Can. J. Math. Vol. 43 (4), 1991 pp. 832–851

REMARKS ON MARTIN'S AXIOM AND THE CONTINUUM HYPOTHESIS

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Martin's axiom and the Continuum Hypothesis are studied here using the notion of a *ccc partition* i.e., a partition of the form

$$[X]^{<\omega} = K_0 \cup K_1, \tag{1}$$

where K_0 has the following properties:

(a) K_0 contains subsets of its elements as well as all singletons of X.

(b) Every uncountable subset of K_0 contains two elements whose union is in K_0.

If we have a finite $m \geq 2$, then $[X]^m = K_0 \cup K_1$ is a ccc partition provided that every uncountable family of finite 0-homogeneous sets has two members whose union is also 0-homogeneous. Let

$$\theta \overset{ccc}{\longrightarrow} (\kappa, \omega_1)^{<\omega}$$

denote the statement that every ccc partition of the form (1) for $X = \theta$ has a 0-homogeneous set of size κ i.e., a subset A of θ of size κ such that $[A]^{<\omega} \subseteq K_0$. Similarly one defines $\theta \overset{ccc}{\longrightarrow} (\kappa, \omega_1)^m$ for a fixed finite dimension m. Note that this relation is a considerable weakening of the ordinary partition relation $\theta \longrightarrow (\kappa, \omega_1)^m$, since a ccc partition can never have an uncountable 1-homogeneous set, so of the two alternatives given by $\theta \longrightarrow (\kappa, \omega_1)^m$ the second one is always impossible. This explains the symbol "ω_1" in our notation. The following are examples of results about this new relation:

$$\mathfrak{c}^+ \overset{ccc}{\longrightarrow} (\mathfrak{c}^+, \omega_1)^{<\omega}, \tag{2}$$

$$\mathfrak{c} \overset{ccc}{\not\longrightarrow} (\mathfrak{c}, \omega_1)^3, \tag{3}$$

$$\mathfrak{b} \overset{ccc}{\not\longrightarrow} (\mathfrak{b}, \omega_1)^2, \tag{4}$$

$$\mathfrak{t} \overset{ccc}{\not\longrightarrow} (\mathfrak{t}, \omega_1)^{<\omega}. \tag{5}$$

(Here $\mathfrak{c}$ denotes the cardinality of the continuum; $\mathfrak{b}$ is the minimal cardinality of an $<^*$-unbounded subset of ω^ω; $\mathfrak{t}$ is the minimal type of a $\subset^*$-decreasing sequence of infinite subsets of ω which can't be extended.) The result (2) can be thought of as a form of the Erdős-Rado-Kurepa result $\mathfrak{c}^+ \longrightarrow (\mathfrak{c}^+, \omega_1)^2$, and, in fact, a direct proof of (2) can be given

Research supported in part by NSF Grant DMS-8505550.
Received by the editors May 29, 1990.
AMS subject classification: 03E05, 03E50, 04A20.
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https://doi.org/10.4153/CJM-1991-048-8 Published online by Cambridge University Press

by copying a standard argument for the Erdős-Rado-Kurepa result: For every $\delta < \mathfrak{c}^+$ of cofinality $> \omega$ fix a maximal $\mathcal{F}_\delta \subseteq K_0 \cap [\delta]^{<\omega}$ such that:

(c) $F \cup \{\delta\} \notin K_0$ for all F in $\mathcal{F}_\delta$,

(d) $F \cup G \notin K_0$ for all $F \neq G$ in $\mathcal{F}_\delta$.

Since every $\mathcal{F}_\delta$ is countable there is stationary $S \subseteq \mathfrak{c}^+$ such that for some $\mathcal{F}$, $\mathcal{F}_\delta = \mathcal{F}$ for all δ in S. It is easily checked that $[S]^{<\omega} \subseteq K_0$.

As in the case of the ordinary partition symbol, $\theta \overset{ccc}{\longrightarrow} (\kappa, \omega_1)^m$ becomes stronger as m grows with $\theta \overset{ccc}{\longrightarrow} (\kappa, \omega_1)^{<\omega}$ being the strongest statement. Hence, $\mathfrak{c}^+ \overset{ccc}{\longrightarrow} (\mathfrak{c}^+, \omega_1)^3$, $\mathfrak{c}^+ \overset{ccc}{\longrightarrow} (\mathfrak{c}^+, \omega_1)^4, \ldots$ are all consequences of (2). Note that their ordinary counterparts are all false giving us an example that shows the difference between the two partition relations. The difference becomes even more striking when one analyzes the well-known partition of Sierpinski's result $\mathfrak{c} \not\rightarrow (\omega_1)^2_2$ obtained by comparing the usual ordering of the reals with a well-ordering. First of all, note that one cannot prove the exact ccc version of Sierpinski's result since MA implies $\mathfrak{c} \overset{ccc}{\longrightarrow} (\kappa, \omega_1)^{<\omega}$ for all $\kappa < \mathfrak{c}$. [Apply MA to the poset of all finite 0-homogeneous sets.] It turned out that the search for ccc versions of $\mathfrak{c} \not\rightarrow (\omega_1)^2_2$ included some key steps in solutions of several well-known open problems of seemingly quite different nature (see [19] and [22]). The relations (3) and (4) of [19] and (5) of [22] are the results of that search. They will essentially be reproduced here in statements (16), (14) and (17) of § 4, respectively.

Let p_ξ, $(\xi < \theta)$, be a sequence of elements of a ccc poset $\mathcal{P}$. Let K_0 be the set of all finite $F \subseteq \theta$ for which there is a p in $\mathcal{P}$ such that $p \leq p_\xi$ for all ξ in F. Let K_1 be the complement of K_0 in $[\theta]^{<\omega}$. Then $[\theta]^{<\omega} = K_0 \cup K_1$ is a ccc partition such that p_ξ, $(\xi \in H)$, is centered in $\mathcal{P}$ iff $[H]^{<\omega} \subseteq K_0$. This shows that $\theta \overset{ccc}{\longrightarrow} (\kappa, \omega_1)^{<\omega}$ is equivalent to the familiar statement (see [1; § 1]) that every ccc poset (space) has precaliber (θ, κ). Thus (2), reduces to the well-known result (see [1;5.12], [10; § 10]) that every ccc poset has precaliber $\mathfrak{c}^+$.

This paper refines and builds on the result of [22] which says that

(6) $\mathrm{MA}_{\aleph_1}$ is equivalent to $\omega_1 \overset{ccc}{\longrightarrow} (\omega_1)^{<\omega}_2$,

i.e., that Martin's axiom is not only a forcing axiom (i.e., a strong Baire Category Theorem; see [12]) but also a Ramsey-type statement which has been considered long before the invention of forcing (see [9], [8], [11], [16]). In particular, we shall prove (§ 2) that MA_θ is Ramsey for θ not necessarily equal to ω_1. The proof of (6) involved a fine analysis of Martin's axiom and required new constructions (in the spirit of [19]) of ccc posets and partitions. The analysis had some applications outside the original scope (see [22], [21]) and this will also be the case with the present paper. For example, using the result of § 2 we shall analyze the Luzin-set axiom discussed by van Douwen and Fleissner [3] and Miller and Prikry [14]. Another problem we shall solve here (§ 1) is the technical problem of specializing Aronszajn trees using only the 2-dimensional Ramsey axiom for ccc partitions (which is a slight weakening of the familiar statement that every ccc poset has Property K; see [4]). The last section (§ 5) contains a definition of a *definable ccc poset* or a *definable ccc partition* which is used to supplement the discussion started in [3]. In § 4 we analyze a Ramsey-theoretic form of the Continuum Hypothesis.

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1. **Aronszajn trees.** The result (6) naturally leads to the following two questions:

(7) Is $MA_{\aleph_1}$ equivalent to $\omega_1 \overset{ccc}{\longrightarrow} (\omega_1)^2_2$?

(8) Is $\omega_1 \overset{ccc}{\longrightarrow} (\omega_1)^3_2$ equivalent to $\omega_1 \overset{ccc}{\longrightarrow} (\omega_1)^2_2$?

Test problems for both (7) and (8) are whether we can code subsets of ω_1 by the reals, or whether we can specialize A-trees using only the 2-dimensional Ramsey axiom for ccc partitions. (It is known (see [22], [21]) that three dimensions are sufficient for both of these two tasks.) In this section we pass the second test. The proof might appear a bit technical but we shall see in the next section that the process of *specializing* is quite relevant to Martin's axiom. It can be considered as part of the general problem of *2-dimensional coding.*

THEOREM 1. *To every Aronszajn tree T we can associate a ccc partition $[T]^2 = K_0 \cup K_1$ such that if there is uncountable X such that $[X]^2 \subseteq K_0$ then T is the union of countably many antichains.*

PROOF. We shall assume $T = \omega_1, <_T$ where $\alpha <_T \beta$ implies $\alpha < \beta$. The association will depend on two objects. The first one is an one-to-one sequence $r_\alpha, (\alpha < \omega_1)$, of elements of $\{0,1\}^\omega$. The second object is an $e\colon [\omega_1]^2 \longrightarrow \omega$ such that:

(e) $e_\alpha = e(\cdot,\alpha)\colon \alpha \longrightarrow \omega$ is one-to-one for all α,

(f) $\{\xi \leq \alpha : e(\xi,\alpha) \neq e(\xi,\beta)\}$ is finite for all $\alpha < \beta$.

