mathbb{R}^n)$, then $\|\Delta^2 f\|_{L^p(\mathbb{R}^n)} \sim \|f^{K,*}_{c,q}\|_{L^p(\mathbb{R}^n)}$, with the positive equivalence constants depending only on c, p, q and n.*

We also consider the corresponding results for higher order Sobolev spaces. To this end, we begin with some notions. For any $\ell \in \mathbb{N}$ and $t \in (0,\infty)$, the *higher order average operator $B_{\ell,t}$* is defined by setting, for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$B_{\ell,t} f(x) := -\frac{2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell-j} B_{jt} f(x). \tag{1.3}$$

Here and hereafter, $\binom{2\ell}{\ell-j}$ for any $j \in \{0,\dots,\ell\}$ denotes the *binomial coefficients*; see also [2,7–9]. Obviously, $B_{1,t} f = B_t f$.

The following combinatorial result is a powerful tool to establish the corresponding results as in Theorem 1.1 for higher order Sobolev spaces. In what follows, we use $\vec{0}_n$ to denote the *origin* of $\mathbb{R}^n$.

Theorem 1.3 *If $\ell \in \mathbb{N}$ and $k \in \{0,\dots,\ell-1\}$, then $\sum_{j=0}^{2\ell}(-1)^j\binom{2\ell}{j}|\ell-j|^{2k} = 0$. Here and hereafter, for any $a \in \mathbb{R}$, $a^0 := 1$. Moreover, if $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then, for any $\ell \in \mathbb{N}$ and $x \in \mathbb{R}^n$,*

$$\sum_{|\alpha|=2\ell} \frac{1}{\alpha!} \partial^\alpha \varphi(x) \fint_{B(\vec{0}_n,1)} y^\alpha \, dy = \frac{a_\ell}{b_\ell} (-\Delta)^\ell \varphi(x),$$

where $\binom{2\ell}{j}$ for any $j \in \{0,\dots,2\ell\}$ denotes the binomial coefficient and

$$\begin{cases} a_\ell := \dfrac{1}{\binom{2\ell}{\ell}} \dfrac{1 \times 3 \times \cdots \times (2\ell-1)}{(n+2)(n+4)\cdots(n+2\ell)}, \\ b_\ell := \dfrac{(-1)^\ell}{\binom{2\ell}{\ell}} \left[\displaystyle\sum_{j=0}^{2\ell} (-1)^j \binom{2\ell}{j} |\ell-j|^{2\ell} \right]. \end{cases} \tag{1.4}$$

Using Theorem 1.3, we can easily extend Theorem 1.1 to high order Sobolev spaces.

Theorem 1.4 *Let $p \in (1,\infty)$ and $\ell \in \mathbb{N}$. Then the results of Theorem 1.1 remains true when $W^{4,p}(\mathbb{R}^n)$, $B_t f$ and t^2 therein are replaced by $W^{2\ell+2,p}(\mathbb{R}^n)$, $B_{\ell,t} f$ and $t^{2\ell}$, respectively.*

This article is organized as follows. The proofs of Theorem 1.1 and Corollary 1.2 are presented in Section 2. To this end, we need to use some ideas from the proof of [8, Theorem 1.1], together with an accurate estimate of $\frac{\varphi(x)-B_t\varphi(x)}{t^2}$ for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $t \in (0,\infty)$. Section 3 is devoted to proving Theorems 1.3 and 1.4. A key step of these proofs is to obtain the pointwise limit $\lim_{t\to 0^+} \frac{\varphi(x)-B_{\ell,t}\varphi(x)}{t^{2\ell}}$ for any $\ell \in \mathbb{N}$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, by using the Fourier transform and the Taylor expansion.

Remark 1.5 (i) Recall that, recently, Alabern et al. [1] used some new square functions associated with ball averages to characterize Sobolev spaces $W^{\alpha,p}(\mathbb{R}^n)$

with any given $p \in (1,\infty)$ and $\alpha \in (0,\infty)$. For example, when $\alpha \in (0,2)$, Alabern et al. [1, Theorem 3] used the following square function

$$S_\alpha(f)(\cdot) := \left[\int_0^\infty \left|\frac{f(\cdot) - B_t f(\cdot)}{t^\alpha}\right|^2 \frac{dt}{t}\right]^{1/2}, \quad \forall\, f \in L^1_{\mathrm{loc}}(\mathbb{R}^n),$$

to characterize the Sobolev space $W^{\alpha,p}(\mathbb{R}^n)$ with $p \in (1,\infty)$ and $\alpha \in (0,2)$ in the following sense: $f \in W^{\alpha,p}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ and $S_\alpha(f) \in L^p(\mathbb{R}^n)$. Its proof strongly relies on the theory of vector-valued Calderón–Zygmund operators and some subtle estimates, which is totally different from the approach used in the proofs of the above theorems. Combining both ideas, it is quite natural to ask whether or not the following square function

$$\widetilde{S}_\alpha(f) := \left[\int_0^\infty \left| t^{3-\alpha} \frac{\partial}{\partial t}\left(\frac{f(\cdot) - B_t f(\cdot)}{t^2}\right)\right|^2 \frac{dt}{t}\right]^{1/2}$$

can characterize the Sobolev space $W^{\alpha,p}(\mathbb{R}^n)$ with any given $p \in (1,\infty)$ and $\alpha \in (0,4)$. To limit its length, we will not pursue this question in this article.

(ii) We point out that the equivalence between (i) and (ii) of Theorem 1.1 (and also of Theorem 1.4) are still true if we replace the underlying Euclidean space $\mathbb{R}^n$ by any open set $\Omega \subset \mathbb{R}^n$, but the other equivalences are still unknown; see Remark 2.4 below for more details.

We end this section with some conventions on some notions and notation. For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $t \in (0,\infty)$, let $\varphi_t(\cdot) := t^{-n}\varphi(\cdot/t)$. For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the *Hardy–Littlewood maximal function* $\mathcal{M}f$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$\mathcal{M}f(x) := \sup_{B \ni x} \fint_B |f(y)|\,dy, \tag{1.5}$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ containing x. Throughout the article, we always let $\mathbb{N} := \{1,2,\dots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. The symbol C denotes a positive constant which may vary from line to line, but is independent of the main parameters. We use the symbol $f \lesssim g$ to denote that there exists a positive constant C such that $f \le Cg$. The symbol $f \sim g$ is used as an abbreviation of $f \lesssim g \lesssim f$. We also use the following convention: If $f \le Cg$ and $g = h$ or $g \le h$, we then write $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$, rather than $f \lesssim g = h$ or $f \lesssim g \le h$. For any $p \in [1,\infty]$, let p' denote the conjugate number of p, that is, $1/p + 1/p' = 1$.

2 Proofs of Theorem 1.1 and Corollary 1.2

The following lemma, which is motivated by [8, Lemma 2.1], plays an important role in our proofs.

