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RADICALS AND IDEMPOTENTS III: q-CENTRAL IDEMPOTENTS

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Abstract

Previously ['Radicals and idempotents I, II', *Comm. Alg.* **49**(1) (2021), 73–84 and **50**(11) (2022), 4791–4804], we have studied the interaction between radicals of rings and idempotents in general or those of particular types, for example, left semicentral. Here we carry out similar investigations for q-central idempotents, that is, those idempotents e satisfying the condition $(ea - eae)(be - ebe) = 0$ for all a, b.

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1. Introduction

In [2], we studied connections between radicals and *corners*, subrings of the form eAe where e is an idempotent of a ring A, including corner-hereditary radical classes, the radical classes $\mathcal{R}$ satisfying the condition

$$e^2 = e \in A \in \mathcal{R} \Rightarrow eAe \in \mathcal{R}$$

and corner-strict $\mathcal{R}$, those satisfying

$$(e^2 = e \in A \;\&\; eAe \in \mathcal{R}) \Rightarrow eAe \subseteq \mathcal{R}(A).$$

In [3], we investigated relative versions of these notions: hereditariness or strictness for corners defined by idempotents of some specified kind, for example, left semicentral idempotents. We also examined radical classes defined by the presence or absence of idempotents of various kinds.

In the present paper we revisit some of these concerns in the context of *q-central idempotents* as defined by Lam [5] for rings with identity. The q-central property is weaker than that of being left or right semicentral, but it is easily seen that all radical classes are hereditary for corners defined by q-central idempotents e since the map

$$A \to eAe : \quad a \mapsto eae \quad \text{for all } a$$

is a homomorphism (endomorphism) of the ring A containing e precisely when e is q-central.

We also investigate the property HPC (hereditary for phantom corners) of radical classes $\mathcal{R}$ such that (in the notation of the Peirce decomposition)

$$(A \in \mathcal{R} \,\&\, A_{11} = eAe \in \mathcal{R}) \Rightarrow A_{22} \in \mathcal{R}.$$

This property also has relative forms such as HPC for left semicentral idempotents.

In [3], it was shown that *normal radicals* have HPC and all radical classes have HPC for left or right semicentral idempotents. Here, we show that all radical classes have HPC for what we call *Peirce trivial idempotents* (definition in the next section).

Now Peirce trivial is a stronger property than q-central and we have been unable to determine whether or not every radical class has HPC for q-central idempotents. However, in the final section, we present the first example (to our knowledge) of a radical class which lacks HPC.

Our notation and terminology are generally standard. Unexplained notions from radical theory can be found in [4]. Note though that as usual, we denote classes by upper case script rather than lower case Greek letters. The symbols $\triangleleft$, $<_l$ and $<_r$ denote ideals, left ideals and right ideals, respectively. The following notation is used for particular radical classes:

$$\mathcal{B}: \text{ prime}; \quad \mathcal{N}: \text{ nil}; \quad \mathcal{G}: \text{ Brown–McCoy}; \quad \mathcal{T}: \text{ torsion}.$$

2. q-central idempotents

Lam [5] calls an idempotent e of a ring A with identity *q-central* if it satisfies the condition

$$eA(1 - e)Ae = 0.$$

(Such idempotents are called *inner-Peirce-trivial* in [1].) The paper [5] contains several characterisations of q-central idempotents, some of which are related to the Peirce decomposition and can be extended without difficulty from rings with identity to rings in general.

The condition $eA(1 - e)Ae = 0$ is equivalent to

$$eA(1 - e) \cdot (1 - e)Ae = 0.$$

The two factors here are terms in the Peirce decomposition and we shall use the nonunital analogue of the second equation in our definition of q-central idempotents.

We first recall the Peirce decomposition. If e is an idempotent in a ring A, let

$$A_{11} = eAe; \quad A_{12} = \{ea - eae : a \in A\};$$
$$A_{21} = \{ae - eae : a \in A\}; \quad A_{22} = \{a - ea - ae + eae : a \in A\}.$$

Then A is the *abelian group* direct sum of A_{11}, A_{12}, A_{21} and A_{22}, and the multiplicative relationships between A_{11}, A_{12}, A_{21} and A_{22} ensure the partial matrix ring $\left[\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right]$ is well defined, and, moreover, is isomorphic to A. Both the additive representation

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and the matrix form can be called the Peirce decomposition of A with respect to e. In what follows, we shall always use the A_{ij} notation $(i, j = 1, 2)$ to describe Peirce decomposition and our use of the same notation for all idempotents should not be seriously ambiguous.