[A simple way to obtain such e is to consider $\rho_1\colon [\omega_1]^2 \longrightarrow \omega$ defined recursively by $\rho_1(\alpha,\beta) = \max\{|C_\beta \cap \alpha|, \rho_1(\alpha, \min(C_\beta \setminus \alpha))\}$ from a sequence $C_\alpha, (\alpha < \omega_1)$, such that $C_{\alpha+1} = \{\alpha\}$ and $\operatorname{tp} C_\alpha = \omega$ and $\sup C_\alpha = \alpha$ for limit α. A simple inductive proof shows that ρ_1 satisfies (e) and (f) except that $(\rho_1)_\alpha$'s are only finite-to-one. We correct this by an obvious *stretching-up* formula (see [20; p. 271, 1.6]). A detailed historical remark concerning the existence of such e is given in [20; pp. 291–2]. We adopt the convention $e(\alpha,\alpha) = 0$ whenever we consider a mapping e with domain some $[\theta]^2 = \{\langle \alpha,\beta \rangle : \alpha < \beta < \theta\}$. For q and r in $\{0,1\}^\omega$ (or ω^ω), $\Delta(q,r)$ is the minimal i such that $q(i) \neq r(i)$; if $q = r$, let $\Delta(q,r) = \infty$. If X is a (finite) set of ordinals and if i is less that its size, $X(i)$ denotes the ith element of X in the increasing order.] The partition

$$[\omega_1]^2 = K_0 \cup K_1 \tag{9}$$

associated to T is defined by

$$\{\alpha,\beta\} \in K_0 \text{ iff } e_\alpha^{-1}(l) \text{ and } e_\beta^{-1}(l) \text{ are } T\text{-incomparable for all } l < \Delta(r_\alpha, r_\beta) \tag{10}$$

for which both $e_\alpha^{-1}(l)$ and $e_\beta^{-1}(l)$ are defined (i.e., $l \in \operatorname{range}(e_\alpha) \cap \operatorname{range}(e_\beta)$) and different.

Suppose X is uncountable and $[X]^2 \subseteq K_0$. For s in $\{0,1\}^{<\omega}$ of length equal to some $l+1$, set

$$A_s = \{\xi < \omega_1 : e(\xi,\alpha) = l \text{ for some } \alpha \text{ in } X \text{ with } s \subset r_\alpha\}.$$

Then A_s is an antichain of T for all s. Since A_s cover T we would be done once we prove the following

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CLAIM. (9) *is a ccc partition.*

PROOF. Let $\mathcal{F}$ be an uncountable family of disjoint finite 0-homogeneous sets of (9). We may assume that the members of $\mathcal{F}$ have some fixed size k.

Consider a limit ordinal $\delta < \omega_1$ and an F in $\mathcal{F}$ with all elements above δ. Let $n = n(F,\delta)$ be the minimal integer such that for all $\alpha < \beta$ in $F \cup \{\delta\}$:

(g) $\Delta(r_\alpha, r_\beta) < n$,

(h) $e(\xi,\alpha) \neq e(\xi,\beta)$ implies $e(\xi,\alpha), e(\xi,\beta) \leq n$ for all $\xi \leq \alpha$.

Let $H = H(\delta,F)$ be the set of all ξ such that $e(\xi,\alpha) \leq n$ for some α in $F \cup \{\delta\}$. Note that $F \cup \{\delta\}$ is a subset of H. Note also that if we take the transitive collapse of H to an integer the sequence $e_\alpha \restriction H, (\alpha \in F)$, collapses to a k-sequence $\bar{s}$ of mappings with integer-domains. Let $\bar{r}$ denote the sequence of restrictions $r \restriction (n+1), (\alpha \in F)$, enumerated in increasing order. This means that every F of $\mathcal{F}$ above δ generates a quadruple

$$\langle H(\delta,F) \cap \delta, \bar{r}(\delta,F), \bar{s}(\delta,F), n(\delta,F) \rangle$$

of parameters. The set of such quadruples is clearly countable, so we can find two elements F_δ and G_δ of $\mathcal{F}$ above δ generating the same quadruple of parameters which we denote by $\langle H_\delta, \bar{r}_\delta, \bar{s}_\delta, n_\delta \rangle$. Moreover, we choose F_δ and G_δ to satisfy the following isomorphism condition with respect to H_δ and the tree T:

(i) If $l \leq n$, if $i < k$, if $\alpha = F_\delta(i)$ and $\beta = G_\delta(i)$, then $e_\alpha^{-1}(l)$ and $e_\beta^{-1}(l)$ have the same T-predecessors in H_δ.

Let $m = m_\delta$ be the minimal integer above n_δ such that $\Delta(r_\alpha, r_\beta) < m$ for all α in F_δ and β in G_δ. Let I_δ be the set of all ξ such that $e(\xi,\alpha) \leq m$ for some α in $F_\delta \cup G_\delta$. Let $\bar{p}_\delta$ be the sequence of restrictions $r_\alpha \restriction m, (\alpha \in F_\delta)$, and let $\bar{q}_\delta$ be the sequence of restrictions $r_\alpha \restriction m, (\alpha \in G_\delta)$, enumerated in increasing order. Let $\bar{t}_\delta$ and $\bar{u}_\delta$ be the transitive collapses of $e_\alpha \restriction I_\delta, (\alpha \in F_\delta)$, and $e_\beta \restriction I_\delta, (\beta \in G_\delta)$, respectively. By the Pressing Down Lemma we can find stationary $S \subseteq \omega_1$, $\langle H, \bar{r}, \bar{s}, n \rangle$ and $\langle I, \bar{p}, \bar{q}, \bar{t}, \bar{u}, m \rangle$ such that for all δ in S:

(j) $\langle H_\delta, \bar{r}_\delta, \bar{s}_\delta, n_\delta \rangle = \langle H, \bar{r}, \bar{s}, n \rangle$,

(k) $\langle I_\delta \cap \delta, \bar{p}_\delta, \bar{q}_\delta, \bar{t}_\delta, \bar{u}_\delta, m_\delta \rangle = \langle I, \bar{p}, \bar{q}, \bar{t}, \bar{u}, m \rangle$.

Moreover, we may assume the following analogue of (i) where k_1 is the size of $I_\gamma \setminus \gamma$ for all γ is S:

(l) For all γ and δ in S and $i < k_1$, if $\eta = I_\gamma(i)$ and $\xi = I_\delta(i)$ then ξ and η have the same T-predecessors in I.

By the standard property of an A-tree there exist $\gamma < \delta$ in S such that

(m) every element of $I_\gamma \setminus \gamma$ is incomparable with every element of $I_\delta \setminus \delta$.

We will show that $[F_\gamma \cup G_\delta]^2 \subseteq K_0$ and this will finish the proof that (9) is a ccc partition. So let $\alpha = F_\gamma(i)$ and $\beta = G_\delta(j)$ be given where $i,j < k$.

CASE 1. $i \neq j$. Pick an $l < \Delta(r_\alpha, r_\beta)$. Note that by the first choice of parameters (see (g)), $l < n$. Assume $e_\alpha^{-1}(l)$ and $e_\beta^{-1}(l)$ are both defined. We shall prove that they are either equal or T-incomparable.

SUBCASE 1[a]. $e_\alpha^{-1}(l) < \gamma$ and $e_\beta^{-1}(l) < \delta$. By the first choice of parameters, $e_\alpha^{-1}(l)$ and $e_\beta^{-1}(l)$ are elements of H which is an initial segment of both $H(\gamma, F_\gamma)$ and $H(\delta, G_\delta)$.

Therefore, the behaviour of e_α and e_β on H is coded by the i^{th} and j^{th} term of $\bar{s}$, respectively. So if $\beta'' = F_\gamma(j)$, we have $e_\beta^{-1}(l) = e_{\beta''}^{-1}(l)$. Since $\{\alpha, \beta''\} \in K_0$, by (10) it follows that $e_\alpha^{-1}(l)$ and $e_\beta^{-1}(l)$ are T-incomparable if they are different.

SUBCASE 1^{b}. $e_\alpha^{-1}(l) < \gamma$ and $e_\beta^{-1}(l) \in I_\delta \setminus \delta$. Let $\beta' = G_\gamma(j)$. Since $e_\alpha^{-1}(l)$ is in $H \subseteq I$ the condition (l) reduces the problem to checking that $e_\alpha^{-1}(l)$ and $e_{\beta'}^{-1}(l)$ are incomparable in T (because the position of $e_\beta^{-1}(l)$ in I_δ is equal to the position of $e_{\beta'}^{-1}(l)$ in I_γ). Now the condition (i) reduces this problem to checking that $e_\alpha^{-1}(l)$ and $e_{\beta''}^{-1}(l)$ are incomparable in T. This follows from $\{\alpha, \beta''\} \in K_0$ and the fact that $l < \Delta(r_\alpha, r_\beta) = \Delta(r_\alpha, r_{\beta'}) = \Delta(r_\alpha, r_{\beta''})$.

SUBCASE 1^{c}. $e_\alpha^{-1}(l) \in I_\gamma \setminus \gamma$ and $e_\beta^{-1}(l) < \delta$. This is essentially symmetric to the previous subcase by going from α to $\alpha' = F_\delta(i)$ and then to $\alpha'' = G_\delta(i)$.

SUBCASE 1^{d}. $e_\alpha^{-1}(l) \in I_\gamma \setminus \gamma$ and $e_\beta^{-1}(l) \in I_\delta \setminus \delta$. The incomparability follows from the condition (m).

CASE 2. $i = j$. Consider again an $l < \Delta(r_\alpha, r_\beta)$. Note that now $l < m$.

SUBCASE 2^{a}. $e_\alpha^{-1}(l) < \gamma$ and $e_\beta^{-1}(l) < \delta$. Then $e_\alpha^{-1}(l)$ and $e_\beta^{-1}(l)$ are elements of I which is an intial segment of both I_γ and I_δ. Therefore the ith term of $\bar{u}$ codes both $e_\beta \restriction I$ and $e_{\beta'} \restriction I$. It follows that $e_\beta^{-1}(l) = e_{\beta'}^{-1}(l)$. If $l \leq n$, then $e_{\beta'}^{-1}(l)$ is in H and since the ith term of $\bar{s}$ codes both $e_\alpha \restriction H$ and $e_{\beta'} \restriction H$ it follows that $e_\alpha^{-1}(l) = e_{\beta'}^{-1}(l)$. If $l > n$, then $e_\alpha^{-1}(l)$ and $e_{\beta'}^{-1}(l)$ are not elements of H, so by (h) we have

$$e_\gamma\big(e_\alpha^{-1}(l)\big) = e_\alpha\big(e_\alpha^{-1}(l)\big) = l = e_{\beta'}\big(e_{\beta'}^{-1}(l)\big) = e_\gamma\big((e_{\beta'}^{-1}(l)\big)$$

Since e_γ is one-to-one, it follows that $e_\alpha^{-1}(l)$ and $e_{\beta'}^{-1}(l)$ are equal, so we are done.