Lemma 2.1 *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and let $\widetilde{C}$ be a positive constant. Then*

$$\lim_{t\to 0^+} \frac{1}{t}\frac{\partial}{\partial t}\left(\frac{\varphi - B_t\varphi}{t^2}\right) = -\frac{1}{4(n+2)(n+4)}\Delta^2\varphi \quad \text{in } \mathcal{S}(\mathbb{R}^n) \tag{2.1}$$

and

$$\lim_{t\to 0^+} \fint_{B(\cdot,t)} \frac{1}{t}\frac{\partial}{\partial t}\left(\frac{\varphi(y) - B_{\widetilde{C}t}\varphi(y)}{t^2}\right) dy = -\frac{\widetilde{C}^4}{4(n+2)(n+4)}\Delta^2\varphi \quad \textit{in } \mathcal{S}(\mathbb{R}^n). \tag{2.2}$$

Proof By the Taylor expansion of φ, for any given $x, y \in \mathbb{R}^n$, we have

$$\begin{aligned}\varphi(y) = \varphi(x) &+ \sum_{1\le|\alpha|\le 4} \frac{1}{\alpha!}\partial^\alpha \varphi(x)(y-x)^\alpha \\ &+ \sum_{|\alpha|=5} \frac{1}{4!}\left[\int_0^1 \partial^\alpha \varphi(x+s(y-x))(1-s)^4\, ds\right](y-x)^\alpha.\end{aligned} \tag{2.3}$$

Fixing $t \in (0,\infty)$ and $x \in \mathbb{R}^n$, and taking the average over $y \in B(x,t)$ on both sides of (2.3), we obtain

$$\begin{aligned}B_t\varphi(x) = \varphi(x) &+ \sum_{1\le|\alpha|\le 4} \frac{1}{\alpha!}\partial^\alpha \varphi(x)\fint_{B(x,t)} (y-x)^\alpha\, dy \\ &+ \sum_{|\alpha|=5} \frac{1}{4!}\int_0^1 (1-s)^4 \fint_{B(x,t)} \partial^\alpha \varphi\big(x+s(y-x)\big)(y-x)^\alpha\, dy\, ds.\end{aligned} \tag{2.4}$$

Via some trivial computations, we find that

$$\begin{aligned}\sum_{1\le|\alpha|\le 4} \frac{1}{\alpha!}\partial^\alpha \varphi(x)\fint_{B(x,t)} (y-x)^\alpha\, dy &= \frac{t^2}{2(n+2)}\Delta\varphi(x) \\ &\quad + t^4 \sum_{|\alpha|=4}\frac{1}{\alpha!}\partial^\alpha\varphi(x)\fint_{B(\vec{0}_n,1)} y^\alpha\, dy \\ &= \frac{t^2}{2(n+2)}\Delta\varphi(x) + \frac{t^4}{8(n+2)(n+4)}\Delta^2\varphi(x).\end{aligned}$$

From this and (2.4), we deduce that

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{\varphi(x) - B_t\varphi(x)}{t^2}\right) \\ &\quad = -\frac{t}{4(n+2)(n+4)}\Delta^2\varphi(x) \\ &\qquad - \frac{\partial}{\partial t}\left(\sum_{|\alpha|=5}\frac{t^3}{4!}\int_0^1 (1-s)^4 \fint_{B(\vec{0}_n,1)} \partial^\alpha\varphi(x+sty)y^\alpha\, dy\, ds\right) \\ &\quad = -\frac{t}{4(n+2)(n+4)}\Delta^2\varphi(x) \\ &\qquad - \sum_{|\alpha|=5}\frac{3t^2}{4!}\int_0^1 (1-s)^4 \fint_{B(\vec{0}_n,1)} \partial^\alpha\varphi(x+sty)y^\alpha\, dy\, ds \\ &\qquad - \sum_{|\alpha|=5}\frac{t^3}{4!}\int_0^1 (1-s)^4 \fint_{B(\vec{0}_n,1)} \sum_{|\gamma|=1}\partial^{\alpha+\gamma}\varphi(x+sty)sy^{\gamma+\alpha}\, dy\, ds.\end{aligned} \tag{2.5}$$

By Fubini's theorem and integration by parts, we conclude that

$$\begin{aligned}
&\sum_{|\alpha|=5}\frac{t^3}{4!}\int_0^1(1-s)^4 \fint_{B(\vec{0}_n,1)}\sum_{|\gamma|=1}\partial^{\alpha+\gamma}\varphi(x+sty)sy^{\gamma+\alpha}\,dy\,ds\\
&\quad=\sum_{|\alpha|=5}\frac{t^2}{4!}\fint_{B(\vec{0}_n,1)}\int_0^1\Big[\sum_{|\gamma|=1}\partial^{\alpha+\gamma}\varphi(x+sty)ty^{\gamma}\Big]s(1-s)^4\,ds\,y^{\alpha}\,dy\\
&\quad=\sum_{|\alpha|=5}\frac{t^2}{4!}\int_0^1(5s-1)(1-s)^3\fint_{B(\vec{0}_n,1)}\partial^{\alpha}\varphi(x+sty)y^{\alpha}\,dy\,ds,
\end{aligned}$$

which, together with (2.5), implies that

$$\begin{aligned}
\text{(2.6)}\qquad \frac{\partial}{\partial t}\left(\frac{\varphi(x)-B_t\varphi(x)}{t^2}\right)&=-\frac{t}{4(n+2)(n+4)}\Delta^2\varphi(x)\\
&\quad-\sum_{|\alpha|=5}\frac{t^2}{12}\int_0^1(1+s)(1-s)^3\\
&\quad\times\fint_{B(\vec{0}_n,1)}\partial^{\alpha}\varphi(x+sty)y^{\alpha}\,dy\,ds.
\end{aligned}$$

Combining this, the definition of quasi-norms $\{\|\cdot\|_{\alpha,\ell}\}_{\alpha\in(\mathbb{Z}_+)^n,\ell\in\mathbb{Z}_+}$ and the fact that, for any $x_1, x_2\in\mathbb{R}^n$, $1+|x_1|\le(1+|x_1+x_2|)(1+|x_2|)$, we know that, for any $\beta\in(\mathbb{Z}_+)^n$, $m\in\mathbb{Z}_+$, $t\in(0,\infty)$ and $x\in\mathbb{R}^n$,

$$\begin{aligned}
&\left|\partial^{\beta}\left(\frac{1}{t}\frac{\partial}{\partial t}\left(\frac{\varphi-B_t\varphi}{t^2}\right)-\left[-\frac{1}{4(n+2)(n+4)}\Delta^2\varphi\right]\right)(x)\right|(1+|x|)^m\\
&\quad=\left|\frac{1}{t}\frac{\partial}{\partial t}\left(\frac{\partial^{\beta}\varphi(x)-B_t\partial^{\beta}\varphi(x)}{t^2}\right)-\left[-\frac{1}{4(n+2)(n+4)}\Delta^2(\partial^{\beta}\varphi)\right](x)\right|(1+|x|)^m\\
&\quad\le\sum_{|\alpha|=5}\frac{t}{12}\int_0^1(1+s)(1-s)^3\fint_{B(\vec{0}_n,1)}|\partial^{\alpha+\beta}\varphi(x+sty)y^{\alpha}|\,dy\,ds(1+|x|)^m\\
&\quad\lesssim t\|\varphi\|_{5+|\beta|,m}\int_0^1(1+s)(1-s)^3\fint_{B(\vec{0}_n,1)}\frac{(1+|x|)^m}{(1+|x+sty|)^m}\,dy\,ds\\
&\quad\lesssim t(1+t)^m\|\varphi\|_{5+|\beta|,m},
\end{aligned}$$

which converges to 0 as $t\to0^+$. This proves (2.1).