We define an idempotent e of a ring A to be *q-central* if

$$A_{12}A_{21} = 0.$$

This is a good point to show the relationship between q-central idempotents and some other types of idempotents. We have the following characterisation:

$$\begin{aligned} q\text{-central} &: A_{12}A_{21} = 0; \\ \text{left semicentral} &: A_{12} = 0; \\ \text{right semicentral} &: A_{21} = 0; \\ \text{central} &: A_{12} = 0 \text{ and } A_{21} = 0. \end{aligned}$$

It will be observed that

$$\text{central} \Rightarrow \text{left semicentral} \Rightarrow q\text{-central}$$

and

$$\text{central} \Rightarrow \text{right semicentral} \Rightarrow q\text{-central}.$$

Indeed, the motivation in [5] for the term 'q-central' is that it is a weakening of 'semicentral' and could thus be thought of as 'quarter-central'.

The next few results contain further characterisations of q-central idempotents, analogous to unital results in [1, 5].

THEOREM 2.1. *For an idempotent e in a ring A, the following are equivalent:*

(i) *e is q-central;*
(ii) *the map $A \to eAe : a \mapsto eae$ is a ring homomorphism;*
(iii) *$eabe = eaebe$ for all $a, b \in A$.*

PROOF. Since $eae \cdot ebe = eaebe$, items (ii) and (iii) are equivalent.

(i) $\Rightarrow$ (ii). For all $a, b \in A$,

$$\begin{aligned} 0 = (ea - eae)(be - ebe) &= eabe - eaebe - eaebe + eaebe \\ &= eabe - eaebe = eabe - eae \cdot ebe. \end{aligned}$$

This argument is reversible, so also (ii) $\Rightarrow$ (i). □

Adapting the terminology used in [1] for rings with identity, we say that an idempotent e in a ring A is *Peirce trivial* if

$$A_{12}A_{21} = 0 \quad \text{and} \quad A_{21}A_{12} = 0.$$

In rings with identity, e is *Peirce trivial* if and only if e and $1 - e$ are both q-central.

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THEOREM 2.2. *For an idempotent e in a ring A, the following conditions are equivalent:*

(i) *e is q-central;*
(ii) *A_{12} is a right ideal of A;*
(iii) *A_{21} is a left ideal of A.*

PROOF. (i) ⇔ (ii). We have

$$A_{12}A_{11} = 0; \quad A_{12}A_{12} = 0; \quad A_{12}A_{22} \subseteq A_{12}; \quad A_{12}A_{21} \subseteq A_{11}.$$

Hence, A_{12} is a right ideal if and only if $A_{12}A_{21} \subseteq A_{12}$. However, $A_{11} \cap A_{12} = 0$ so A_{12} is a right ideal if and only if $A_{12}A_{21} = 0$, that is, e is q-central.

Similarly, (i) ⇔ (iii). □

There is an analogous result for Peirce trivial idempotents.

THEOREM 2.3. *For an idempotent e in a ring A, the following are equivalent:*

(i) *e is Peirce trivial;*
(ii) *$A_{12} \lhd A$;*
(iii) *$A_{21} \lhd A$;*
(iv) *$A_{12} + A_{21} \lhd A$;*
(v) *$A_{12} + A_{21}$ is an ideal of square zero.*

PROOF. (i) ⇒ (ii) and (iii). Since e is Peirce trivial,

$$A_{12}A_{21} = 0; \quad A_{21}A_{12} = 0.$$

Thus,

$$AA_{12} = A_{11}A_{12} + A_{12}A_{12} + A_{21}A_{12} + A_{22}A_{12} \subseteq A_{12} + 0 + 0 + 0$$

and since A_{12} is a right ideal, $A_{12} \lhd A$. Similarly, $A_{21} \lhd A$.

Clearly, (ii) and (iii) ⇒ (iv).

