

Group Gradings on Associative Algebras with Involution

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Abstract. In this paper we describe the group gradings by a finite abelian group G of the matrix algebra $M_n(F)$ over an algebraically closed field F of characteristic different from 2, which respect an involution (involution gradings). We also describe, under somewhat heavier restrictions on the base field, all G -gradings on all finite-dimensional involution simple algebras.

1 Introduction

In this paper we deal with associative algebras with involution $*$ over an algebraically closed field F . Our objective is to classify all group gradings of finite-dimensional $*$ -simple algebras which are involution gradings, *i.e.*, compatible with an involution $*$. Recall that each such algebra is either simple, that is, a matrix algebra M_n of some order n over F , or the sum of two ideals each isomorphic to the same M_n for some n .

It should be pointed out that the involution gradings of M_n over a field F of characteristic zero have been dealt with in [3]. In a recent work [4] it was pointed out that a (silent) assumption of [3] that the elementary and fine components of a graded algebra are respected by a graded involution is generally not true. Using the approach of the present work (see Subsection 3.1 below), it became possible in [4] to suggest a remedy to this problem. In this paper we present our original proof of the classification of involution gradings in the cases of elementary and fine gradings over any algebraically closed field and a final result in the case of mixed gradings. The restriction on the characteristic of the base field is relaxed to an arbitrary characteristic different from 2.

In the case of non-simple involution simple algebras we give a complete classification of involution gradings in the case of algebraically closed field of characteristic 0 or coprime to the order of the grading group.

2 Definitions

Given a group G and an algebra R over a field F , we say that R is G -graded if

$$(2.1) \quad R = \bigoplus_{g \in G} R_g,$$

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where each R_g is a vector space and $R_g R_h \subset R_{gh}$ for all $g, h \in G$. A subspace $V \subset R$ is called *graded* if $V = \bigoplus_{g \in G} (V \cap R_g)$. A G -graded algebra without non-zero proper graded ideals is called *graded simple*.

Recall that an antiautomorphism $*$ of order 2 on R is called an involution. An involution ideal or $*$ -ideal I of R is any ideal with $I^* = I$ and a $*$ -simple or involution simple algebra is an algebra with involution without non-zero proper $*$ -ideals. It is well known that a $*$ -simple algebra can be simple or the sum of two simple ideals. In the first case any involution $*$ of $R = M_n$ is given by $X^* = \Phi^{-1t} X \Phi$ where ${}^t X$ is the transpose of the matrix X and Φ is a non-degenerate symmetric or skew-symmetric matrix. If Φ is symmetric, $*$ is said to be of transpose type and if Φ is skew-symmetric, $*$ is of symplectic type (and $n = 2m$ is even). By applying an inner automorphism of M_n , one can write $X^* = {}^t X$ for the transpose involution and, in the case of symplectic involution, one can assume $\Phi = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, where I is the identity matrix of order m .

If R is a non-simple involution algebra, not necessarily finite dimensional, then there exist a simple algebra A such that $R \cong A \oplus A^{\text{op}}$ where A^{op} is the opposite algebra of A . The involution is then the exchange involution $(x, y)^* = (y, x)$ and the product is given by $(x, y)(x', y') = (xx', y'y')$, for $x, y, x', y' \in A$.

If R is an algebra with involution $*$, an involution grading of R by a group G is a decomposition (2.1) where $(R_g)^* = R_g$.

3 Involution Gradings on Matrix Algebras

Involution gradings on the algebra M_n of $n \times n$ matrices over an algebraically closed field of characteristic zero have been studied in [3]. The approach of that paper made use of the connection between group gradings and group actions by automorphisms on an algebra, which imposes restrictions on the base field of coefficients: we must have enough many roots of degree $d = |G|$. Here we suggest different techniques which allow extending the results of [3] to the case when the characteristic of the base field is different from 2, but otherwise arbitrary. Notice that by a result of [3], $\text{Supp } R$, the support of R , always generates an abelian subgroup of G . Hence, without loss of generality, we may restrict ourselves to the case when G is an abelian group.