Next we show (2.2). For any $\varphi\in\mathcal{S}(\mathbb{R}^n)$, by the Taylor expansion, we obtain, for any $t\in(0,\infty)$ and $x\in\mathbb{R}^n$,

$$\varphi(x)-B_t\varphi(x)=-\sum_{|\alpha|=2}t^2\int_0^1(1-s)\fint_{B(\vec{0}_n,1)}\partial^{\alpha}\varphi(x+sty)y^{\alpha}\,dy\,ds.$$

From this, (2.6), and integration by parts, we deduce that, for any $\beta \in (\mathbb{Z}_+)^n$, $m \in \mathbb{Z}_+$, $t \in (0,\infty)$, and $x \in \mathbb{R}^n$,

$$\begin{aligned}
&\Bigg|\partial^\beta\Bigg(\fint_{B(\cdot,t)} \frac{1}{t}\frac{\partial}{\partial t}\left(\frac{\varphi(y) - B_{\widetilde{C}t}\varphi(y)}{t^2}\right) dy \\
&\qquad - \left[-\frac{\widetilde{C}^4}{4(n+2)(n+4)}\Delta^2\varphi(\cdot)\right]\Bigg)(x)\Bigg|(1+|x|)^m \\
&\quad \le \frac{\widetilde{C}^4}{4(n+2)(n+4)}\left|\partial^\beta\left(\Delta^2\varphi(\cdot) - \fint_{B(\cdot,t)}\Delta^2\varphi(y)\,dy\right)(x)\right|(1+|x|)^m \\
&\qquad + \left|\partial^\beta\left(\sum_{|\alpha|=5}\frac{\widetilde{C}^5 t}{12}\fint_{B(\cdot,t)}\int_0^1 (1+s)(1-s)^3 \fint_{B(\vec{0}_n,1)} \partial^\alpha\varphi(y+\widetilde{C}stz)z^\alpha\,dz\,ds\,dy\right)(x)\right| \\
&\qquad\quad \times (1+|x|)^m \\
&\quad \lesssim \sum_{|\alpha|=2} t^2\int_0^1 (1-s)\fint_{B(\vec{0}_n,1)}\left|\partial^{\alpha+\beta}(\Delta^2\varphi)(x+sty)y^\alpha\right|dy\,ds(1+|x|)^m \\
&\qquad + t(1+|x|)^m\sum_{|\alpha|=5}\fint_{B(\vec{0}_n,1)}\int_0^1(1+s)(1-s)^3\fint_{B(\vec{0}_n,1)} \\
&\qquad\quad \times\left|\partial^{\alpha+\beta}\varphi(x+ty+\widetilde{C}stz)z^\alpha\right|dz\,ds\,dy \\
&\quad \lesssim t^2(1+t)^m\|\varphi\|_{4+|\beta|,m} + t\left[1+(1+\widetilde{C})t\right]^m\|\varphi\|_{5+|\beta|,m},
\end{aligned}$$

which converges to 0, as $t \to 0^+$. This finishes the proof of Lemma 2.1. ∎

Lemma 2.2 *Let $p \in (1,\infty)$, $1/p + 1/p' = 1$ and $f \in L^p(\mathbb{R}^n)$.*

(i) *Let $g \in L^{p'}(\mathbb{R}^n)$. Then, for almost every $t \in (0,\infty)$, $\langle \frac{\partial}{\partial t}(B_t f), g\rangle = \langle f, \frac{\partial}{\partial t}(B_t g)\rangle$.*
(ii) *Let $\phi \in C_c^\infty(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n}\phi(x)dx = 1$. Then, for almost every $t \in (0,\infty)$ and $x \in \mathbb{R}^n$,*

$$\frac{\partial}{\partial t}\big(B_t f(x)\big) = \lim_{k\to\infty}\frac{\partial}{\partial t}\big(B_t(\phi_{2^{-k}} * f)(x)\big).$$

Proof By switching to polar coordinates, we know that, for any $t \in (0,\infty)$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}
B_t f(x) &= \frac{1}{|B(x,t)|}\int_{B(0,t)} f(x+y)\,dy \\
&= \frac{1}{v_n t^n}\int_0^t\int_{\mathbb{S}^{n-1}} f(x+\rho\theta)\rho^{n-1}\,d\theta\,d\rho,
\end{aligned} \tag{2.7}$$

where v_n denotes the volume of the unit ball and $\mathbb{S}^{n-1}$ the unit sphere. For any $x \in \mathbb{R}^n$ and $\rho \in (0,\infty)$, let $F(x,\rho) := \rho^{n-1}\int_{\mathbb{S}^{n-1}} f(x+\rho\theta)\,d\theta$. Now, we claim that, for almost every $t \in (0,\infty)$ and any $x \in \mathbb{R}^n$,

$$\frac{\partial}{\partial t}\int_0^t\int_{\mathbb{S}^{n-1}} f(x+\rho\theta)\rho^{n-1}\,d\theta\,d\rho = F(x,t). \tag{2.8}$$

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Indeed, by the Hölder inequality, we obtain, for any $t_1, t_2 \in (0,\infty)$ with $t_1 < t_2$ and $x \in \mathbb{R}^n$,

$$\int_{t_1}^{t_2} \left|F(x,\rho)\right| d\rho \le \int_{\{y\in\mathbb{R}^n: t_1\le|y|\le t_2\}} \left|f(x+y)\right| dy$$
$$\le \|f\|_{L^p(\mathbb{R}^n)} \left|\{y\in\mathbb{R}^n:\ t_1\le|y|\le t_2\}\right|^{1/p'} < \infty,$$

which implies that, for any $x \in \mathbb{R}^n$, $F(x,\cdot) \in L^1_{\mathrm{loc}}(0,\infty)$ and (2.8) holds true. This proves the above claim. From this claim and (2.7), we deduce that, for almost every $t \in (0,\infty)$ and any $x \in \mathbb{R}^n$,

$$\frac{\partial}{\partial t}\big(B_t f(x)\big) = -\frac{n}{v_n t^{n+1}} \int_{B(0,t)} f(x+y)\,dy + \frac{1}{v_n t} \int_{\mathbb{S}^{n-1}} f(x+t\theta)\,d\theta. \tag{2.9}$$

Repeating the above steps, we also obtain, for almost every $t \in (0,\infty)$ and any $x \in \mathbb{R}^n$,

$$\frac{\partial}{\partial t}\big(B_t g(x)\big) = -\frac{n}{v_n t^{n+1}} \int_{B(0,t)} g(x+y)\,dy + \frac{1}{v_n t} \int_{\mathbb{S}^{n-1}} g(x+t\theta)\,d\theta.$$

From this, (2.9), and Fubini's theorem, it follows that, for almost every $t \in (0,\infty)$,

$$\begin{aligned}
\left\langle \frac{\partial}{\partial t}(B_t f),\, g\right\rangle &= \int_{\mathbb{R}^n} \left[-\frac{n}{v_n t^{n+1}} \int_{B(x,t)} f(y)\,dy + \frac{1}{v_n t}\int_{\mathbb{S}^{n-1}} f(x+t\theta)\,d\theta\right] g(x)\,dx \\
&= -\frac{n}{v_n t^{n+1}} \int_{\mathbb{R}^n} f(y) \int_{B(y,t)} g(x)\,dx\,dy \\
&\quad + \frac{1}{v_n t} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}^n} f(x+t\theta) g(x)\,dx\,d\theta \\
&= \left\langle f,\, \frac{\partial}{\partial t}(B_t g)\right\rangle.
\end{aligned}$$

This proves (i).