(iv) ⇒ (i). If $A_{12} + A_{21} \lhd A$, then

$$A_{12}A_{21} \subseteq (A_{12} + A_{21})A_{21} \subseteq A_{12} + A_{21}.$$

However, $A_{12}A_{21} \subseteq A_{11}$, so $A_{12}A_{21} = 0$. Similarly, $A_{21}A_{12} = 0$, so e is Peirce trivial.

Finally, if $A_{12} + A_{21} \lhd A$, then

$$(A_{12} + A_{21})^2 \subseteq A_{12}A_{12} + A_{12}A_{21} + A_{21}A_{12} + A_{21}A_{21} = 0,$$

so (iv) ⇒ (v). □

Using Theorems 2.1, 2.2 and 2.3, we can see that for *semiprime rings*, all our centrality properties coincide. For if e is a q-central idempotent of a semiprime ring A, then A_{12} and A_{21} are one-sided ideals of square zero, so that $A_{12} = 0 = A_{21}$ and e is therefore *central*.

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3. Connections with radicals

Our first result is almost immediate.

THEOREM 3.1. *Every radical class $\mathcal{R}$ is hereditary, but not necessarily very hereditary, for corners defined by q-central idempotents.*

PROOF. If $e \in A \in \mathcal{R}$ and e is a q-central idempotent, then eAe is a homomorphic image of A and hence is in $\mathcal{R}$. By [3, Theorem 2.1], radical classes need not be very hereditary for corners defined by semicentral idempotents and hence for q-central idempotents. □

THEOREM 3.2 (See [3, Proposition 2.3]). *If $\mathcal{S}$ is the semisimple class corresponding to a radical class $\mathcal{R}$ containing all nilpotent rings, then $\mathcal{S}$ is hereditary for corners defined by q-central idempotents and hence $\mathcal{R}$ in strict for such corners.*

PROOF. All rings in $\mathcal{S}$ are semiprime, so q-central = central for idempotents in rings in $\mathcal{S}$. The result now follows from [3, Theorem 2.1] and [2, Corollary 3.10]. □

PROPOSITION 3.3. *Any nonzero q-central idempotent in a prime ring is an identity.*

PROOF. Let e be a nonzero q-central idempotent in a prime ring A. Then as A is semiprime, e is central, so that $A = eAe \oplus I$ for an ideal I. Since A is prime and $e \neq 0$, we have $I = 0$ and thus e is an identity for $eAe = A$. □

In [3], we studied radical classes defined by the presence or absence of idempotents of various kinds. This we shall now do for q-central idempotents, first showing that these satisfy some conditions considered in [3, page 4795]:

I(i) if e is a q-central idempotent of A and $I \triangleleft A$, then $e + I$ is q-central in A/I;

I(ii) if $e \in I \triangleleft A$ and e is a q-central idempotent in A, then e is a q-central idempotent in I;

I(iii) if $e \in I \triangleleft A$ and e is a q-central idempotent in I, then e is a q-central idempotent in A.

Items $I(i)$ and $I(ii)$ are easily verified. If e is a q-central idempotent in I, then for all $a, b \in A$, we have $eabe = e(ea)(be)e = e(ea)e \cdot e(be)e = eae \cdot ebe$.

By [3, Theorem 3.3], we have the following result.

THEOREM 3.4. *Let $\mathcal{R}_q$ be the class of rings of which every nonzero homomorphic image has a nonzero q-central idempotent. Then $\mathcal{R}_q$ is a radical class and by items $I(i)$, $I(ii)$ and $I(iii)$, its semi-simple class is*

$$\mathcal{S}_q = \{A : A \text{ has no nonzero } q\text{-central idempotent}\}.$$

Our definition of $\mathcal{R}_q$ is analogous to the definitions, in [3, Examples 3.6 and 3.7], of $\mathcal{R}_c$ (respectively $\mathcal{R}_{lsc}$) as the class of rings, each nonzero homomorphic image of which has a nonzero central (respectively left semicentral) idempotent.

Let $\mathcal{R}^{(q)} = \{A :$ no homomorphic image of A has a nonzero q-central idempotent$\}$. By [3, Theorem 3.8], and using items $I(i)$–$I(iii)$, we have the following result.

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THEOREM 3.5. *The class $\mathcal{R}^{(q)}$ is a radical class and is the upper radical class defined by the class of subdirectly irreducible rings whose hearts contain nonzero q-central idempotents.*

By Proposition 3.3, a q-central idempotent of a heart must be an identity. Thus, using [3, Example 3.10] gives the following corollary.