Let us first recall a theorem in [5] according to which any grading of $R = M_n$ over an algebraically closed field F can be given as follows. We have that $R = A \otimes B$ where each of A and B is graded and isomorphic to a matrix algebra, say, $A \cong M_k$, $B \cong M_l$, where the grading on M_k is elementary while that on M_l is fine. Now in [3] it is proved that in the case R admits an elementary involution grading, we must have that the k -tuple $\tau = (g_1, \dots, g_k)$ defining the elementary grading should satisfy certain conditions, while the fine involution grading is always the tensor product of several (-1) -gradings on M_2 . We do not give full details here because, as it turns out, the same results also hold in the case of arbitrary fields of characteristic different from 2, and we postpone its full formulation until a later point.

3.1 Elementary Involution Gradings

Following the argument in [7], we write the n -tuple which defines our elementary grading as $\tau = (g_1^{(k_1)}, \dots, g_m^{(k_m)})$ where $g_i \neq g_j$ for $i \neq j$, $k_1, \dots, k_m > 0$. The k -tuple

(k_1, \dots, k_m) defines a partition of the matrices in R into blocks as follows.

Let us set

$$\varepsilon_1 = E_{11} + \dots + E_{k_1 k_1}, \dots, \varepsilon_m = E_{k_{m-1}+1, k_{m-1}+1} + \dots + E_{k_m k_m},$$

where the E_{ij} 's are the usual matrix units. Then $\varepsilon_1, \dots, \varepsilon_m$ form a system of pairwise orthogonal idempotents of R with $1 = \varepsilon_1 + \dots + \varepsilon_m$. Let us set $A_i = \varepsilon_i R \varepsilon_i$, which is the i -th diagonal block and let us write $A = A_1 \oplus \dots \oplus A_m$, which is the identity component R_e of the grading we are dealing with. By our hypothesis, $A^* = A$. Let us write the involution defined on R as $X^* = \Phi^{-1} {}^t X \Phi$ and consider the mapping $\varphi: X \rightarrow \Phi^{-1} X \Phi$. Then $\varphi(A) = A$. Indeed, every element X in A has the form $X = {}^t Y$ for some $Y \in A$. Then $\varphi(X) = \varphi({}^t Y) = Y^* \in A$.

Since φ is an automorphism of A we have that $\varphi(A_i) = A_{\sigma(i)}$ for a suitable permutation σ of $1, 2, \dots, m$. Let ψ be the inner automorphism of R given by conjugation by the permutation matrix S which permutes the blocks A_i according to σ . Therefore the automorphism $\chi = \psi^{-1} \varphi$ leaves every block A_i invariant $\chi(A_i) = A_i$, $i = 1, \dots, m$.

Now the restriction of χ to A_i is an automorphism of this matrix algebra so that there exists a $k_i \times k_i$ -matrix T_i such that $\chi(X) = T_i^{-1} X T_i$ for any $X \in A_i$. If we let $T = \text{diag}(T_1, \dots, T_m)$, then the action of φ will coincide with conjugation by TS . Thus the conjugation by Φ and TS coincide on A .

If $\Psi = \Phi^{-1} TS$ and $\Psi = [\Psi_{ij}]$, then for any $X = \text{diag}(X_1, \dots, X_m) \in A$ we have $X\Psi = \Psi X$ or $X_i \Psi_{ij} = \Psi_{ij} X_j$ for any $1 \leq i, j \leq m$. If $i = j$, then $X_i \Psi_{ii} = \Psi_{ii} X_i$ and it follows that $\Psi_{ii} = \lambda_i I$ is a scalar matrix, for some nonzero scalar λ_i . If $i \neq j$, then choosing $X_i = I_{k_i}$ and $X_j = 0$ we immediately obtain $\Psi_{ij} = 0$. So $\Psi = \text{diag}(\lambda_1 I_{k_1}, \dots, \lambda_m I_{k_m})$ is a diagonal matrix. Since $\Phi = TS\Psi^{-1}$, then it follows that $\Phi = [\Phi_{ij}]$ is a block matrix such that in each column of blocks and in each row of blocks we have exactly one nonzero block.