SUBCASE 2^{b}. $e_\alpha^{-1}(l) < \gamma$ and $e_\beta^{-1}(l) \in I_\delta \setminus \delta$. Similarly as in the previous subcase we have $e_\alpha^{-1}(l) = e_{\beta'}^{-1}(l) \in I$. Since the ith term of $\bar{u}$ codes both $e_\beta \restriction I$ and $e_{\beta'} \restriction I$ we must have that $e_\beta^{-1}(l)$ is equal to $e_{\beta'}^{-1}(l)$, a contradiction. So, the Subcase 2^{b} never occurs.

SUBCASE 2^{c}. $e_\alpha^{-1}(l) \in I_\gamma \setminus \gamma$ and $e_\beta^{-1}(l) < \delta$. This is symmetric to the previous subcase.

SUBCASE 2^{d}. $e_\alpha^{-1}(l) \in I_\gamma \setminus \gamma$ and $e_\beta^{-1}(l) \in I_\delta \setminus \delta$. The incomparability of $e_\alpha^{-1}(l)$ and $e_\beta^{-1}(l)$ follows from (m). This finishes the proof of Theorem 1.

The proof of Theorem 1 gives a general procedure of associating to every partition of the form

$$[\omega_1]^2 = C_0 \cup C_1 \tag{11}$$

another partition

$$[\omega_1]^2 = K_0 \cup K_1 \tag{12}$$

such that every uncountable 0-homogeneous set for (12) gives a decomposition of ω_1 into countably many 0-homogeneous sets for (11). [$\{\alpha, \beta\} \in K_0$ iff $\{e_\alpha^{-1}(l), e_\beta^{-1}(l)\} \in$

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C_0 for all $l < \Delta(r_\alpha, r_\beta)$ for which $e_\alpha^{-1}(l)$ and $e_\beta^{-1}(l)$ are defined and different.] We are especially interested under which conditions on (11) the partition (12) is ccc. Note that it is not sufficient to assume only that (11) is ccc. For example, let C_0 be the comparability graph of a Souslin tree. The proof of Theorem 1 needs the following property of (11) (which corresponds to the condition (m)):

(n) For every uncountable family $\mathcal{F}$ of disjoint finite subsets of ω_1 there exist $F \neq G$ in $\mathcal{F}$ such that $\{\alpha, \beta\} \in C_0$ for all α in F and β in G.

Clearly, the incomparability graph of an Aronszajn tree satisfies (n). A requirement closely related to (n) is the condition that the poset $\mathcal{P}$ of all finite 0-homogeneous sets for (11) is *powerfully ccc* i.e., that all finite powers of $\mathcal{P}$ are ccc. Note that $\omega_1 \overset{ccc}{\longrightarrow} (\omega_1)_2^2$ is an immediate consequence of the classical statement (considered in [9], [8]) that every ccc poset $\mathcal{P}$ has property K (i.e., that every uncountable subset of $\mathcal{P}$ contains an uncountable subset of pairwise compatible elements). On the other hand, $\omega_1 \overset{ccc}{\longrightarrow} (\omega_1)_2^2$ implies that every powerfully ccc poset $\mathcal{P}$ has property K. [If p_ξ, $(\xi < \omega_1)$, is a sequence of elements of $\mathcal{P}$ define $[\omega_1]^2 = K_0 \cup K_1$ by letting $\{\xi, \eta\}$ in K_0 iff p_ξ and p_η are compatible in $\mathcal{P}$. Let $\mathcal{F}$ be a disjoint family of finite 0-homogeneous set all of the same fixed size n. To every F we associate the element of $\mathcal{P}^k$ where $k = n + \binom{n}{2}$ which first lists p_ξ, $(\xi \in F)$, in increasing order and which on the $\{i,j\}$th place has an extension of p_ξ and p_η where ξ is the ith and η the jth element of F in the increasing enumeration. Since $\mathcal{P}^k$ is ccc we can find F and G in $\mathcal{F}$ for which the corresponding elements of $\mathcal{P}^k$ are compatible. It is easily checked that this means that $F \cup G$ is 0-homogeneous. Thus $[\omega_1]^2 = K_0 \cup K_1$ is a ccc partition. Let H be an uncountable 0-homogeneous set. Then p_ξ, $(\xi \in H)$, is a sequence of pairwise compatible elements.] Powerfully ccc posets have proved to be sufficient in any known application of Martin's axiom. In fact, it is still unknown whether Martin's axiom for powerfully ccc posets is actually the same as the full Martin's axiom. (Note that while a Souslin tree T is not powerfully ccc it can be destroyed using a powerfully ccc poset i.e., the poset of all finite antichains of T.)

2. **Martin's axiom.** The purpose of this section is to extend (6) by proving that higher instances of MA are also Ramsey. The proof uses the existence of a $c: \theta^2 \longrightarrow \omega$ which is not constant on the product of any two infinite subsets of θ. This statement is in the literature usually denoted by $\binom{\theta}{\theta} \not\rightarrow \binom{\omega}{\omega}_\omega^{1,1}$. Note that $\binom{\omega_1}{\omega_1} \not\rightarrow \binom{\omega}{\omega}_\omega^{1,1}$ so that (6) will be a special case of our result.

THEOREM 2. *Assume* $\binom{\theta}{\theta} \not\rightarrow \binom{\omega}{\omega}_\omega^{1,1}$. *Then* $\theta \overset{ccc}{\longrightarrow} (\theta, \omega_1)^{<\omega}$ *implies* MA_θ *for powerfully ccc posets.*

Our interest in the assumption is primarily based on its upward absoluteness. This will be explained and used in later sections of this paper. In particular, the next section explains this assumption when $\theta = \omega_2$ in the context of $\mathrm{MA}_{\aleph_2}$. By the proof of Theorem 3.3 of [22], the proof of Theorem 2 reduces to showing that every powerfully ccc poset $\mathcal{P}$ of size $\leq \theta$ is σ-centered assuming only the Ramsey axiom for ccc partitions. [This is so because the poset of Theorem 1.5 of [22] (note the typo on p. 407, l. 11 of [22]: 1.4 should be 1.5) is also powerfully ccc.] To prepare for this we need two lemmas.

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LEMMA 1. $\binom{\theta}{\theta} \not\to \binom{\omega}{\omega}^{1,1}_{\omega}$ *implies* $\theta \not\to [\theta]^{<\omega}_{\theta}$.

PROOF. Suppose that $\theta \longrightarrow [\theta]^{<\omega}_{\theta}$ and let $c \colon \theta^2 \longrightarrow \omega$ be given. By coding a submodel of some large enough H_κ as an algebra on θ and using the fact that $\theta \longrightarrow [\theta]^{<\omega}_{\theta}$ we can get an elementary submodel M of H_κ of size θ such that $c \in M$ and $\theta \not\subseteq M$ (see [3],[6]). Let α be the minimal ordinal of θ not in M. Choose $X \subseteq \theta \cap M$ of size α^+ such that for some n, $c(\alpha, \xi) = n$ for all ξ in X. By elementarity of M, α is a limit ordinal and for every finite $F \subseteq X$ the set A_F of all $\beta < \alpha$ such that $c(\beta, \xi) = n$ for all ξ in F is unbounded in α. Since X has size bigger than α, there is $\alpha_0 < \alpha$ such that $X_0 = \{\xi \in X : c(\alpha_0, \xi) = n\}$ has size α^+. Let $\xi_0 = \min X_0$. Then $A_{\{\xi_0, \xi\}}$ is unbounded in α for all ξ in X_0, so we can choose α_1 above α_0 and $X_1 \subseteq X_0$ of size α^+ such that α_1 is in $A_{\{\xi_0, \xi\}}$ for all ξ in X_1. Now choose ξ_1 in X_1 above ξ_0. Proceeding in this way we construct two increasing sequences $\{\alpha_i\} \subseteq \alpha$ and $\{\xi_i\} \subseteq X$ such that $c(\alpha_i, \xi_j) = n$ for all i and j. Since c was arbitrary this shows that $\binom{\theta}{\theta} \longrightarrow \binom{\omega}{\omega}^{1,1}_{\omega}$ finishing the proof.

LEMMA 2. *Suppose $c \colon \theta^2 \longrightarrow \omega$ is not constant on the product of any two infinite sets. Then for every sequence $F_\xi, (\xi < \omega_1)$ of disjoint finite subsets of θ and $m < \omega$ there exist $\xi \neq \eta$ such that $c(\alpha, \beta) > m$ for all α in F_ξ and β in F_η.*

PROOF. Suppose the Lemma is false for some $\{F_\xi\}$ and m. We may assume that the F_ξ have some fixed size n. Moreover, we may assume that for every $i < n$, $F_\xi(i), (\xi < \omega_1)$, is strictly increasing. Let $\mathcal{U}$ be a uniform ultrafilter on ω_1 and let $\mathcal{V}$ be a uniform ultrafilter on ω. Then we can find $i, j < n$ and $k \leq m$ such that

$$a.e._{\mathcal{U}}\ \xi \quad a.e._{\mathcal{V}}\ \eta \quad c\big(F_\eta(j), F_\xi(i)\big) = k$$

This means that if $X = \{F_\eta(j) : \eta < \omega\}$ then there is uncountable subset Y of θ_1 such that for every finite $H \subseteq Y$ the set X_H of α's in X such that $c(\alpha, \beta) = k$ for all β in H is infinite. Since X is countable there must be α_0 in X such that $Y_0 = \{\beta \in Y : c(\alpha_0, \beta) = k\}$ is uncountable. Let $\beta_0 = \min Y_0$. Then we can find α_1 in $X_{\{\beta_0\}}$ above α_1 such that $Y_1 = \{\beta \in Y_0 : c(\alpha_1, \beta) = k\}$ is uncountable, etc. This procedure gives us two increasing sequences $\{\alpha_i\} \subseteq X$ and $\{\beta_i\} \subseteq Y$ such that $c(\alpha_i, \beta_j) = k$ for all i and j contradicting the assumption of the Lemma.