To show (ii), we first observe that by (2.9), the Minkowski inequality, the Hölder inequality, and Fubini's theorem, for almost every $t \in (0,\infty)$,

$$\begin{aligned}
&\left\|\frac{\partial}{\partial t}(B_t f)\right\|_{L^p(\mathbb{R}^n)} \\
&\lesssim \left\{\int_{\mathbb{R}^n} \left[\int_{B(0,t)} |f(x+y)|\,dy\right]^p dx\right\}^{1/p} + \left\{\int_{\mathbb{R}^n} \left[\int_{\mathbb{S}^{n-1}} |f(x+t\theta)|\,d\theta\right]^p dx\right\}^{1/p} \\
&\lesssim \left[\int_{\mathbb{R}^n} \int_{B(0,t)} |f(x+y)|^p\,dy\,dx\right]^{1/p} + \left[\int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} |f(x+t\theta)|^p\,d\theta\,dx\right]^{1/p} \\
&\lesssim \|f\|_{L^p(\mathbb{R}^n)} < \infty.
\end{aligned}$$

That is, $\frac{\partial}{\partial t}(B_t f) \in L^p(\mathbb{R}^n)$ for almost every $t \in (0,\infty)$. From this and the fact that $\{\phi_{2^{-k}}\}_{k\in\mathbb{N}}$ is an approximation to the identity, it follows that, for almost every $t \in (0,\infty)$ and $x \in \mathbb{R}^n$,

$$\lim_{k\to\infty} \phi_{2^{-k}} * \left(\frac{\partial}{\partial t}(B_t f)\right)(x) = \frac{\partial}{\partial t}\big(B_t f(x)\big).$$

Combining this, (2.9), and Fubini's theorem, we know that, for almost every $t \in (0,\infty)$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}
&\frac{\partial}{\partial t}\big(B_t f(x)\big)\\
&= \lim_{k\to\infty}\int_{\mathbb{R}^n}\phi_{2^{-k}}(x-y)\frac{\partial}{\partial t}\big(B_t f(y)\big)\,dy\\
&= \lim_{k\to\infty}\int_{\mathbb{R}^n}\phi_{2^{-k}}(x-y)\left[-\frac{n}{v_n t^{n+1}}\int_{B(0,t)} f(y+z)\,dz + \frac{1}{v_n t}\int_{\mathbb{S}^{n-1}} f(y+t\theta)\,d\theta\right]dy\\
&= \lim_{k\to\infty}\left[-\frac{n}{v_n t^{n+1}}\int_{B(0,t)}\phi_{2^{-k}} * f(x+z)\,dz + \frac{1}{v_n t}\int_{\mathbb{S}^{n-1}}\phi_{2^{-k}} * f(x+t\theta)\,d\theta\right].
\end{aligned}$$

From this and (2.9) with f replaced by $\phi_{2^{-k}} * f$, we deduce that, for almost every $t \in (0,\infty)$ and $x \in \mathbb{R}^n$,

$$\frac{\partial}{\partial t}\big(B_t f(x)\big) = \lim_{k\to\infty}\frac{\partial}{\partial t}\big(B_t(\phi_{2^{-k}} * f)(x)\big).$$

This finishes the proof of Lemma 2.2. ∎

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 We first show (i) $\Rightarrow$ (ii). Let $f \in W^{4,p}(\mathbb{R}^n)$ and

$$g_1 := \frac{-1}{4(n+2)(n+4)}\Delta^2 f.$$

For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, from Lemmas 2.1 and 2.2, it follows that there exists a set $E \subset (0,\infty)$ of measure zero such that

$$\begin{aligned}
\lim_{t\in(0,\infty)\setminus E,\, t\to 0^+}\left\langle \frac{1}{t}\frac{\partial}{\partial t}\left(\frac{f-B_t f}{t^2}\right),\varphi\right\rangle &= \lim_{t\to 0^+}\left\langle f,\ \frac{1}{t}\frac{\partial}{\partial t}\left(\frac{\varphi - B_t\varphi}{t^2}\right)\right\rangle\\
&= \left\langle f,\ \frac{-1}{4(n+2)(n+4)}\Delta^2\varphi\right\rangle = \langle g_1,\ \varphi\rangle.
\end{aligned}$$

This proves (ii).

We now show (ii) $\Rightarrow$ (iii). Assume that (ii) is satisfied. By Lemmas 2.1 and 2.2, we know that there exists a set $E \subset (0,\infty)$ of measure zero such that, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}
\langle \Delta^2 f,\varphi\rangle = \langle f,\Delta^2\varphi\rangle &= \lim_{t\to 0+}\left\langle f, -\frac{1}{4(n+2)(n+4)t}\frac{\partial}{\partial t}\left(\frac{\varphi - B_t\varphi}{t^2}\right)\right\rangle\\
&= \lim_{t\in(0,\infty)\setminus E,\, t\to 0+}\left\langle -\frac{1}{4(n+2)(n+4)t}\frac{\partial}{\partial t}\left(\frac{f-B_t f}{t^2}\right),\varphi\right\rangle.
\end{aligned}$$

Combining this and (ii), we conclude that $\Delta^2 f = -\frac{1}{4(n+2)(n+4)} g_1 \in L^p(\mathbb{R}^n)$ in $\mathcal{S}'(\mathbb{R}^n)$. Since $f \in L^p(\mathbb{R}^n)$, it follows that $f \in W^{4,p}(\mathbb{R}^n)$.

We now claim that, for any $f \in C^\infty(\mathbb{R}^n)\cap W^{4,p}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, and $t \in (0,\infty)$,

$$\left|\frac{1}{t}\frac{\partial}{\partial t}\left(\frac{f(x)-B_t f(x)}{t^2}\right)\right| \le \frac{5}{72}\mathcal{M}\left(\sum_{|\alpha|=4}|\partial^\alpha f|\right)(x), \tag{2.10}$$

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where $\mathcal{M}$ denotes the Hardy–Littlewood maximal function as in (1.5). Indeed, by the Taylor expansion of f of order 4 and integration by parts, we obtain, for any $t \in (0,\infty)$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}
&\left|\frac{\partial}{\partial t}\left(\frac{f(x)-B_t f(x)}{t^2}\right)\right| \\
&\quad = \left|\frac{\partial}{\partial t}\left(\sum_{|\alpha|=4}\frac{t^2}{3!}\int_0^1 (1-s)^3 \fint_{B(\vec{0}_n,1)} \partial^\alpha f(x+sty) y^\alpha \, dy\, ds\right)\right| \\
&\quad = \left|\sum_{|\alpha|=4}\frac{t}{6}\int_0^1 (1+s)(1-s)^2 \fint_{B(\vec{0}_n,1)} \partial^\alpha f(x+sty) y^\alpha \, dy\, ds\right| \\
&\quad \le \frac{t}{6}\int_0^1 (1+s)(1-s)^2 \fint_{B(x,st)} \left[\sum_{|\alpha|=4} |\partial^\alpha f(y)|\right] dy\, ds \le \frac{5t}{72}\mathcal{M}\left(\sum_{|\alpha|=4}|\partial^\alpha f|\right)(x),
\end{aligned}$$

which proves (2.10).