Being the matrix of a symmetric or skew-symmetric bilinear form, Φ is either symmetric or skew-symmetric. This implies that $\Phi_{ij} = \pm {}^t \Phi_{ji}$. Since Φ also is non-degenerate, it follows that all blocks Φ_{ij} must be square matrices. By a change of bases in the spaces R_g , which does not change the grading, as in [7], we may assume that

$$\Phi = \text{diag} \left(\underbrace{I, \dots, I}_s, \underbrace{\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}}_r \right),$$

where $s + 2r = n$, in the symmetric case and

$$\Phi = \text{diag} \left(\underbrace{\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \dots, \begin{bmatrix} [r]0 & I \\ -I & 0 \end{bmatrix}}_r \right),$$

where $2r = n$, in the skew-symmetric case.

If we rearrange the elements of the defining m -tuple τ , we then have

$$\tau = (g_1^{(t_1)}, \dots, g_s^{(t_s)}, g_{s+1}^{(u_1)}, g_{s+2}^{(u_1)}, \dots, g_{s+2r-1}^{(u_r)}, g_{s+2r}^{(u_r)})$$

in the symmetric case and $\tau = (g_1^{(u_1)}, g_2^{(u_1)}, \dots, g_{2r-1}^{(u_r)}, g_{2r}^{(u_r)})$ in the skew symmetric case.

In order to obtain the relations among the components of τ we let X_{ij} denote a block matrix of order n whose only possible nonzero block is in the (i, j) position. The degree of this matrix in our grading is $g_i^{-1}g_j$. Now if we apply the involution, the resulting matrix should have the same grading.

Suppose first that Φ is a symmetric matrix. Direct calculations, using the explicit form of Φ , as above, show that

$$\begin{aligned} X_{ij}^* &= ({}^tX)_{ji}, \quad \text{for } 1 \leq i, j \leq s, \\ X_{i,s+2j}^* &= ({}^tX)_{s+2j-1,i}, \quad \text{for } 1 \leq i \leq s, 1 \leq j \leq r, \\ X_{s+2i,s+2j}^* &= ({}^tX)_{s+2j-1,s+2i-1}, \quad \text{for } 1 \leq i, j \leq r. \end{aligned}$$

Hence we deduce that $g_i^{-1}g_j = g_j^{-1}g_i$ or $g_i^2 = g_j^2$, and

$$g_i^2 = g_{s+2j-1}g_{s+2j}, \quad g_{s+2i-1}g_{s+2i} = g_{s+2j-1}g_{s+2j}.$$

Therefore, we have the equalities:

$$g_1^2 = \dots = g_s^2 = g_{s+1}g_{s+2} = \dots = g_{s+2r-1}g_{s+2r}.$$

In the skew-symmetric case the calculation is quite similar, producing the equalities $g_1g_2 = \dots = g_{2r-1}g_{2r}$.

Obviously these conditions on Φ and τ are also sufficient for the elementary grading to be an involution grading. Thus we have the following results. Recall that for R an algebra with involution $*$, $H(R, *)$ denotes the Jordan subalgebra of symmetric elements and $K(R, *)$ the Lie subalgebra of skew-symmetric elements.

Lemma 3.1 *Let $R = M_n$ be equipped with an involution $*$ defined by a symmetric non-degenerate bilinear form. Let G be an abelian group and let R be equipped with an elementary involution G -grading defined by an n -tuple (g_1, \dots, g_n) . Then $g_1^2 = \dots = g_m^2 = g_{m+1}g_{m+1} = \dots = g_{m+1}g_{m+2l}$ for some $0 \leq l \leq \frac{n}{2}$ and $m + 2l = n$. The involution $*$ is defined by $X^* = (\Phi^{-1})^t X \Phi$ where*

$$\Phi = \begin{pmatrix} I_m & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix}$$

with I_s the $s \times s$ identity matrix. Moreover, $K(R, *)$ consists of all matrices of the type

$$\begin{pmatrix} P & S & T \\ -{}^tT & A & B \\ -{}^tS & C & -{}^tA \end{pmatrix},$$

where ${}^tP = -P \in M_m$, $A, {}^tB = -B$, ${}^tC = -C \in M_l$, $S, T \in M_{m \times l}$. Hence $H(R, *)$ consists of all matrices of the type

$$\begin{pmatrix} P & S & T \\ {}^tT & A & B \\ {}^tS & C & {}^tA \end{pmatrix}$$

where ${}^tP = P \in M_m$, $A, {}^tB = B$, ${}^tC = C \in M_l$, $S, T \in M_{m \times l}$.