COROLLARY 3.6. *$\mathcal{R}^{(q)} = \mathcal{G}$ $(= \mathcal{R}^{(lsc)} = \mathcal{R}^{(c)})$, where $\mathcal{R}^{(c)}$ and $\mathcal{R}^{(lsc)}$ are the analogues of $\mathcal{R}^{(q)}$ for central and left semicentral idempotents, respectively.*

In [3], we saw that $\mathcal{R}_c$ is the lower radical class defined by the rings with identity (Example 3.6) and $\mathcal{R}_{lsc}$ is the lower radical class defined by all rings with left identities (Example 3.7). It could be interesting to find a characterisation of $\mathcal{R}_q$ as a lower radical class (or even as an upper radical class). The best we can do at present is to find an upper radical class which is an upper bound of $\mathcal{R}_q$.

PROPOSITION 3.7. *If e is a nonzero q-central idempotent of a ring A, then A has a prime ideal I such that $e + I$ is an identity for A/I.*

PROOF. Since A contains a nonzero idempotent element, it is not a prime radical ring. Now,

$$e \notin \mathcal{B}(A) = \bigcap \{P : P \text{ is a prime ideal of } A\}.$$

Hence, there is a prime ideal I for which $e \notin I$. Then, A/I is a prime ring and $e + I$ is a q-central idempotent of an A/I by item $I(i)$, so by Proposition 3.3, $e + I$ is an identity for A/I. □

COROLLARY 3.8. *With the hypothesis of Proposition 3.7, A has a maximal ideal M such that $e + M$ is an identity of A/M.*

PROOF. In Proposition 3.7, A/I has a maximal ideal K/I such that $A/K \cong (A/I)(K/I)$ has $e + K$ as an identity. (Let K/I be maximal with respect to exclusion of $e + I$.) □

The Brown–McCoy radical class $\mathcal{G}$ is hereditary, so its upper radical class $U(\mathcal{G})$ is

$$\{R : R \text{ has no nonzero homomorphic image in } \mathcal{G}\}.$$

THEOREM 3.9. *We have $\mathcal{R}_q \subseteq U(\mathcal{G})$.*

PROOF. If $A \in \mathcal{R}_q$ and B is a nonzero homomorphic image of A, then B has a nonzero q-central idempotent and hence, by Corollary 3.8, a homomorphic image which is a simple ring with identity. However, then $B \notin \mathcal{G}$, so $A \in U(\mathcal{G})$. □

The containment in Theorem 3.9 is proper, as we shall see in Section 5. This section will also contain a counterexample to the converse of Proposition 3.7, that is, if a prime, even simple prime, factor ring has an identity which lifts to an idempotent, the latter need not be q-central.

4. The phantom corners problem

In [3, page 4194], we defined, for a radical class $\mathcal{R}$, the property

$$\text{(HPC)} \quad (e^2 = e \in A \in \mathcal{R} \ \&\ A_{11} = eAe \in \mathcal{R}) \Rightarrow A_{22} \in \mathcal{R}.$$

This can be relativised in the obvious way. For example, normal radical classes have HPC and all radical classes have HPC for left or right semicentral idempotents [3, pages 4794–4795].

We now examine HPC for q-central idempotents. Note that as all radical classes are hereditary for q-central idempotents, HPC for q-central idempotents requires that $A_{22} \in \mathcal{R}$ for all $A \in \mathcal{R}$ and for all q-central idempotents.

Proposition 4.1. *Every radical class $\mathcal{R}$ has HPC for Peirce trivial idempotents.*

Proof. Let $\mathcal{R}$ be a radical class and e a Peirce trivial idempotent of a ring $A \in \mathcal{R}$. By Theorem 2.3, $A_{12} + A_{21} \lhd A$ and so $A_{11} \oplus A_{12} \cong A/(A_{12} + A_{21}) \in \mathcal{R}$, whence $A_{22} \in \mathcal{R}$. □

We have been unable to determine whether all radical classes have HPC for all q-central idempotents, but there are affirmative answers in the presence of certain hypotheses.