For more general $f \in W^{4,p}(\mathbb{R}^n)$, we argue by an approximation method. To this end, let $\phi \in C_c^\infty(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \phi(x)dx = 1$. Then $\phi_{2^{-k}} * f \in C^\infty(\mathbb{R}^n) \cap W^{4,p}(\mathbb{R}^n)$ for any $k \in \mathbb{N}$. Since $\{\phi_{2^{-k}}\}_{k\in\mathbb{N}}$ is an approximation to the identity, it then follows that, for almost every $x \in \mathbb{R}^n$, $\lim_{k\to\infty}(\phi_{2^{-k}} * f)(x) = f(x)$. By Lemma 2.2 and (2.10), we know that, for almost every $t \in (0,\infty)$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}
&\left|\frac{1}{t}\frac{\partial}{\partial t}\left(\frac{f(x)-B_t f(x)}{t^2}\right)\right| \\
&\quad = \lim_{k\to\infty}\left|\frac{1}{t}\frac{\partial}{\partial t}\left(\frac{(\phi_{2^{-k}} * f)(x)-B_t(\phi_{2^{-k}} * f)(x)}{t^2}\right)\right| \\
&\quad \le \frac{5}{72}\mathcal{M}\left(\sum_{|\alpha|=4}\sup_{k\in\mathbb{N}}|\phi_{2^{-k}} * \partial^\alpha f|\right)(x) \le \frac{5}{72}\mathcal{M}\left(\sum_{|\alpha|=4}\mathcal{M}(|\partial^\alpha f|)\right)(x),
\end{aligned}$$

where $\mathcal{M}$ denotes the Hardy–Littlewood maximal function as in (1.5). Now, letting

$$g_2 := \frac{5}{72}\mathcal{M}\left(\sum_{|\alpha|=4}\mathcal{M}(|\partial^\alpha f|)\right),$$

we then deduce from the boundedness of $\mathcal{M}$ on $L^p(\mathbb{R}^n)$, with $p \in (1,\infty)$, that

$$\|g_2\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{|\alpha|=4}\|\partial^\alpha f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{W^{4,p}(\mathbb{R}^n)},$$

which proves (iii).

The proof of (iii) $\Rightarrow$ (iv) is trivial.

Observe that (v) follows directly from (iv) with $C_3 := C_1$ and $g_4 := \mathcal{M}g_3$, where $\mathcal{M}$ denotes the Hardy–Littlewood maximal function as in (1.5).

We now show (v) $\Rightarrow$ (i). To show (i), for any $f \in L^p(\mathbb{R}^n)$, $t \in (0,\infty)$ and $x \in \mathbb{R}^n$, let

$$G(x,t) := \fint_{B(x,t)} \frac{1}{t}\frac{\partial}{\partial t}\left(\frac{f(y)-B_{C_3 t}f(y)}{t^2}\right) dy.$$

Then, by (1.2), we know that there exists a set $E \subset (0,\infty)$ of measure zero such that

$$\sup_{t\in(0,\infty)\setminus E} \left\| G(\cdot,t)\right\|_{L^p(\mathbb{R}^n)} \le \|g_4\|_{L^p(\mathbb{R}^n)} < \infty.$$

From this and the Banach-Alaoglu theorem (see [17, p. 70, Theorem 3.17]), it follows that there exist a sequence $\{t_j\}_{j=1}^{\infty}$ of positive integers converging to zero and a function $h \in L^p(\mathbb{R}^n)$ such that $\|h\|_{L^p(\mathbb{R}^n)} \le \|g_4\|_{L^p(\mathbb{R}^n)}$ and, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\lim_{j\to\infty} \left\langle G(\cdot,t_j),\, \varphi\right\rangle = \langle h, \varphi\rangle. \tag{2.11}$$

On another hand, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, by Fubini's theorem, we know that, for almost every $t \in (0,\infty)$,

$$\begin{aligned}
\left\langle G(\cdot,\, t),\, \varphi\right\rangle &= \frac{1}{t}\int_{\mathbb{R}^n} \varphi(x) \fint_{B(\vec{0}_n,t)} \frac{\partial}{\partial t}\left(\frac{f(x+y) - B_{C_3 t} f(x+y)}{t^2}\right) dy\, dx\\
&= \frac{1}{t}\fint_{B(\vec{0}_n,t)} \int_{\mathbb{R}^n} \frac{\partial}{\partial t}\left(\frac{f(x+y) - B_{C_3 t} f(x+y)}{t^2}\right) \varphi(x)\, dx\, dy\\
&= \left\langle f, \fint_{B(\cdot,t)} \frac{1}{t}\frac{\partial}{\partial t}\left(\frac{\varphi(y) - B_{C_3 t}\varphi(y)}{t^2}\right) dy\right\rangle.
\end{aligned}$$

Combining this, Lemma 2.1 and (2.11), we conclude that

$$\langle h, \varphi\rangle = \left\langle f,\, -\frac{C_3^4}{4(n+2)(n+4)}\Delta^2\varphi\right\rangle = \left\langle -\frac{C_3^4}{4(n+2)(n+4)}\Delta^2 f,\, \varphi\right\rangle.$$

This implies that $\Delta^2 f = -\frac{4(n+2)(n+4)}{C_3^4} h \in L^p(\mathbb{R}^n)$ and hence $f \in W^{4,p}(\mathbb{R}^n)$.

Finally, we deduce from above proof that all the functions $\{g_i\}_{i=1}^4$ can be chosen so that, for any $i \in \{1,2,3,4\}$, $\|g_i\|_{L^p(\mathbb{R}^n)} \sim \|\Delta^2 f\|_{L^p(\mathbb{R}^n)}$ with the positive equivalence constants independent of f. This finishes the proof of Theorem 1.1. ∎

Remark 2.3 By the proof of (v) ⇒ (i) of Theorem 1.1, we find that the fact that (1.2) holding true for almost every t in a small right neighborhood of zero is enough to guarantee that Theorem 1.1(i) holds true. Thus, for any $K \in (0,\infty]$, if we replace $t \in (0,\infty)$ by $t \in (0,K)$ in (iii), (iv), and (v) of Theorem 1.1, these conclusions still hold true.

Remark 2.4 Observe that in the proof of (ii) ⇒ (iii) of Theorem 1.1, we actually first prove (ii) ⇒ (i). Thus, the equivalence between (i) and (ii) of Theorem 1.1 can be proved independently of the proofs of (iii), (iv) and (v) of Theorem 1.1. Moreover, it is easy to show that this equivalence is also true if we replace the underlying Euclidean space $\mathbb{R}^n$ by an open set Ω of $\mathbb{R}^n$, because in (ii) we are only concerned with the limiting behavior when $t \to 0^+$ and $B(x,t) \subset \Omega$ if $x \in \Omega$ and $t \in (0,\infty)$ is small enough. It is still unclear whether or not Theorem 1.1(i) is equivalent, respectively, to (iii), (iv), or (v) of Theorem 1.1 when the underlying space is an open set of $\mathbb{R}^n$; the main difficulty is that the present proofs for other equivalences strongly depend on the symmetry of the balls,which is not available for any ball of any open set of $\mathbb{R}^n$.

We also obtain the following corollary; its proof is similar to that of Theorem 1.1, the details being omitted.

Corollary 2.5 *For any $t \in (0,\infty)$, $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let $H_f(x,t) := \frac{1}{t}\frac{\partial}{\partial t}$ $(\frac{f(x)-B_t f(x)}{t^2})$. Let $p \in (1,\infty)$. Then the following statements are mutually equivalent:*

(i) $f \in W^{4,p}(\mathbb{R}^n)$.