We are ready now to start the proof of Theorem 2. The proof consists of associating to every powerfully ccc poset $\mathcal{P}$ of size θ a ccc partition

$$[\theta]^{<\omega} = K_0 \cup K_1 \tag{13}$$

such that every set $X \subseteq \theta$ of size θ with the property $[X]^{<\omega} \subseteq K_0$ gives us a decomposition of $\mathcal{P}$ into countably many centered sets. We shall need three objects in order define (13) from $\mathcal{P}$. The first object is a $c \colon \theta^2 \longrightarrow \omega$ which is not constant on the product of any two infinite sets. The second object (obtained from the first using Lemma 1) is a $d \colon [\theta]^{<\omega} \longrightarrow \mathcal{P}$ such that $d''[X]^{<\omega} = \mathcal{P}$ for all $X \subseteq \theta$ of size θ. We shall also use a one-to-one sequence $r_\alpha, (\alpha < \theta)$, of elements of $\{0, 1\}^\omega$ (Note that $\theta \leq \mathfrak{c}$.) The mapping c determines a new $c^* \colon [\theta]^{<\omega} \longrightarrow \omega$ by

$$c^*(F) = \max\{c(\alpha, \beta) : \langle \alpha, \beta \rangle \in F^2\}.$$

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For F in $[\theta]^{<\omega}$ and $\bar{s} = \langle s_0, \dots, s_k \rangle$ in $(\{0,1\}^{<\omega})^{<\omega}$, let $\mathcal{P}_F(\bar{s})$ denote the set of all elements of $\mathcal{P}$ of the form $d(\alpha_0, \dots, \alpha_k)$ for some $\alpha_0 < \cdots < \alpha_k$ in F with the property

$$r_{\alpha_i} \restriction c^*(\alpha_0, \dots, \alpha_k) = s_i$$

for all $i \leq k$. The partition (13) is defined by letting an F in K_0 iff for every $\bar{s}$ in $(\{0,1\}^{<\omega})^{<\omega}$, the set $\mathcal{P}_F(\bar{s})$ is consistent in $\mathcal{P}$ i.e., there is a condition in $\mathcal{P}$ extending every element of $\mathcal{P}_F(\bar{s})$. Note that $\mathcal{P}_F(\bar{s}) = \emptyset$ for all but finitely many $\bar{s}$ promising that the proof of the following crucial claim may not be as hopeless as it looks.

CLAIM. (13) *is a ccc partition.*

PROOF. Suppose we have an uncountable Δ-system $\mathcal{F}$ with root F_0 of finite 0-homogeneous sets of some fixed size n. Refining $\mathcal{F}$ we may assume that for some m and all F:

(o) $c(\alpha, \beta) < m$ and $\Delta(r_\alpha, r_\beta) < m$ for all α and β in F.

Since the property of c is preserved by any forcing extension, working in such an extension by $\mathcal{P}^{\bar{k}}$, where $\bar{k}$ is the cardinality of $(\{0,1\}^{<m})^{\leq n}$, and making a further refinement, we may assume that

(p) $\mathcal{P}_F(\bar{s}) \cup \mathcal{P}_G(\bar{s})$ is consistent for all F and G in $\mathcal{F}$ and $\bar{s}$ in $(\{0,1\}^{<m})^{\leq n}$.

Choose two uncountable $\mathcal{F}_0, \mathcal{F}_1 \subseteq \mathcal{F}$ and $\bar{m} > m$ such that

(q) for every F in $\mathcal{F}_0$ and G in $\mathcal{F}_1$, $\Delta(r_\alpha, r_\beta) < \bar{m}$ for $\alpha \neq \beta$ in $F \cup G$.

By Lemma 2 (applied to an uncountable disjoint family of sets of the form $F \cup G$ for F in $\mathcal{F}_0$ and G in $\mathcal{F}_1$) there exist F in $\mathcal{F}_0$ and G in $\mathcal{F}_1$ such that

(r) $c(\alpha, \beta) > \bar{m}$ for $\alpha \in F \setminus F_0$ and $\beta \in G \setminus F_0$.

We will show that $E = F \cup G$ is in K_0 finishing the proof that (13) is ccc. So let $\bar{s}$ be a given sequence from $(\{0,1\}^{<\omega})^{<\omega}$. Note that we may assume that for some l, every term of $\bar{s}$ has length l. Let $k+1$ be the length of $\bar{s}$, i.e., $\bar{s} = \langle s_0, \dots, s_k \rangle$.

CASE 1. $l < m$. We will show that in this case

$$\mathcal{P}_E(\bar{s}) = \mathcal{P}_F(\bar{s}) \cup \mathcal{P}_G(\bar{s})$$

which is consistent by (p). To see this, consider $\alpha < \cdots < \alpha_k$ in E. If $\{\alpha_i\} \subseteq F$ or $\{\alpha_i\} \subseteq G$, $d(\alpha_0, \dots, \alpha_k)$ would be in the right hand side of the equality and so we would be done. But if this does not happen there would be i and j such that $\alpha_i \in F \setminus F_0$ and $\alpha_j \in G \setminus F_0$ which by (r) would give us

$$l = c^*(\alpha_0, \dots, \alpha_k) \geq c(\alpha_i, \alpha_j) > \bar{m} > m,$$

a contradiction.

CASE 2. $m \leq l < \bar{m}$. Consider again $\alpha_0 < \cdots < \alpha_k$ from E. By (o) and the definition of c^*, it is not the case that $\{\alpha_i\} \subseteq F$ or $\{\alpha_i\} \subseteq G$. So we can find i and j such that $\alpha_i \in F \setminus F_0$ and $\alpha_j \in G \setminus F_0$ giving us a similar contradiction as in the previous case. This shows that $\mathcal{P}_E(\bar{s})$ is empty and we are done.

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CASE 3. $\bar{m} \leq l$. Note that by (q) in this case for every $i \leq k$ there can be at most one α in E for which r_α extends s_i. This means that if $\alpha_0 < \cdots < \alpha_k$ in E has the properties $r_{\alpha_i} \supset s_i$ for all i and $c^*(\alpha_0, \ldots, \alpha_k) = l$, the sequence $\alpha_0 < \cdots < \alpha_k$ would be unique with these properties, and so $d(\alpha_0, \ldots, \alpha_k)$ would be the only element of $\mathcal{P}_E(\bar{s})$. Hence in the present case $\mathcal{P}_E(\bar{s})$ is at most a singleton and therefore consistent. This finishes the proof that (13) is a ccc partition.

Assume X is a subset of θ of size θ such that $[X]^{<\omega} \subseteq K_0$. For a sequence $\bar{s}$ in $(\{0,1\}^{<\omega})^{<\omega}$ of some length $k+1$, let $\mathcal{P}_X(\bar{s})$ be the set of all elements of $\mathcal{P}$ of the form $d(\alpha_0, \ldots, \alpha_k)$ for some $\alpha_0 < \cdots < \alpha_k$ in X such that

$$r_{\alpha_i} \upharpoonright c^*(\alpha_0, \ldots, \alpha_k) = s_i$$

for all $i \leq k$. Then by the definition of K_0, $\mathcal{P}_X(\bar{s})$ is centered in $\mathcal{P}$ for every $\bar{s}$. By the choice of d, the sets $\mathcal{P}_X(\bar{s})$ cover $\mathcal{P}$ showing that $\mathcal{P}$ is σ-centered. This finishes the proof of Theorem 2.

Note that in the above proof, much as in § 1, we have performed a certain *specializing* procedure. This time we have specialized (made σ-centered) a certain ccc poset by a ccc partition. The key combinatorial object needed for this process is certainly the partition c. Note that the property of c stated in Lemma 2 is the only property needed in the proof of Theorem 2. Hence the specializing procedure (and, therefore, the reduction of MA_θ to the Ramsey-type statement) can be performed in a wide variety of cases. The more interesting is the question whether any assumption (besides $\theta < \mathfrak{c}$) is really needed in Theorem 2. We shall get a better idea about this question in §§4 and 5 where we connect it to the question whether some natural weakenings of CH are in fact equivalent to CH. Another interesting problem which emerges from the proof of Theorem 2 asks for conditions on the partitions of the form $[\theta]^m = C_0 \cup C_1$ (with θ not necessarily equal to ω_1) which would enable us to construct another ccc partition specializing it in the sense of § 1. It should be clear that the proof of Theorem 2 does produce such a partition having the form $[\theta]^{2m} = K_0 \cup K_1$ assuming that, moreover, we have a $d\colon [\theta]^2 \longrightarrow \theta$ such that $d''[X]^2 = \theta$ for all $X \subseteq \theta$ of size θ. (Put an F of $[\theta]^{2m}$ in K_0 if for all s_0 and s_1 in $(\{0,1\}^{<\omega})^{<\omega}$ the set of all ordinals of the form $d(\alpha_0, \alpha_1)$ for $\alpha_0 < \alpha_1$ in F with the property $r_{\alpha_i} \upharpoonright c^*(\alpha_0, \alpha_1) = s_i$ for $i < 2$ is C_0-homogeneous. Of course, it is also necessary to assume that the poset $\mathcal{P}$ of all finite $F \subseteq \theta$ with property $[F]^m \subseteq C_0$ is powerfully ccc.) Such a d exists for many small cardinals such as $\omega_1, \omega_2, \omega_3, \ldots$ (see [20]).

3. **Chang's Conjecture.** Recall (see [6; § 19]) that Chang's Conjecture (CC) is the statement that every model of the form $\langle \omega_2, \omega_1, <, \ldots \rangle$ for a countable first-order language has an uncountable elementary submodel whose intersection with ω_1 is countable. In this section we shall prove the following theorem which explains the assumption of Theorem 2 in the context of $\mathrm{MA}_{\aleph_2}$.

THEOREM 3. *Under* $\mathrm{MA}_{\aleph_2}$, $\binom{\omega_2}{\omega_2} \longrightarrow \binom{\omega}{\omega}^{1,1}_{\omega}$ *is equivalent to Chang's Conjecture.*

The converse implication does not need $\mathrm{MA}_{\aleph_2}$ so we state it as a simple lemma.

https://doi.org/10.4153/CJM-1991-048-8 Published online by Cambridge University Press

LEMMA 3. *CC implies* $\binom{\omega_2}{\omega_2} \longrightarrow \binom{\omega}{\omega}^{1,1}_{\omega}$.