Proposition 4.2. *Every left or right hereditary radical class has HPC for q-central idempotents.*

Proof. Let $\mathcal{R}$ be a left-hereditary radical class and e a q-central idempotent of a ring $A \in \mathcal{R}$. Then the map

$$A \to eAe = A_{11} : \quad a \mapsto eae$$

is a ring homomorphism. Its kernel as a homomorphism of abelian groups is $\left[\begin{smallmatrix} 0 & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right]$, so this is an ideal. Now,

$$\begin{bmatrix} 0 & A_{12} \\ 0 & A_{22} \end{bmatrix} <_e \begin{bmatrix} 0 & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \lhd \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cong A,$$

so $\left[\begin{smallmatrix} 0 & A_{12} \\ 0 & A_{22} \end{smallmatrix}\right] \in \mathcal{R}$. Since

$$\begin{bmatrix} 0 & A_{12} \\ 0 & 0 \end{bmatrix} \lhd \begin{bmatrix} 0 & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & A_{12} \\ 0 & A_{22} \end{bmatrix} \Big/ \begin{bmatrix} 0 & A_{12} \\ 0 & 0 \end{bmatrix} \cong A_{22},$$

we have $A_{22} \in \mathcal{R}$ as required. For the right hereditary case, use $\left[\begin{smallmatrix} 0 & 0 \\ A_{21} & A_{22} \end{smallmatrix}\right]$. □

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REMARK 4.3.

(i) The requirement that $\mathcal{R}$ be left or right hereditary is weaker than the requirement that it be normal.
(ii) Whether $\mathcal{N}$ is normal or not depends on the Köthe problem, but $\mathcal{N}$ is certainly left and right hereditary.
(iii) Radical classes which are left or right hereditary are corner hereditary for all idempotents by [2, Proposition 3.1].

PROPOSITION 4.4. *If a radical class $\mathcal{R}$ contains all nilpotent rings, that is, if $\mathcal{B} \subseteq \mathcal{R}$, then $\mathcal{R}$ has HPC for q-central idempotents.*

PROOF. Let e be a q-central idempotent of $A \in \mathcal{R}$. Then A_{12} (respectively A_{21}) is a nilpotent right (respectively left) ideal of A by Theorem 2.2, so $A_{12} + A_{21} \subseteq \mathcal{B}(A)$. Hence,

$$\begin{aligned} A/\mathcal{B}(A) &= (A_{11} + A_{12} + A_{21} + A_{22})/\mathcal{B}(A) \\ &= (A_{11} + A_{22} + \mathcal{B}(A))/\mathcal{B}(A) \\ &\cong (A_{11} + A_{22})/((A_{11} + A_{22}) \cap \mathcal{B}(A)). \end{aligned}$$

Now,

$$(A_{11} + A_{22}) \cap \mathcal{B}(A) \in \mathcal{B} \subseteq \mathcal{R}$$

as $\mathcal{B}$ is strongly hereditary, while

$$(A_{11} + A_{22})/((A_{11} + A_{22}) \cap \mathcal{B}(A)) \cong A/\mathcal{B}(A) \in \mathcal{R},$$

so

$$A_{11} \oplus A_{22} = A_{11} + A_{22} \in \mathcal{R},$$

whence $A_{22} \in \mathcal{R}$. □

REMARK 4.5.

(i) Although $\mathcal{B} \subseteq \mathcal{G}$, $\mathcal{G}$ is neither left nor right hereditary.
(ii) The radical class $\mathcal{T}$ is left and right hereditary, but $\mathcal{B} \nsubseteq \mathcal{T}$.
(iii) The results just proved show that if $\mathcal{R}$ and $\mathcal{U}$ are radical classes, $\mathcal{R}$ is left or right hereditary and $\mathcal{B} \subseteq \mathcal{U}$, then $\mathcal{R} \cap \mathcal{U}$ has HPC for q-central idempotents. For instance, $\mathcal{T} \cap \mathcal{G}$ has HPC for q-central idempotents.

5. A useful ring

In this section, we shall study the partial matrix ring $\left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{smallmatrix}\right]$. We do not know whether all radical classes have HPC for q-central idempotents. In fact, we do not have any examples of radical classes without HPC, global or specialised. The next result gives an example of nonsatisfaction of HPC.