(ii) *$f \in L^p(\mathbb{R}^n)$ and there exist a set $E \subset (0,\infty)$ of measure 0 and $g \in L^p(\mathbb{R}^n)$ such that*

$$\liminf_{t\in(0,\infty)\setminus E,\, t\to 0^+} H_f(\cdot,t) = g \quad \text{in } \mathcal{S}'(\mathbb{R}^n);$$

(iii) *$f \in L^p(\mathbb{R}^n)$ and there exist $g \in L^p(\mathbb{R}^n)$ and a sequence $\{t_k\}_{k\in\mathbb{N}}$ of positive numbers such that $\lim_{k\to\infty} t_k = 0$ and*

$$\lim_{k\to\infty} H_f(\cdot,t_k) = g \quad \text{in } \mathcal{S}'(\mathbb{R}^n);$$

(iv) *$f \in L^p(\mathbb{R}^n)$ and there exists a set $E \subset (0,\infty)$ of measure 0 such that*

$$\sup_{t\in(0,\infty)\setminus E} \left\| H_f(\cdot,t)\right\|_{L^p(\mathbb{R}^n)} =: C_4 < \infty.$$

In (ii) *and* (iii), *the function g can be chosen such that $\|g\|_{L^p(\mathbb{R}^n)}$ is equivalent to $\|\Delta^2 f\|_{L^p(\mathbb{R}^n)}$, with the positive equivalence constants independent of f. This also holds true for C_4 in* (iv).

Proof of Corollary 1.2 If $f \in W^{4,p}(\mathbb{R}^n)$, then, by Theorem 1.1(ii), we know that there exists a function $g \in L^p(\mathbb{R}^n)$ such that $\|g\|_{L^p(\mathbb{R}^n)} \lesssim \|\Delta^2 f\|_{L^p(\mathbb{R}^n)}$ and, for almost every $t \in (0,\infty)$ and $x \in \mathbb{R}^n$,

$$\left|\frac{\partial}{\partial t}\left(\frac{f(x)-B_t f(x)}{t^2}\right)\right| \le t g(x).$$

From this, it follows that, for almost every $x \in \mathbb{R}^n$,

$$f^{K,*}_{c,q}(x) \lesssim \operatorname*{ess\,sup}_{t\in(0,K)} \left[\fint_{B(x,t)} |g(y)|^q\,dy\right]^{\frac{1}{q}} \lesssim \left[\mathcal{M}\left(|g|^q\right)(x)\right]^{\frac{1}{q}},$$

where $\mathcal{M}$ denotes the Hardy–Littlewood maximal function as in (1.5). This, combined with the boundedness of the maximal operator $\mathcal{M}$ on $L^{p/q}(\mathbb{R}^n)$ with $q \in [1,p)$, further implies that

$$\|f^{K,*}_{c,q}\|_{L^p(\mathbb{R}^n)} \lesssim \left\|\left[\mathcal{M}(|g|^q)\right]^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^n)} \sim \|\mathcal{M}(|g|^q)\|^{\frac{1}{q}}_{L^{p/q}(\mathbb{R}^n)} \lesssim \|g\|_{L^p(\mathbb{R}^n)} \lesssim \|\Delta^2 f\|_{L^p(\mathbb{R}^n)}.$$

Conversely, suppose that $f^{K,*}_{c,q} \in L^p(\mathbb{R}^n)$. Then, by the definition of $f^{K,*}_{c,q}$ and the Hölder inequality, we conclude that, for almost every $t \in (0,K)$ and $x \in \mathbb{R}^n$,

$$\fint_{B(x,t)} \frac{\partial}{\partial t}\left(\frac{f(y)-B_t f(y)}{t^2}\right) dy \le t f^{K,*}_{c,q}(x).$$

From this, Theorem 1.1 and Remark 2.3, we deduce that $f \in W^{4,p}(\mathbb{R}^n)$ and

$$\|f\|_{W^{4,p}(\mathbb{R}^n)} \lesssim \|f_{c,q}^{K,*}\|_{L^p(\mathbb{R}^n)} < \infty.$$

This finishes the proof of Corollary 1.2. ∎

3 Proofs of Theorems 1.3 and 1.4

In this section, we characterize the higher order Sobolev spaces $W^{2\ell+2,p}(\mathbb{R}^n)$, with $\ell \in \mathbb{N}$ and $p \in (1,\infty)$, by means of $B_{\ell,t}$ as in (1.3). To this end, we need the following technical lemma, which is from [9, Lemma 2.1].

Lemma 3.1 *For any $\ell \in \mathbb{N}$, $t \in (0,\infty)$, $f \in \mathcal{S}(\mathbb{R}^n)$, and $\xi \in \mathbb{R}^n$, it holds true that*

$$(B_{\ell,t}f)^\wedge(\xi) = \left[1 - A_\ell(t|\xi|)\right]\widehat{f}(\xi),$$

where

$$A_\ell(s) := \gamma_n \frac{4^\ell}{\binom{2\ell}{\ell}} \int_0^1 (1-u^2)^{\frac{n-1}{2}} \left(\sin\frac{us}{2}\right)^{2\ell} du, \quad \forall\, s \in \mathbb{R},$$

with $\gamma_n := [\int_0^1 (1-u^2)^{\frac{n-1}{2}}\, du]^{-1}$ and $\binom{2\ell}{\ell}$ being the binomial coefficients.

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3 Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. From Lemma 3.1 and the fact that, for any $f \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, $f(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \widehat{f}(\xi)e^{ix\xi}\,d\xi$, we deduce that, for any $\ell \in \mathbb{N}$, $t \in (0,\infty)$, and $x \in \mathbb{R}^n$,

$$\varphi(x) - B_{\ell,t}\varphi(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} A_\ell(t|\xi|)\widehat{\varphi}(\xi)e^{ix\xi}\,d\xi.$$

Combining this with the fact that, for any $\ell \in \mathbb{N}$ and $\xi \in \mathbb{R}^n$, $[(-\Delta)^\ell\varphi]^\wedge(\xi) = |\xi|^{2\ell}\widehat{\varphi}(\xi)$, we obtain, for any $\ell \in \mathbb{N}$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}
(3.1)\qquad &\lim_{t\to 0^+} \frac{\varphi(x) - B_{\ell,t}\varphi(x)}{t^{2\ell}} \\
&= (2\pi)^{-n/2}\int_{\mathbb{R}^n} \lim_{t\to 0^+} \frac{A_\ell(t|\xi|)}{t^{2\ell}}\widehat{\varphi}(\xi)e^{ix\xi}\,d\xi \\
&= \gamma_n \frac{4^\ell}{\binom{2\ell}{\ell}} \int_0^1 (1-u^2)^{\frac{n-1}{2}} \left(\frac{u}{2}\right)^{2\ell} du (2\pi)^{-n/2}\int_{\mathbb{R}^n} \widehat{\varphi}(\xi)|\xi|^{2\ell}e^{ix\xi}\,d\xi \\
&= a_\ell(-\Delta)^\ell\varphi(x),
\end{aligned}$$

where a_ℓ is as in (1.4).