PROOF. Let $c\colon \omega_2 \times \omega_2 \longrightarrow \omega$ be a given partition and by CC choose an uncountable elementary submodel M of $\langle \omega_2, \omega_1, <, c\rangle$ such that $\delta = M \cap \omega_1$ is countable. Choose a k such that the set X_0 of all ξ in M such that $c(\delta, \xi) = k$ is uncountable. By the elementarity of M, for every finite $F \subseteq X_0$ there is arbitrarily large $\alpha < \delta$ such that $c(\alpha, \xi) = k$ for all ξ in F. So we can proceed as in the proof of Lemma 1 to construct increasing sequences $\{\alpha_i\} \subseteq \delta$ and $\{\xi_i\} \subseteq X_0$ such that $c(\alpha_i, \xi_j) = k$. This finishes the proof.

In the rest of this section we shall assume the negation of CC in order to construct a ccc poset $\mathcal{P}$ forcing a $c\colon \omega_2 \times \omega_2 \longrightarrow \omega$ which is not constant on the product of any two infinite subsets of ω_2. This will finish the proof of Theorem 3. We first need a Lemma that will reformulate the negation of CC to a convenient form. We shall need a convention that whenever we consider an a with domain $[\theta]^2$ for some θ then writing $a(\alpha, \beta)$ implicitly yields the fact $\alpha < \beta$. When we don't have any requirement on the ordering between α and β we use $a\{\alpha, \beta\}$ to denote either $a(\alpha, \beta)$ or $a(\beta, \alpha)$ depending whether $\alpha < \beta$ or $\beta < \alpha$.

LEMMA 4. *CC fails iff there is* $a\colon [\omega_2]^2 \longrightarrow \omega_1$ *such that for every uncountable disjoint family* $\mathcal{F}$ *of finite subsets of* ω_2 *and every* $\delta < \omega_1$ *there exist* $F \neq G$ *in* $\mathcal{F}$ *such that* $a\{\alpha, \beta\} > \delta$ *for every* α *in* F *and* β *in* G.

PROOF. Suppose $\mathcal{A} = \langle \omega_2, \omega_1, <, \ldots\rangle$ witnesses the negation of CC. Expanding $\mathcal{A}$, we may assume that it contains every integer as a predicate and also a function $b\colon [\omega_2]^2 \longrightarrow \omega_1$ such that $b(\cdot, \alpha)\colon \alpha \longrightarrow |\alpha|$ is one-to-one for all α. For $\alpha < \beta < \omega_2$ let $\mathcal{S}_{\alpha\beta}$ be the Skolem closure of $\{\alpha, \beta\}$ in $\mathcal{A}$, and let

$$a(\alpha, \beta) = \sup(\mathcal{S}_{\alpha,\beta} \cap \omega_1).$$

Note that $a(\alpha, \beta) \geq b(\alpha, \beta)$ for all α and β. Suppose that a does not satisfy the conclusion of the Lemma for some F_ξ, $(\xi < \omega_1)$, and δ. We may assume that F_ξ have the same fixed size n. Refining still further, we may assume $F_\xi(i)$, $(\xi < \omega_1)$, is increasing for all $i < n$. Let $\mathcal{U}$ be an uniform ultrafilter on ω_1. Then we can choose $i, j \leq n$ such that

(s) $a.e._{\mathcal{U}}\ \xi\ a.e._{\mathcal{U}}\ \eta\quad a\{F_\xi(i), F_\eta(j)\} \leq \delta$.

CASE 1. $\sup_\xi F_\xi(i) \leq \sup_\xi F_\xi(j)$. By (s) and the fact that $\mathcal{A}$ is a witness for the negation of CC there exist $\xi_0 < \cdots < \xi_k < \omega_1$ such that if $\alpha_\ell = F_{\xi_\ell}(i)$ for $\ell = 0, \ldots, k$, then $\delta < h(\alpha_0, \ldots, \alpha_k) < \omega_1$ for some Skolem function h of $\mathcal{A}$ and such that the set

$$X = \{\eta < \omega_1 : a\big(\alpha_\ell, F_\eta(j)\big) \leq \delta \text{ for all } \ell \leq k\}$$

is in $\mathcal{U}$. In particular, X is uncountable. Choose an η in X such that $\beta = F_\eta(j)$ is above α_k. Let α in $\{\alpha_0, \ldots, \alpha_k\}$ be such that $b(\alpha, \beta) \geq b(\alpha_\ell, \beta)$ for all $\ell \leq k$. Then $\alpha_0, \ldots, \alpha_k$ are all in $\mathcal{S}_{\alpha\beta}$, and so $a(\alpha, \beta) \geq h(\alpha_0, \ldots, \alpha_k) > \delta$, a contradiction.

CASE 2. $\sup_\xi F_\xi(i) > \sup_\eta F_\eta(j)$. By (s) there is ξ above $\sup_\eta F_\eta(j)$ such that

$$Y = \{\eta < \omega_1 : a\big(F_\eta(j), F_\xi(i)\big) \leq \delta\}$$

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is in $\mathcal{U}$. Since $b\left(\cdot, F_\xi(i)\right)$ is one-to-one there is η in Y such that $b\left(F_\eta(j), F_\xi(i)\right) > \delta$ But this contradicts the fact that a dominates b. (See the remark after the definition of a.) This finishes the proof.

From now on, we shall fix an a satisfying Lemma 4. Moreover we shall need another $e\colon [\omega_2]^2 \longrightarrow \omega_1$ such that for all $\alpha < \beta < \gamma < \omega_2$ (see [20; §2, p. 274, 1.18]):

(t) $e(\cdot, \alpha)\colon \alpha \longrightarrow \omega_1$ is countable-to-one

(u) $e(\alpha, \gamma) \leq \max\{e(\alpha, \beta), e(\beta, \gamma)\}$.

(v) $e(\alpha, \beta) \leq \max\{e(\alpha, \gamma), e(\beta, \gamma)\}$.

By increasing a we shall assume that $a(\alpha, \beta) \geq e(\alpha, \beta)$ for all α and β. Finally we are ready to define our poset $\mathcal{P}$ as the set of all $p\colon [D_p]^2 \longrightarrow \omega$, where D_p is a finite subset of ω_2, satisfying the following key condition with respect the two partitions a and e:

(w) If α, β and γ are three distinct elements of D_p with γ smaller than both α and β, then $e(\gamma, \alpha) > a(\alpha, \beta)$ implies $p(\gamma, \alpha) \neq p(\gamma, \beta)$.

(Note that if $e(\gamma, \alpha) > a(\alpha, \beta) \geq e(\alpha, \beta)$, the conditions (u) and (v) easily give $e(\gamma, \alpha) = e(\gamma, \beta)$, so there is no ambiguity in (w)). We let $p \leq q$ if p extends q and $p(\xi, \alpha) \neq p(\xi, \beta)$ for all $\alpha < \beta$ in D_q and $\xi\,(< \alpha)$ in $D_p \setminus D_q$. The crucial part of the proof is contained in the following claim.

CLAIM. *$\mathcal{P$ is a ccc poset.*

PROOF. Let $p_\xi, (\xi < \omega_1)$, be a given sequence of elements of $\mathcal{P}$. Let D_ξ be the domain of p_ξ. We may assume that D_ξ form a Δ-system with root D, and that for all ξ and η, p_ξ and p_η are isomorphic with respect to the order isomorphism of D_ξ and D_η which fixes D. Let F_ξ be the set of all $e(\alpha, \beta)$ for $\alpha < \beta$ in D_ξ. Refining further, we may assume that F_ξ form a Δ-system with root F. Choose $\delta < \omega_1$ above every element of F and every ordinal of the form $a(\alpha, \beta)$ for $\alpha < \beta$ in D. By removing countably many ξ, we may assume that $F_\xi \setminus F$ is above δ for all ξ. By the property of a from Lemma 4, there exist $\xi \neq \eta$ such that

$$a\{\alpha, \beta\} > \delta \text{ for } \alpha \in D_\xi \setminus D \text{ and } \beta \in D_\eta \setminus D.$$

Extend $p_\xi \cup p_\eta$ to a mapping $p\colon [D_\xi \cup D_\eta]^2 \longrightarrow \omega$ which is one-to-one on new pairs avoiding the old values. Checking the following two facts will finish the proof.

$p \leq p_\xi$ and $p \leq p_\eta$: By symmetry, we may check only $p \leq p_\xi$. Suppose $\alpha < \beta$ are in D_ξ and $\gamma < \alpha$ is in $D_\eta \setminus D_\xi = D_\eta \setminus D$. If one of α and β is in $D_\xi \setminus D, p(\gamma, \alpha) \neq p(\gamma, \beta)$ follows from the behaviour of p on new pairs. This means that we may assume that α and β are in D, and so we have to check that $p_\eta(\gamma, \alpha) \neq p_\eta(\gamma, \beta)$. This will follow from (w) if we show that $e(\gamma, \alpha) > a(\alpha, \beta)$. For this it suffices to show that $e(\gamma, \alpha)$ is in $F_\eta \setminus F$. Otherwise, since all p_ζ are isomorphic we would have that $e(D_\zeta(i), \alpha) \in F$ for all $\zeta < \omega_1$ where i is the place of γ in D_η. This contradicts the condition (t).

p satisfies (w): Suppose α, β and γ are three different elements of $D_\xi \cup D_\eta$ with γ smaller than both α and β such that $e(\gamma, \alpha) > a(\alpha, \beta)$. We have already noted that (u), (v) and the fact $a(\alpha, \beta) \geqq e(\alpha, \beta)$ give us the equality $e(\gamma, \alpha) = e(\gamma, \beta)$. We have to prove that $p(\gamma, \alpha) \neq p(\gamma, \beta)$. By the choice of p on a new pairs we may assume that

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$\{\gamma,\alpha\}$ and $\{\gamma,\beta\}$ are in $[D_\xi]^2 \cup [D_\eta]^2$. Since p_ξ and p_η both satisfy (w) the only case needed to be checked is when $\gamma \in D$ and (say) $\alpha \in D_\xi \setminus D$ and $\beta \in D_\eta \setminus D$. The equality $e(\gamma,\alpha) = e(\gamma,\beta)$ and the fact that $F_\xi \cap F_\eta = F$ show that $e(\gamma,\alpha)$ must be in F which is bounded by δ. But $a\{\alpha,\beta\} > \delta$, a contradiction. This finishes the proof that $\mathcal{P}$ is a ccc poset.