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THEOREM 5.1. *The upper radical class $U(\{2\mathbb{Z}\})$ defined by $\{2\mathbb{Z}\}$ is strict and corner-hereditary but does not satisfy HPC.*

PROOF. As all accessible subrings of $2\mathbb{Z}$ are ideals, $U(\{2\mathbb{Z}\})$ is the class of rings which do not have nonzero homomorphisms to $\{2\mathbb{Z}\}$. Since

$$U(\{2\mathbb{Z}\}) = U(\{\text{accessible subrings of } 2\mathbb{Z}\}) = U(\{\text{subrings of } 2\mathbb{Z}\}),$$

we see that $U(\{2\mathbb{Z}\})$ is strict.

All rings with identity are in $U(\{2\mathbb{Z}\})$; in particular, all corners of all rings are in $U(\{2\mathbb{Z}\})$ which is therefore corner-hereditary.

We now consider the Peirce ring $\left[\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right]$ of $\left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{smallmatrix}\right]$. Let $e = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Then,

$$A_{11} = e\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}e = \begin{bmatrix} \mathbb{Z} & 0 \\ 0 & 0 \end{bmatrix}.$$

Also, for all $m, n, x, y \in \mathbb{Z}$,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} m & n \\ 2x & 2y \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} m & n \\ 2x & 2y \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix}.$$

It follows that $A_{12} = \left[\begin{smallmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{smallmatrix}\right]$. Similar calculations complete the demonstration that $\left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{smallmatrix}\right]$ itself is the Peirce decomposition matrix ring of our ring.

Let $f : \left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{smallmatrix}\right] \to 2\mathbb{Z}$ be a ring homomorphism. Then as noted above, $f(A_{11}) = 0$. Now,

$$A_{11}A_{12} = A_{12} \quad \text{and} \quad A_{21}A_{11} = A_{21},$$

so $f(A_{12}) = f(A_{11})f(A_{12}) = 0$ and $f(A_{21}) = f(A_{21})f(A_{11}) = 0$. Finally, $A_{21}A_{12} = A_{22}$, so

$$f(A_{22}) = f(A_{21})f(A_{12}) = 0.$$

Thus, $f = 0$ and $\left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{smallmatrix}\right]$ is in $U(\{2\mathbb{Z}\})$ as is its corner

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

However, $A_{22} = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 2\mathbb{Z} \end{smallmatrix}\right] \cong 2\mathbb{Z} \notin U(\{2\mathbb{Z}\})$. □

PROPOSITION 5.2. *The ring $\left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{smallmatrix}\right]$ has no nonzero q-central idempotents.*

PROOF. The idempotents are the matrices $\left[\begin{smallmatrix} 1 & 0 \\ z & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & w \\ 0 & 0 \end{smallmatrix}\right]$, where $z \in 2\mathbb{Z}$ and $w \in \mathbb{Z}$. The case $z = 0$ $(= w)$ gives $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, which we shall continue to call e. We shall prove that none of these idempotents is q-central by testing their 'multiplicative property' on the matrices $\left[\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0 & 1 \\ 2 & 2 \end{smallmatrix}\right]$ and their product $\left[\begin{smallmatrix} 2 & 2 \\ 0 & 2 \end{smallmatrix}\right]$.

Now, for all (even) z,

$$\begin{aligned} &\begin{bmatrix} 1 & 0 \\ z & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ z & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ z & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ z & 0 \end{bmatrix} \\ &\qquad = \begin{bmatrix} z & 0 \\ z^2 & 0 \end{bmatrix}\begin{bmatrix} z & 0 \\ z^2 & 0 \end{bmatrix} = \begin{bmatrix} z^2 & 0 \\ z^3 & 0 \end{bmatrix}, \end{aligned}$$

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while

$$\begin{bmatrix} 1 & 0 \\ z & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ z & 0 \end{bmatrix} = \begin{bmatrix} 2+2z & 0 \\ 2z+2z^2 & 0 \end{bmatrix}.$$

Since the equation $z^2 = 2 + 2z$ has no even integer solutions, the products are unequal.

Also, for all w,

$$\begin{bmatrix} 1 & w \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & w \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & w \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & w \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 2w & 2w^2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2w & 2w^2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4w^2 & 4w^3 \\ 0 & 0 \end{bmatrix},$$

while

$$\begin{bmatrix} 1 & w \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & w \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2w \\ 0 & 0 \end{bmatrix}.$$

However, the equation $2 = 4w^2$ has no integer solutions, so again the products are unequal.