The last lemma deals with the case of an elementary grading compatible with an involution defined by a skew-symmetric non-degenerate bilinear form.

Lemma 3.2 *Let $R = M_n$, $n = 2k$, be equipped with an involution $*$ defined by a skew-symmetric non-degenerate bilinear form. Let G be an abelian group and suppose R is equipped with an elementary involution G -grading defined by an n -tuple (g_1, \dots, g_n) . Then $g_1g_{k+1} = \dots = g_kg_{2k}$, and the involution $*$ is defined by $X^* = (\Phi^{-1}) {}^tX\Phi$ where*

$$\Phi = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

I is the $k \times k$ identity matrix. Hence $K(R, *)$ consists of all matrices of the type

$$\begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix}, \quad \text{with } A, B, C, \in M_k, \quad {}^tB = B, {}^tC = C,$$

while $H(R, *)$ consists of all matrices of the type

$$\begin{pmatrix} A & B \\ C & {}^tA \end{pmatrix}, \quad \text{with } A, B, C, \in M_k, \quad {}^tB = -B, {}^tC = -C.$$

3.2 Fine Involution Gradings

The case of fine gradings is actually very simple. Indeed, as shown in [5], no such grading can appear on M_n if n is divisible by the characteristic p of the base field F . Now if such gradings do appear, this means that $(n, p) = 1$. But the support S of a fine grading is always a subgroup of order n^2 . Now we can repeat the argument in [1] using S because under these conditions $\widehat{S} \cong S$. As a consequence, the following holds.

Proposition 3.3 *Let F be a field of characteristic different from 2 and suppose that $R = M_n$ is endowed with an involution and a fine grading by a group G that are compatible to each other. Then R is isomorphic, as a graded algebra with involution to the tensor product of graded involution invariant subalgebras $A_1 \otimes \dots \otimes A_q$ where each A_i is isomorphic to a 2×2 -matrix algebra from the list in Lemma 3.4 below. The grading on R is the direct product of gradings on A_1, \dots, A_q and the involution is given by $(a_1 \otimes \dots \otimes a_q)^* = a_1^* \otimes \dots \otimes a_q^*$, where $a_i \in A_i$. In other words, the involution is defined by the bilinear form with matrix $\Phi = \Phi_1 \otimes \dots \otimes \Phi_q$ where Φ_i defines the involution of A_i and is given in the list of Lemma 3.4.*

Lemma 3.4 Let $R = M_2$ be endowed with an involution $*$ corresponding to a symmetric or skew-symmetric non-degenerate bilinear form with matrix Φ . The (-1) -grading of M_2 is an involution grading if and only if one of the following holds:

(i) Φ is skew-symmetric,

$$\Phi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K(R, *) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right\}, \quad H(R, *) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right\};$$

and

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix}^* = \begin{pmatrix} t & -y \\ -z & x \end{pmatrix};$$

(ii) Φ is symmetric,

$$\Phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad K(R, *) = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \right\}, \quad H(R, *) = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \right\};$$

and

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix}^* = \begin{pmatrix} t & y \\ z & x \end{pmatrix};$$

(iii) Φ is symmetric,

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K(R, *) = \left\{ \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \right\}, \quad H(R, *) = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right\};$$

and

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix}^* = \begin{pmatrix} x & z \\ y & t \end{pmatrix};$$

(iv) Φ is symmetric,

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad K(R, *) = \left\{ \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} \right\}, \quad H(R, *) = \left\{ \begin{pmatrix} a & b \\ -b & c \end{pmatrix} \right\};$$

and

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix}^* = \begin{pmatrix} x & -z \\ -y & t \end{pmatrix}.$$

3.3 Mixed Gradings

Now suppose that $R = M_n$ is endowed with a “mixed” involution G -grading, that is neither elementary nor fine. We consider the decomposition $R = C \otimes D$ into the product of a component $C \cong M_p$ with elementary grading and $D \cong M_q$ with fine grading. As is shown in [4], we do not need to have $C^* = C$ and $D^* = D$. But in the majority of cases this is still true. For example, as is shown in [4], this is true if the Sylow 2-subgroup of G is cyclic. If we combine the results of the previous two sections, then we obtain the following description of such involution gradings on matrix algebras M_n over an arbitrary algebraically closed field of characteristic different from two.