Forcing internally by $\mathcal{P}$, we get a $g\colon [\omega_2]^2 \longrightarrow \omega$ such that

$$\{\xi < \alpha : g(\xi,\alpha) = g(\xi,\beta)\}$$

is finite for all $\alpha < \beta < \omega_2$. Let $c(\alpha,\beta) = 2g(\alpha,\beta)+2$ for $\alpha < \beta$, $c(\alpha,\beta) = 2g(\alpha,\beta)+1$ for $\alpha > \beta$, and $c(\alpha,\beta) = 0$ for $\alpha = \beta$. It is straightforward to check that c cannot be constant on the product of any two infinite subsets of ω_2. This completes the proof.

By comparing [0; pp. 593–4] and [7; p. 163] we may conclude that some of the arguments from the proof of Theorem 3 must have been known to Baumgartner and Taylor,and Galvin. For example, Theorem 3 (or its proof) seems to be relevant to some questions asked in [0; p. 607]. Functions $a\colon [\omega_2]^2 \longrightarrow \omega_1$ with the unboundedness property of Lemma 4 have been also considered in some other places in the literature (see [15],[13]). For example, in [15; p. 141] Galvin constructed such an a starting from a chain in $\omega_1^{\omega_1}$, $<^*$ of type ω_2+1, an assumption that is stronger than the negation of CC. Theorem 2.10 of [13] claims that we even have an equivalence in Galvin's Theorem provided the partition a has the following stronger unboundedness property: for every uncountable disjoint family $\mathcal{F}$ of finite subsets of ω_2 and for every $\delta < \omega_1$ there exist uncountable $\mathcal{G} \subseteq \mathcal{F}$ such that for every $F \neq G$ in $\mathcal{G}$, $a\{\alpha,\beta\} > \delta$ for α in F and β in G. Note that in the context of $MA_{\aleph_1}$ the stronger unboundedness property is in fact equivalent to the unboundedness property of Lemma 4 since the poset of finite approximation to $\mathcal{G}$ is ccc. Hence, looking at the model of $MA_{\aleph_1}$ in which CC is false but in which $\omega_1^{\omega_1}$, $<^*$ does not have chains of type ω_2+1, we see that Theorem 2.10 of [13] contradicts our Lemma 4. The model can be obtained by adding, in the standard way, a $\square$-sequence to a model of CC. In fact, there is no need to further force $MA_{\aleph_1}$ (though, we can) since it is easily shown that if $\langle C_\alpha : \alpha < \omega_2\rangle$ is a $\square$-sequence and if $\rho\colon [\omega_2]^2 \longrightarrow \omega_1$ is defined (see [20; §2]) recursively by

$$\rho(\alpha,\beta) = \sup\Big\{\mathrm{tp}(C_\beta \cap \alpha), \rho\big(\alpha, \min(C_\beta \setminus \alpha)\big), \rho(\xi,\alpha) : \xi \epsilon C_\beta \cap \alpha\Big\}$$

($\rho(\alpha,\alpha) = 0$ for all α), then ρ already has the stronger unboundedness property.

The problem of forcing $\binom{\theta}{\theta} \not\rightarrow \binom{\omega}{\omega}_\omega^{1,1}$ for other values of θ is an interesting technical problem which does not seem to have been considered in the literature.

4. **A form of the Continuum Hypothesis.** In this section we consider the following consequence of CH:

(Σ) *Every ccc poset of size at most $\mathfrak{c}$ is the union of $\aleph_1$ centered subsets.*

A formulation of Σ in terms of ccc partitions has the following form.

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(Σ) *For every ccc partition $[X]^{<\omega} = K_0 \cup K_1$ on a set X of size at most $\mathfrak{c}$, the set X is the union of $\aleph_1$ 0-homogeneous subsets.*

By considering only partitions of some fixed finite power $[X]^m$, the corresponding weaker (and more interesting) statement will be denoted by Σ^m. The statement Σ has a strong influence on the reals very similar to that of CH. We shall now list some of the consequences using known ccc partitions. In every case we shall reproduce a short definition of the partition because of a later analysis of its definability properties. In what follows, $\mathbb{R}$ denotes the Baire space ω^ω.

(14)Σ^2*implies* $\mathfrak{b} = \omega_1$.

Choose an $<^*$-increasing and $<^*$-unbounded sequence $A = \{f_\xi : \xi < \mathfrak{b}\}$ of elements of ω^ω. Assume that every f_ξ is increasing. For f and g in A, set $f \leq g$ if $f(i) \leq g(i)$ for all i. We shall need the following two facts from [19; Lemmas 13 and 16]

(x) The poset of all finite pairwise $\leq$-incomparable subsets of A is ccc.

(y) A contains no subset of size $\mathfrak{b}$ of pairwise $\leqq$-incomparable elements.

The partition $[A]^2 = K_0 \cup K_1$ is defined by letting $\{f, g\}$ in K_0 if they are $\leq$-incomparable. The statement Σ^2 and the property (y) imply that $\mathfrak{b}$ must be equal to ω_1.

(15)Σ^3 *implies that there is an ω_1-scale in ω^ω.*

For this we define $[\mathbb{R}]^3 = K_0 \cup K_1$ by: $\{f, g, h\} \in K_0$ if two of the integers $\Delta(f, g)$, $\Delta(f, h)$ and $\Delta(g, h)$ are different. Then (see [19; § 3, Theorem 7]) this is a ccc partition with the property that every 0-homogeneons set is bounded in ω^ω. Hence Σ^3 gives us a subset of ω^ω of size $\aleph_1$ which bounds every element of ω^ω.

(16) *Σ implies that every $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is the union of $\aleph_1$ continuous subfunctions. Hence the continuum has cofinality ω_1.*

Define $[\mathbb{R}]^3 = K_0 \cup K_1$ by letting $\{x, y, z\}$ in K_0 if for every a, b, c in $\{x, y, z\}$,

$$\Delta(a, c) < \Delta(b, c) \text{ implies } \Delta(f(a), f(c)) < \Delta(f(b), f(c)).$$

(Recall that $\Delta(x, x) = \infty$ for all x). Then (see [19; p. 366]) this is a ccc partition with the property that f is continuous on every 0-homogeneous subset of $\mathbb{R}$. This gives (16).

It would be of interest to find more applications of Σ^m for some finite m, and in particular of Σ^2.

(17) *Σ implies that every nonprincipal ultrafilter $\mathcal{U}$ on ω has π-character $\aleph_1$ i.e., there is a family $\mathcal{F}$ of size $\aleph_1$ of infinite subsets of ω such that every element of $\mathcal{U}$ includes one of $\mathcal{F}$.*

For a finite subset F of $\mathcal{U}$, let a_F be the intersection of F and let Δ_F be the set of all $\min(a \Delta b)$ for a and b in F. The partition $[\mathcal{U}]^{<\omega} = K_0 \cup K_1$ is defined by letting an F in K_0 if $a_F \cap k$ has size at least the size of $\Delta_F \cap k$ for all $k < \omega$. Then (see [22;

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Theorem 1.3]) this is a ccc partition with the property that every 0-homogeneous set has infinite intersection. This gives (17). A similar idea applied to the family of all dense open subsets of $\{0,1\}^{<\omega}$ (ordered by the reverse inclusion) gives the following fact.

(18) *Σ implies that there is a family $\mathcal{F}$ of $\aleph_1$ nowhere dense sets of reals such that every nowhere dense set is included in a member of $\mathcal{F}$.*

To see this, let O be the family of all dense open subsets of $\{0,1\}^{<\omega}$ and for $O \neq P$ in O let $\Delta(O,P)$ be the minimal n such that $O \cap \{0,1\}^n \neq P \cap \{0,1\}^n$. For $X \subseteq O$ let O_X be the intersection of X, and let Δ_X be the set of all $\Delta(O,P)$ for $O \neq P$ in X. Finally define $[O]^{<\omega} = K_0 \cup K_1$ by letting a finite $X \subseteq O$ in K_0 iff for all $n < m$ in Δ_X and s in $\{0,1\}^n$ there is t in $\{0,1\}^m \cap O_X$ such that $s \subseteq t$. Then as in (17) one shows that the defined partition is ccc and has the property that O_X is dense open for every 0-homogeneous X. Applying Σ to $[O]^{<\omega} = K_0 \cup K_1$ gives us (18).

(19) *Σ implies that the measure algebra has a dense subset $\mathcal{F}$ of size $\aleph_1$ and there is a family $\mathcal{G}$ of $\aleph_1$ null sets such that every null set is included in an element of $\mathcal{G}$.*

To see this, let $\mathcal{A}_\varepsilon$ be the family of all compact sets of reals of measure $> \varepsilon$ (for some fixed $\varepsilon > 0$). Put a finite subset F of $\mathcal{A}_\varepsilon$ in K_0 if its intersection has measure $> \varepsilon$. Then $[\mathcal{A}_\varepsilon]^{<\omega} = K_0 \cup K_1$ is a ccc partition (see [12; pp. 167–8]), and applying Σ we get a family $\mathcal{F}$ of $\aleph_1$ compact positive sets such that every positive set includes a member of $\mathcal{F}$. Let $\mathcal{G}$ be the family of complements of sets of the form $\mathbb{Q} + F$ for F in $\mathcal{F}$. Then every null set is included in a member of $\mathcal{G}$.

This means that Σ gives us Luzin and Sierpinski sets of reals and other similar consequences of CH. The purpose of this section is not only to list applications of Σ to known ccc partitions but also to give the following result involving ccc partitions of a quite different kind.

THEOREM 4. *Σ implies* $\binom{\omega_2}{\omega_2} \longrightarrow \binom{\omega}{\omega}^{1,1}_{\omega}$.

PROOF. Suppose $\binom{\omega_2}{\omega_2} \not\longrightarrow \binom{\omega}{\omega}^{1,1}_{\omega}$. Then $\mathfrak{c} \geq \omega_2$ so $\omega_2 \stackrel{ccc}{\longrightarrow} (\omega_2, \omega_1)^{<\omega}$ is a consequence of Σ. By Theorem 2, $MA_{\aleph_2}$ holds for powerfully ccc posets. This means that the dense set of the measure algebra from (19), and therefore the whole measure algebra, is σ-centered, a contradiction. This finishes the proof.