This proves that $\left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{smallmatrix}\right]$ has no nonzero q-central idempotents. □

Corollary 5.3. $\left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{smallmatrix}\right] \notin \mathcal{R}_q$.

Proposition 5.4. $\left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{smallmatrix}\right] \in U(\mathcal{G})$.

Proof (outline). The calculations involved in showing this result are lengthy, so we content ourselves with an outline. We have

$$\begin{bmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{bmatrix} \lhd \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{bmatrix}$$

and

$$\begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{bmatrix} \Big/ \begin{bmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{bmatrix} \cong \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix},$$

which maps onto $\mathbb{Z}_2$. Hence,

$$\begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{bmatrix} \Big/ \begin{bmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{bmatrix} \notin \mathcal{G}.$$

One shows that the ideals of $\left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{smallmatrix}\right]$ have the form $\left[\begin{smallmatrix} p\mathbb{Z} & q\mathbb{Z} \\ 2r\mathbb{Z} & 2s\mathbb{Z} \end{smallmatrix}\right]$, where $p, q, r, s \in \mathbb{Z}$, and there are certain relationships between p, q, r and s. It then turns out that each $\left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{smallmatrix}\right]/\left[\begin{smallmatrix} p\mathbb{Z} & q\mathbb{Z} \\ 2r\mathbb{Z} & 2s\mathbb{Z} \end{smallmatrix}\right]$ can be mapped onto $\left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{smallmatrix}\right]/\left[\begin{smallmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{smallmatrix}\right]$ or something similar and hence is not in $\mathcal{G}$. Since no nonzero homomorphic image of $\left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{smallmatrix}\right]$ is in $\mathcal{G}$, we have $\left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{smallmatrix}\right] \in U(\mathcal{G})$. □

By Theorem 3.9, $\mathcal{R}_q \subseteq U(\mathcal{G})$. The next corollary follows from Proposition 5.4 and Corollary 5.3.

Corollary 5.5. *We have* $\left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{smallmatrix}\right] \in U(\mathcal{G}) \setminus \mathcal{R}_q$, *so* $\mathcal{R}_q \subsetneq U(\mathcal{G})$.

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Our final result shows that the converse to Proposition 3.7 is false.

PROPOSITION 5.6. *If a simple factor ring has an identity which lifts to an idempotent, the latter need not be q-central.*

PROOF. The ring

$$\begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{bmatrix} \Big/ \begin{bmatrix} 2\mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{bmatrix} \cong \mathbb{Z}_2$$

and under this homomorphism, $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] \mapsto 1$. However, $\left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{smallmatrix}\right]$ has no nonzero q-central idempotents. □

In [2, Corollary 3.12], we showed that if a radical class $\mathcal{R}$ is very corner hereditary, then both $\mathcal{R}$ and its semisimple class $\mathcal{S}$ are corner-hereditary. The status of the converse is unknown. We can get a partial converse using material from this section.

PROPOSITION 5.7. *Let $\mathcal{R}$ be a radical class with semi-simple class $\mathcal{S}$ and suppose $\mathcal{R}$ contains all rings with identity. If $\mathcal{R}$ and $\mathcal{S}$ are both corner-hereditary, then $\mathcal{R}$ is very corner-hereditary.*

PROOF. Let e be an idempotent in a ring A. Then $eAe \in \mathcal{R}$. As $\mathcal{S}$ is corner-hereditary, $\mathcal{R}$ is corner strict by [2, Corollary 3.10], so $eAe \subseteq \mathcal{R}(A)$. However, then $\mathcal{R}(eAe) = eAe = eAe \cap \mathcal{R}(A)$, so $\mathcal{R}$ is very corner-hereditary. □

EXAMPLE 5.8. As $U(2\mathbb{Z})$ is the upper radical class defined by the set of subrings of $2\mathbb{Z}$, which is strongly hereditary, the corresponding semi-simple class is strongly hereditary and hence corner-hereditary. All corners of all rings are in $\mathcal{R}$, so $\mathcal{R}$ is corner-hereditary. It follows that $U(2\mathbb{Z})$ is very corner-hereditary.

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