On another hand, via a straightforward calculation, we conclude that, for any $\ell \in \mathbb{N}$ and $t \in (0,\infty)$,

$$(3.2)\qquad \varphi(x) - B_{\ell,t}\varphi(x) = \frac{(-1)^\ell}{\binom{2\ell}{\ell}} \sum_{j=0}^{2\ell} (-1)^j \binom{2\ell}{j} B_{|\ell-j|t}\varphi(x), \quad \forall\, x \in \mathbb{R}^n,$$

where $B_0\varphi(x) := \varphi(x)$. Let $E_\ell := \{\alpha \in (\mathbb{Z}_+)^n : |\alpha| = 2, 4, \dots, 2\ell\}$. From the Taylor expansion of φ, it follows that for any $\ell \in \mathbb{N}$, $j \in \{0, 1, \dots, 2\ell\}$, $t \in (0, \infty)$, and $x \in \mathbb{R}^n$,

$$B_{|\ell-j|t}\varphi(x) = \varphi(x) + \sum_{\alpha\in E_\ell} (|\ell-j|t)^{|\alpha|}\frac{1}{\alpha!}\partial^\alpha\varphi(x)\fint_{B(\vec{0}_n,1)} y^\alpha\,dy$$
$$+ \sum_{|\alpha|=2\ell+1} \frac{(|\ell-j|t)^{2\ell+1}}{(2\ell)!}\int_0^1 (1-s)^{2\ell}\fint_{B(\vec{0}_n,1)} \partial^\alpha\varphi(x+sty)y^\alpha\,dy\,ds,$$

which, together with (3.2) and the fact that $\sum_{j=0}^{2\ell}(-1)^j\binom{2\ell}{j} = 0$ further implies that

$$\begin{aligned}(3.3)\qquad &\varphi(x) - B_{\ell,t}\varphi(x)\\ &= \frac{(-1)^\ell}{\binom{2\ell}{\ell}}\sum_{j=0}^{2\ell}(-1)^j\binom{2\ell}{j}\Big[\sum_{\alpha\in E_\ell}(|\ell-j|t)^{|\alpha|}\frac{1}{\alpha!}\partial^\alpha\varphi(x)\fint_{B(\vec{0}_n,1)} y^\alpha\,dy\Big]\\ &\quad + \frac{t^{2\ell+1}}{(2\ell)!}\frac{(-1)^\ell}{\binom{2\ell}{\ell}}\Big[\sum_{j=0}^{2\ell}(-1)^j\binom{2\ell}{j}|\ell-j|^{2\ell+1}\Big]\\ &\quad\times\int_0^1(1-s)^{2\ell}\fint_{B(\vec{0}_n,1)}\Big[\sum_{|\alpha|=2\ell+1}\partial^\alpha\varphi(x+sty)y^\alpha\Big]\,dy\,ds\\ &=: \mathrm{I}_1(x,t) + \mathrm{I}_2(x,t).\end{aligned}$$

Now we estimate I_1. Via a straightforward calculation, we find that, for any $\ell \in \mathbb{N}$, $t \in (0, \infty)$, and $x \in \mathbb{R}^n$,

$$\begin{aligned}(3.4)\quad \mathrm{I}_1(x,t) &= \frac{(-1)^\ell}{\binom{2\ell}{\ell}}\sum_{\alpha\in E_\ell} t^{|\alpha|}\frac{\partial^\alpha\varphi(x)}{\alpha!}\fint_{B(\vec{0}_n,1)} y^\alpha\,dy\Big[\sum_{j=0}^{2\ell}(-1)^j\binom{2\ell}{j}|\ell-j|^{|\alpha|}\Big]\\ &= \frac{(-1)^\ell}{\binom{2\ell}{\ell}}\sum_{k=1}^{\ell} t^{2k}\Big[\sum_{j=0}^{2\ell}(-1)^j\binom{2\ell}{j}|\ell-j|^{2k}\Big]\Big[\sum_{|\alpha|=2k}\frac{\partial^\alpha\varphi(x)}{\alpha!}\fint_{B(\vec{0}_n,1)} y^\alpha\,dy\Big].\end{aligned}$$

Since $\varphi \in \mathcal{S}(\mathbb{R}^n)$, it follows that, for any ℓ, $m \in \mathbb{N}$, $t \in (0, \infty)$ and $x \in \mathbb{R}^n$, $|\mathrm{I}_2(x,t)| \lesssim t^{2\ell+1}\|\varphi\|_{2\ell+1,m}$. By this, we further conclude that, for any $\ell \in \mathbb{N}$ and $x \in \mathbb{R}^n$, $\lim_{t\to 0^+}\frac{\mathrm{I}_2(x,t)}{t^{2\ell}} = 0$. Combining this, (3.3) and (3.4), we know that, for any $\ell \in \mathbb{N}$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}&\lim_{t\to 0^+}\frac{\varphi(x) - B_{\ell,t}\varphi(x)}{t^{2\ell}}\\ &\quad= \lim_{t\to 0^+}\frac{(-1)^\ell}{\binom{2\ell}{\ell}}\sum_{k=1}^{\ell-1}\frac{1}{t^{2\ell-2k}}\Big[\sum_{j=0}^{2\ell}(-1)^j\binom{2\ell}{j}|\ell-j|^{2k}\Big]\Big[\sum_{|\alpha|=2k}\frac{\partial^\alpha\varphi(x)}{\alpha!}\fint_{B(\vec{0}_n,1)} y^\alpha\,dy\Big]\\ &\quad\quad+ \frac{(-1)^\ell}{\binom{2\ell}{\ell}}\Big[\sum_{j=0}^{2\ell}(-1)^j\binom{2\ell}{j}|\ell-j|^{2\ell}\Big]\Big[\sum_{|\alpha|=2\ell}\frac{\partial^\alpha\varphi(x)}{\alpha!}\fint_{B(\vec{0}_n,1)} y^\alpha\,dy\Big],\end{aligned}$$

which, together with (3.1) and the arbitrariness of $\varphi \in \mathcal{S}(\mathbb{R}^n)$, further implies that, for any $\ell \in \mathbb{N}$, $k \in \{1, \dots, \ell-1\}$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, $\sum_{j=0}^{2\ell}(-1)^j\binom{2\ell}{j}|\ell-j|^{2k} = 0$ and

$$a_\ell(-\Delta)^\ell\varphi(x) = \frac{(-1)^\ell}{\binom{2\ell}{\ell}}\left[\sum_{j=0}^{2\ell}(-1)^j\binom{2\ell}{j}|\ell-j|^{2\ell}\right]\left[\sum_{|\alpha|=2\ell}\frac{\partial^\alpha\varphi(x)}{\alpha!}\fint_{B(\vec{0}_n,1)} y^\alpha\,dy\right],$$

where a_ℓ is as in (1.4). This finishes the proof of Theorem 1.3. ∎

The crucial tools used to establish the characterizations of higher order Sobolev spaces is the following lemma.