COROLLARY 1. *If $O^\#$ does not exist then Σ implies that there is no inner model satisfying* $MA_{\aleph_2}$.

PROOF. Suppose there is an inner model M such that $\omega_2^M = \omega_2$ and $M \models MA_{\aleph_2}$. By Theorem 4 and upwards absoluteness, we must have $M \models \binom{\omega_2}{\omega_2} \longrightarrow \binom{\omega}{\omega}^{1,1}_{\omega}$. By Theorem 3, we have $M \models$ CC and, in particular, $M \models O^\#$ exists (see [6; p. 212]). Since $O^\#$ is upward absolute (see [6; p. 138]), this completes the proof.

It is still open whether Σ is in fact equivalent to CH. The Corollary 1 gives some restrictions and limits to a possible model of Σ and non-CH. Such a model-candidate

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will be discussed in the next section where we consider some stronger forms of Σ. The present form of Σ has its roots in the following problem of A. Blaszczyk:

(20) Is there a compact ccc space of weight and density equal to $\mathfrak{c}$?

By considering a basis of a given compact space as a poset ordered by reverse inclusion, a positive answer to (20) would mean that Σ is equivalent to CH. A negative answer to (20) would, in particular, mean that there is a model of set theory and a cardinal $\theta < \mathfrak{c}$ such that every productively ccc poset of size at most $\mathfrak{c}$ is the union of θ centered subsets. Assuming θ to be minimal possible (i.e., equal to ω_1) does not seem to make a difference.

5. **The Luzin-set Axiom.** Let X be a topological space and let S be a subset of X. Then S is a *Luzin subset* of X if S is uncountable but $S \cap N$ is countable for every nowhere dense $N \subseteq X$. In [3], van Douwen and Fleissner have considered the following consequence of CH:

(21) *Every compact ccc space of π-weight at most $\mathfrak{c}$ has a Luzin subset.*

By looking at the Stone space of the regular open algebra of a given ccc poset it follows that the statement Σ of §4 is an immediate consequence of (21). In order to make (21) useful one needs to reformulate it in terms of partially ordered sets. The first such attempt was made in [3] and [14] where the following form has been suggested (and denoted by UFA):

(22) *Every ccc poset $\mathcal{P}$ of size at most $\mathfrak{c}$ has a Luzin family of filters, i.e., an uncountable family of filters such that every dense open subset of $\mathcal{P}$ intersects all but countably many of them.*

The following fact shows that (22) may not be the right formulation of (21).

PROPOSITION 1. (22) *is equivalent to CH.*

PROOF. It should be clear that (22) is a consequence of CH so let us prove the converse. For this we shall use the $\mathfrak{b}$-sequence A of (14) and its ccc partition $[A]^2 = K_0 \cup K_1$. We shall assume that, moreover, A has the following property:

(z) For every finite 0-homogeneous subset B of A there are arbitrarily large $\xi < \mathfrak{b}$ such that $B \cup \{f_\xi\}$ is still 0-homogeneous.

It is easily seen that such an A can be obtained either by an inductive construction or by refining the A of (14). Fix also an $e\colon [\mathfrak{b}^+]^2 \longrightarrow \mathfrak{b}$ such that $e(\cdot, \alpha)\colon \alpha \longrightarrow \mathfrak{b}$ is one-to-one for all α. Let $\mathcal{P}$ be the set of all pairs $p = \langle B_p, F_p \rangle$, where B_p is a finite 0-homogeneous subset of A and F_p is a finite subset of $\mathfrak{b}^+$ such that for every $\alpha < \beta$ in F_p there is an f_ξ in B_p such that $\xi \geq e(\alpha, \beta)$. We let $p \leq q$ iff $B_p \supseteq B_q$ and $F_p \supseteq F_q$. A standard Δ-system argument plus the properties (x) and (z) of A show that $\mathcal{P}$ is a ccc poset. If (22) applies to $\mathcal{P}$ it would give us, in particular, a filter $\mathcal{F} \subseteq \mathcal{P}$ of size $\mathfrak{b}^+$. By the definition of $\mathcal{P}$, the union of B_p for p in $\mathcal{F}$ will be a 0-homogenous subset of A of size $\mathfrak{b}$ contradicting (y).

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This means that $\mathfrak{b} = \mathfrak{c}$. Applying (22) to the poset of all finite 0-homogeneous subsets of A, we get $\mathfrak{b} = \omega_1$. This completes the proof.

An examination of the above poset $\mathcal{P}$ shows that one of the reasons it cannot have large filters is that two compatible elements of $\mathcal{P}$ may not have the greatest lower bound ($g\ell b$) in $\mathcal{P}$. Another (deeper) reason for this is that providing an extension of two compatible elements of $\mathcal{P}$ needs a reference to two quite complex objects A and e. Note that $\mathcal{P}$ does have large centered subsets which suggests that the following might be the right formulation of (21) in terms of partially ordered sets.

(23) *Every ccc poset $\mathcal{P}$ of size at most $\mathfrak{c}$ has a Luzin family of centered subsets i.e., an uncountable family of centered sets such that every dense open subset of $\mathcal{P}$ intersects all but countably many of them.*

Note that if $\mathcal{P}$ is closed under $g\ell b$ it does not really matter whether in (23) we talk about filters or centered sets. However, it should be noted that many interesting posets are not closed under $g\ell b$. The proof (via the Stone space of the regular open algebra of $\mathcal{P}$) that (21) and (23) are equivalent statements is left to the reader. They will be from now on called the *Luzin-set Axiom.* A typical application of this axiom (which we do not know to be also a consequence of Σ) is the following construction reproduced here from [3; 4.6]. Let $\mathbb{R}$ be the Baire space ω^ω with an open countable basis $\mathcal{B}$ closed under finite unions and let X be a subset of $\mathbb{R}$. Let $\mathcal{P}$ be the set of all pairs $\langle B, F\rangle$ where B is an element of $\mathcal{B}$ and F is a finite subset of $\mathbb{R}\setminus X$ such that $B \cap F = \emptyset$. Set $\langle A, F\rangle \leq \langle B, G\rangle$ iff $A \supseteq B$ and $F \supseteq G$. Let $\mathcal{P}^*$ be the set of all p in $\mathcal{P}^\omega$ such that $p(i) = \langle \emptyset, \emptyset\rangle$ for all but finitely many i, ordered coordinatewise. Then $\mathcal{P}^*$ is a σ-centered poset such that the existence of a Luzin family of centered subsets of $\mathcal{P}^*$ yields that X is the union of $\aleph_1$ G_{ω_1} subsets of $\mathbb{R}$ (and also the intersection of $\aleph_1$ F_{ω_1} subsets of $\mathbb{R}$). Thus, we have

(17) *The Luzin-set Axiom for σ-centered posets implies that every set of reals is ω_1-Borel. Hence, the Luzin-set Axiom for σ-centered posets implies that $2^{\mathfrak{c}} = 2^{\aleph_1}$.*

The problem with $g\ell b$ encountered in Proposition 1 also occurs in the notions of *definable* or *weakly definable* posets discussed in [3]. For this reason we shall rewrite and supplement the discussion of [3]. The *definable* of [3] seems to mean that the domain, the ordering, and the incomparability relation of a given poset $\mathcal{P}$ are all definable by absolute formulas. If the condition is so restrictive, the natural model (see below) does indeed show the consistency with non-CH of the corresponding Luzin-set Axiom even in its strong form (22). The number of such posets is small ($\leq \mathfrak{c}$) which makes the corresponding axiom rather weak (though still useful; see [3]). It seems more reasonable to drop the requirement that the domain of the poset is definable. This would increase the number of posets (to $2^{\mathfrak{c}}$) and make the corresponding Luzin-set Axiom much stronger. (A similar evolution of an axiom occurred in the case of the Open Coloring Axiom (see [21]) which in the unrestricted form turned out to be quite useful.) However, if we remove the restriction that the domain of a poset $\mathcal{P}$ is definable the problem of filter versus

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centered set does occur so we can expect only the form (23) of the axiom. More important than this is the fact that it does not seem to be sufficient (as suggested in [3]) that only the two place compatibility relation is definable, we need definability of the n-place compatibility relation $\mathrm{Cmp}^n_{\mathcal{P}}(p_0, \ldots, p_{n-1})$ (meaning that there is a p such that $p \leq p_i$ for all $i < n$) for all $2 \leq n < \omega$. Thus we shall say that a poset

$$\mathcal{P} = \langle P, \leqq_{\mathcal{P}}, \mathrm{Cmp}^n_{\mathcal{P}} \rangle_{n<\omega}$$

is definable iff there exist $\leqq$ and $\mathrm{Cmp}^n, (n < \omega)$, in $L(\mathbb{R})$ (or, more generally, in $\mathrm{HOD}({}^{\omega}\mathrm{Ord})$) such that

$$\leq_{\mathcal{P}} = \leq \upharpoonright P \text{ and } \mathrm{Cmp}^n_{\mathcal{P}} = \mathrm{Cmp}^n \upharpoonright P^n$$

for all n. The corresponding statement (23) will be called the *Definable Luzin-set Axiom*. Note that formulating the definable form of Σ i.e., defining the notion of a definable partition of the form $[X]^{<\omega} = K_0 \cup K_1$ (or $[X]^m = K_0 \cup K_1$) is much simpler and natural (much in the spirit of OCA). All we have to say is that there is $\hat{K}_0$ in $L(\mathbb{R})$ (or in $\mathrm{HOD}({}^{\omega}\mathrm{Ord})$) such that $K_0 = \hat{K}_0 \cap [X]^{<\omega}$ (or $\hat{K}_0 \cap [X]^m$). The reader should not have any difficulty in checking that the partitions of (14)–(19) are all definable by very simple formulas. In fact, under the right interpretations of X as a set of reals, all of them involve open $\hat{K}_0$. Note also that (24) involves a definable poset. In fact, every σ-centered poset $\mathcal{P}$ can naturally be represented to have domain a set of reals while $\leq_{\mathcal{P}}$ and $\mathrm{Cmp}^n_{\mathcal{P}}, (n < \omega)$, are restrictions of Borel relations of $\mathbb{R}$. This means that (14)–(19) are consequence of the definable form of Σ and that (24) is the consequence of the definable form of the Luzin-set axiom (23). (The reader may also check (see [3]) that applying the definable Luzin-set axiom to more standard posets one can get (14)–(19) with less work.) Another class of posets $\mathcal{P}$ that allow to be represented with sets of reals as their domains and with $\leq_{\mathcal{P}}$ and $\mathrm{Cmp}^n_{\mathcal{P}}$ as restrictions of G_δ relations of $\mathbb{R}$ is the class of all posets of size at most $\mathfrak{p}$. This is an immediate consequence of the well-known fact that every relation on $\mathfrak{p}$ allows an almost disjoint coding (see [12] and [4; 21F and 21Ne]). The posets involved in the proof of Theorem 4 are not definable in the above sense. In fact, we shall now see that $\binom{\omega_2}{\omega_2} \rightarrow \binom{\omega}{\omega}^{1,1}_{\omega}$ is not a consequence of the definable form of the axiom. This follows from Corollary 1 and the proof of the following result which should provide a good test of the reader's understanding of the above definitions and which should be compared with the corresponding discussion in [3; §6].

THEOREM 5. *The Definable Luzin-set Axiom and the negation of the Continuum Hypothesis are consistent relative to the consistency of a weakly compact cardinal.*

PROOF. Let our ground model be the constructible universe where we fix a weakly compact cardinal κ. Let $\mathrm{Coll}\,(\omega, < \kappa)$ be the standard poset that collapses κ to ω_1 and in its forcing extension let $\dot{Q}_\alpha, (\alpha \leq \omega_1)$, be the standard finite-support ccc iteration such that $\dot{Q}_0 = \{\emptyset\}$ and $\dot{Q}_\alpha \models \mathrm{MA}\ \&\ \mathfrak{c} = \aleph_{\alpha+2}$ for $0 < \alpha < \omega_1$. To simplify the notation, let V^α denote the forcing extension of $\mathrm{Coll}\,(\omega_1, < \kappa) * \dot{Q}_\alpha$. Thus, V^0 is the forcing extension of $\mathrm{Coll}(\omega_1, < \kappa)$.

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Suppose we have a definable ccc poset $\dot{\mathcal{P}}$ in V^{ω_1}. Let φ and $\varphi_n, (n < \omega)$, be formulas with parameters either reals or ordinals such that for $\dot{p}$ and $\dot{q}$ in the domain $\dot{P}$ of $\dot{\mathcal{P}}$,

$$\dot{p} \leq_{\dot{\mathcal{P}}} \dot{q} \text{ iff } \mathrm{L}(\dot{\mathbb{R}}) \models \varphi(\dot{p}, \dot{q}, \ldots),$$

and also that for every $2 \leq n < \omega$ and $\dot{p}_0, \ldots, \dot{p}_{n-1}$ in $\dot{P}$,

$$\dot{p}_0, \ldots, \dot{p}_{n-1} \text{ are compatible in } \dot{\mathcal{P}} \text{ iff } \mathrm{L}(\mathbb{R}) \models \varphi_n(\dot{p}_0, \ldots, \dot{p}_n, \ldots).$$

By weak compactness of κ the names of parameters occuring in φ and $\varphi_n, (n < \omega)$ can be included in a regular subposet of Coll $(\omega, < \kappa) * \dot{Q}_{\omega_1}$ of size $< \kappa$. So, factoring we may assume that all parameters are in the ground model. Now we shall need the following well-known result of Kunen (see [18], [5]) which is stated here in the form that will be used. (The $\mathcal{S}$ below can be any κcc poset that collapses κ to ω_1.)

LEMMA 5. *If κ is a weakly compact cardinal then for every κcc poset $\mathcal{S}$ of the form Coll $(\omega, < \kappa) * \dot{Q}$, the $L(\dot{\mathbb{R}})$ of $\mathcal{S}$ and the $L(\dot{\mathbb{R}})$ of Coll $(\omega, < \kappa)$ satisfy the same sentences with parameters from the ground model.*

We are now ready to finish the proof of Theorem 5. The domain $\dot{P}$ of $\dot{\mathcal{P}}$ may not be definable but it is certainly included in a set from $L(\dot{\mathbb{R}})$ i.e., the field of the relation defined by φ. (For concreteness assume that $\dot{P}$ is, in fact, a subset of $\dot{\mathbb{R}}$.) Let $\dot{P}_\alpha = \dot{P} \cap V^\alpha$ for $0 < \alpha < \omega_1$. In V^α, let $\dot{\mathcal{R}}_\alpha$ be the poset of all finite subsets $\{\dot{p}_0, \ldots, \dot{p}_{n-1}\}$ of $\dot{P}_\alpha$ such that $\varphi_n(\dot{p}_0, \ldots, \dot{p}_{n-1}, \ldots)$ is satisfied by $L(\dot{\mathbb{R}})$ of V^α. The ordering of $\dot{\mathcal{R}}_\alpha$ is the reverse inclusion. Note that

$$V^\alpha \models \mathrm{L}(\dot{\mathbb{R}}) \models \varphi_n(\dot{p}_0, \ldots, \dot{p}_{n-1}, \ldots) \text{ iff}$$
$$V^\beta \models \mathrm{L}(\dot{\mathbb{R}}) \models \varphi_n(\dot{p}_0, \ldots, \dot{p}_{n-1}, \ldots)$$

for all β such that $\alpha \leq \beta \leq \omega_1$. To see this, first absorb $\dot{p}_0, \ldots, \dot{p}_{n-1}$ in a small ($< \kappa$) forcing extension and then apply Lemma 5. This means that $\dot{\mathcal{R}}_\alpha \subseteq \dot{\mathcal{R}}_\beta$ for $\alpha \leq \beta$ and that every $\dot{\mathcal{R}}_\alpha$ is a ccc poset being a subset of the ccc poset $\dot{\mathcal{R}}_{\omega_1}$ of all finite consistent subsets of $\dot{\mathcal{P}}$. Applying MA in $V^{\alpha+1}$ to $\dot{\mathcal{R}}_{\alpha+1}$ and the family of all maximal antichains of $\dot{\mathcal{R}}_{\alpha+1}$ which happen to be in V^α gives us a filter $\dot{\mathcal{F}}_\alpha$ of $\dot{\mathcal{R}}_{\alpha+1}$. Then $\{\cup \dot{\mathcal{F}}_\alpha : \alpha < \omega_1\}$ is a Luzin set of centered subsets of $\dot{\mathcal{P}}$ since every maximal antichain of $\dot{\mathcal{P}}$ occurs in some V^α for $\alpha < \omega_1$. This finishes the proof.

The crucial difficulty encountered in the above proof (which does not occur for the more restrictive notion of *definable* in [3]) is that if we have, for example, $\dot{p}_0$ and $\dot{p}_1$ in $\dot{P}_\alpha$ which are compatible in $\dot{\mathcal{P}}$ i.e., for which $\varphi_2(\dot{p}_0, \dot{p}_1, \ldots)$ holds in $L(\dot{\mathbb{R}})$ then finding the common extension of $\dot{p}_0$ and $\dot{p}_1$ involves reference to the indefinable object $\dot{P}$. Thus, the restriction of $\dot{\mathcal{P}}$ to $\dot{P}_\alpha$ may not be a poset at all! This explains the need for $\dot{\mathcal{R}}_\alpha$ whose definition uses the compatibility relations of $\dot{\mathcal{P}}$ in all dimensions. Note also that while $\dot{\mathcal{F}}_\alpha$ is a filter of $\dot{\mathcal{R}}_{\alpha+1}$, in general, its union is only a centered subset of $\dot{\mathcal{P}}$.

Note that in the proof of Theorem 5 the fact that the ordering $\leqq_{\dot{\mathcal{P}}}$ of our poset $\dot{\mathcal{P}}$ is definable has not been used, so the definition of a definable poset can be relaxed slightly. However, at the moment we don't see any advantage of this change.

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Since $\mathbf{\Sigma}^1_2$-statements are absolute ([17]) the weakly compact cardinal (and the first extension by Coll $(\omega, < \kappa)$) is not needed in showing that Q_{ω_1} forces the Luzin-set axiom for $\mathbf{\Sigma}^1_2$-definable posets. Note that many standard ccc posets do belong to this class. For example, the partitions (posets) of (14)–(19) as well as all σ-centered posets (e.g., (24)) and all *ccc* posets of size at most $\mathfrak{p}$ belong to this class. Unfortunately, the iteration $Q_\alpha, (\alpha < \omega_1)$, can never force the full Luzin-set Axiom (23) nor even the statement Σ of § 4 regardless of the ground model we start with. This has been first shown by Merrill [13] exploiting heavily the fact that this is a finite-support iteration which therefore adds cofinally many Cohen reals. Note that Corollary 1 is saying that if $O^\#$ does not exist then no matter how you iterate as long as you preserve ω_2 and have sufficient amount of MA in at least one of the intermediate models the resulting limit will fail to force Σ. This makes it quite unlikely that one can produce a model of Σ plus non-CH without using a substantial large cardinal assumption.

6. **Questions.** The following is a list of open problems concerning the subject matter of this paper. It should be noted that almost all of them are well-known open problems scattered through the literature under various terminologies. Of course, they vary in importance and difficulty and generally they are less technical than they appear to be.

(25) Is MA equivalent to MA restricted to powerfully ccc posets?

(26) Let $\mathcal{U}$ be a family of infinite sets of integers closed under finite intersections. Does there exist a ccc partition $[\mathcal{U}]^2 = K_0 \cup K_1$ (or $[\mathcal{U}]^m = K_0 \cup K_1$ for some finite m) such that 0-homogeneous sets have infinite intersections?

(27) Is Σ equivalent to CH?

(28) Does Σ imply Chang's Conjecture if CH fails?

(29) Does Σ imply $2^{\aleph_1} = 2^{\mathfrak{c}}$?

(30) Do we need a weakly compact cardinal for the consistency of the definable Luzin-set Axiom with non-CH?

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