Lemma 3.2 *Let $\ell\in\mathbb{N}$, $\varphi\in\mathcal{S}(\mathbb{R}^n)$, $x\in\mathbb{R}^n$ and $\widetilde{C}$ be a given positive constant. Then*

$$\lim_{t\to 0^+}\frac{1}{t}\frac{\partial}{\partial t}\left(\frac{\varphi-B_{\ell,t}\varphi}{t^{2\ell}}\right) = 2\frac{(-1)^\ell}{\binom{2\ell}{\ell}}\left[\sum_{j=0}^{2\ell}(-1)^j\binom{2\ell}{j}|\ell-j|^{2\ell+2}\right]\frac{a_{\ell+1}}{b_{\ell+1}}(-\Delta)^{\ell+1}\varphi \quad \text{in } \mathcal{S}(\mathbb{R}^n) \tag{3.5}$$

and

$$\lim_{t\to 0^+}\fint_{B(\cdot,t)}\frac{1}{t}\frac{\partial}{\partial t}\left(\frac{\varphi(y)-B_{\ell,\widetilde{C}t}\varphi(y)}{t^{2\ell}}\right)dy = 2\widetilde{C}^{2\ell+2}\frac{(-1)^\ell}{\binom{2\ell}{\ell}}\left[\sum_{j=0}^{2\ell}(-1)^j\binom{2\ell}{j}|\ell-j|^{2\ell+2}\right]\frac{a_{\ell+1}}{b_{\ell+1}}(-\Delta)^{\ell+1}\varphi \quad \text{in } \mathcal{S}(\mathbb{R}^n), \tag{3.6}$$

where $\binom{2\ell}{j}$ for any $j\in\{0,\dots,2\ell\}$ denotes the binomial coefficients and $a_{\ell+1}$ and $b_{\ell+1}$ are as in (1.4) with ℓ replaced by $\ell+1$.

Proof For any $\ell\in\mathbb{N}$, $t\in(0,\infty)$ and $x\in\mathbb{R}^n$, by (3.2) and the Taylor expansion of φ, we obtain

$$\begin{aligned}\varphi(x)-B_{\ell,t}\varphi(x) &= \frac{(-1)^\ell}{\binom{2\ell}{\ell}}\sum_{j=0}^{2\ell}(-1)^j\binom{2\ell}{j}\left[\sum_{\alpha\in E_{\ell+1}}(|\ell-j|t)^{|\alpha|}\frac{1}{\alpha!}\partial^\alpha\varphi(x)\fint_{B(\vec{0}_n,1)}y^\alpha\,dy\right]\\ &\quad+\frac{t^{2\ell+3}}{(2\ell+2)!}\frac{(-1)^\ell}{\binom{2\ell}{\ell}}\left[\sum_{j=0}^{2\ell}(-1)^j\binom{2\ell}{j}|\ell-j|^{2\ell+3}\right]\\ &\quad\times\int_0^1(1-s)^{2\ell+2}\fint_{B(\vec{0}_n,1)}\left[\sum_{|\alpha|=2\ell+3}\partial^\alpha\varphi(x+sty)y^\alpha\right]dy\,ds.\end{aligned}$$

From this and Theorem 1.3, it follows that, for any $\ell\in\mathbb{N}$, $t\in(0,\infty)$, and $x\in\mathbb{R}^n$,

$$\begin{aligned}&\varphi(x)-B_{\ell,t}\varphi(x)\\ &\quad= a_\ell(-\Delta)^\ell\varphi(x)t^{2\ell}+\frac{(-1)^\ell}{\binom{2\ell}{\ell}}\left[\sum_{j=0}^{2\ell}(-1)^j\binom{2\ell}{j}|\ell-j|^{2\ell+2}\right]\frac{a_{\ell+1}}{b_{\ell+1}}(-\Delta)^{\ell+1}\varphi(x)t^{2\ell+2}\\ &\quad\quad+\frac{t^{2\ell+3}}{(2\ell+2)!}\frac{(-1)^\ell}{\binom{2\ell}{\ell}}\left[\sum_{j=0}^{2\ell}(-1)^j\binom{2\ell}{j}|\ell-j|^{2\ell+3}\right]\\ &\quad\quad\times\int_0^1(1-s)^{2\ell+2}\fint_{B(\vec{0}_n,1)}\left[\sum_{|\alpha|=2\ell+3}\partial^\alpha\varphi(x+sty)y^\alpha\right]dy\,ds,\end{aligned}$$

where a_ℓ, $a_{\ell+1}$, and $b_{\ell+1}$ are as in (1.4). Combining this and integration by parts, we conclude that, for any $\ell \in \mathbb{N}$, $t \in (0,\infty)$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}\frac{1}{t}\frac{\partial}{\partial t}&\left(\frac{\varphi(x)-B_{\ell,t}\varphi(x)}{t^{2\ell}}\right)\\ &= 2\frac{(-1)^\ell}{\binom{2\ell}{\ell}}\left[\sum_{j=0}^{2\ell}(-1)^j\binom{2\ell}{j}|\ell-j|^{2\ell+2}\right]\frac{a_{\ell+1}}{b_{\ell+1}}(-\Delta)^{\ell+1}\varphi(x)\\ &\quad+\frac{2t}{(2\ell+2)!}\frac{(-1)^\ell}{\binom{2\ell}{\ell}}\left[\sum_{j=0}^{2\ell}(-1)^j\binom{2\ell}{j}|\ell-j|^{2\ell+3}\right]\\ &\quad\times\int_0^1(1+\ell s)(1-s)^{2\ell+1}\fint_{B(\vec{0}_n,1)}\left[\sum_{|\alpha|=2\ell+3}\partial^\alpha\varphi(x+sty)y^\alpha\right]dy\,ds.\end{aligned}$$

Similarly to the proof of Lemma 2.1, we conclude that (3.5) and (3.6) hold true. This finishes the proof of Lemma 3.2. ∎

Proof of Theorem 1.4 With Lemma 2.1 replaced by Lemma 3.2, similarly to the proof of Theorem 1.1, we can show Theorem 1.4, the details being omitted. ∎

From the proofs of Theorems 1.1 and 1.4, we deduce that (3.5) and (3.6) in Lemma 3.2 are the key tools used to establish the characterizations of Sobolev spaces $W^{2\ell+2,p}(\mathbb{R}^n)$ with $\ell \in \mathbb{N}$ and $p \in (1,\infty)$. We can gain some inspiration from this. Let $\ell \in \mathbb{N}$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. From Theorem 1.3 and Lemma 3.2, we deduce that, for any $k \in \{0,1,\dots,2\ell\}$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}\lim_{t\to0^+}\frac{1}{t}\frac{\partial^{2\ell+1-k}}{\partial t^{2\ell+1-k}}&\left(\frac{\varphi-B_{\ell,t}\varphi}{t^k}\right)\\ &=(2\ell+2-k)!\frac{(-1)^\ell}{\binom{2\ell}{\ell}}\left[\sum_{j=0}^{2\ell}(-1)^j\binom{2\ell}{j}|\ell-j|^{2\ell+2}\right]\frac{a_{\ell+1}}{b_{\ell+1}}(-\Delta)^{\ell+1}\varphi\end{aligned}$$

in $\mathcal{S}(\mathbb{R}^n)$ as well as an analogue of (3.6). Thus, similarly to the proofs of Theorems 1.1 and 1.4, we can characterize the higher order Sobolev spaces $W^{2\ell+2,p}(\mathbb{R}^n)$ by means of $\frac{1}{t}\frac{\partial^{2\ell+1-k}}{\partial t^{2\ell+1-k}}(\frac{f-B_{\ell,t}f}{t^k})$ as follows, the details being omitted.

Theorem 3.3 *Let $p \in (1,\infty)$, $\ell \in \mathbb{N}$ and $k \in \{0,1,\dots,2\ell\}$. Then the conclusions of Theorem 1.1 remain true when $W^{4,p}(\mathbb{R}^n)$ and $\frac{\partial}{\partial t}(\frac{f-B_tf}{t^2})$ therein are replaced, respectively, by $W^{2\ell+2,p}(\mathbb{R}^n)$ and $\frac{\partial^{2\ell+1-k}}{\partial t^{2\ell+1-k}}(\frac{f-B_{\ell,t}f}{t^k})$.*

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