Theorem 3.5 Let $R = M_n = \bigoplus_{g \in G} R_g$ be a matrix algebra over an algebraically closed field of characteristic different from 2, graded by a group G , and let $\text{Supp } R$ generate G . Consider the decomposition $R = C \otimes D$ into the tensor product of a subalgebra $C \cong M_m$ with elementary grading and $D \cong M_q$ with fine grading. Let $*$: $R \rightarrow R$ be a graded involution such that $C^* = C$ and $D^* = D$. Then G is abelian, $n = 2^k m$ and R as a G -graded algebra with involution is isomorphic to the tensor product $R^{(0)} \otimes R^{(1)} \otimes \dots \otimes R^{(k)}$, where

- (i) $R^{(0)}, \dots, R^{(k)}$ are graded subalgebras stable under the involution $*$;
- (ii) $R^{(0)} = M_m \cong C$ is as in Lemma 3.2 if $*$ is skew-symmetric on $R^{(0)}$, or as in Lemma 3.1 if $*$ is symmetric on $R^{(0)}$;
- (iii) $R^{(1)} \otimes \dots \otimes R^{(k)} \cong D$ is a $T = T_1 \times \dots \times T_k$ -graded algebra and any $R^{(i)}, 1 \leq i \leq k$, is a $T_i \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ -graded algebra as in Lemma 3.4.
- (iv) The involution on the basis elements of R is given canonically by

$$(Y \otimes X_{t_1} \otimes \dots \otimes X_{t_k})^* = Y^* \otimes X_{t_1}^* \otimes \dots \otimes X_{t_k}^* = \text{sgn}(t)(Y^* \otimes X_{t_1} \otimes \dots \otimes X_{t_k}),$$

where $Y \in R^{(0)}$, X_{t_i} are the elements of the basis of the canonical (-1) -grading of $M_2, i = 1, \dots, k, t = t_1 \dots t_k \in T, \text{sgn}(t) = \pm 1$, depending on the cases in Lemma 3.4.

We conclude this section by quoting a theorem from [4] giving the description of involutions on matrix algebra in the case of general mixed gradings.

Theorem 3.6 Let $\varphi: X \rightarrow \Phi^{-1} {}^t X \Phi$ be an involution compatible with a grading of a matrix algebra R by a finite abelian group G . Let $C \cong M_m$ be the elementary and $D \cong M_q$ the fine component of the grading.

Then $q = 2^k$ for some integral k and $D \cong D^{(1)} \otimes \dots \otimes D^{(k)}$ where each $D^{(i)}$ is isomorphic to M_2 with one of -1 -gradings of Lemma 3.4, $T_i = \text{Supp } D^{(i)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, and $T = \text{Supp } D = T_1 \times \dots \times T_k$.

Also, $C_e^* = C_e$. Let

$$(3.1) \quad C_e = B_1 \oplus \dots \oplus B_m$$

be the decomposition of C_e as the sum of $*$ -simple ideals.

Then, after a G -graded conjugation, we can reduce Φ to the form

$$(3.2) \quad \Phi = S_1 \otimes X_{t_1} + \dots + S_m \otimes X_{t_m},$$

where S_i is one of the matrices $I, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, or $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, and each X_{t_i} is a matrix spanning $D_{t_i}, t_i \in T$. The defining tuple of the elementary grading on C should satisfy the following condition. We assume that the first l summands in (3.2) correspond to those B_i in (3.1) which are simple and the remaining $m - l$ to B_j which are not simple. Let the dimension of a simple B_i be equal to p_i and that of a non-simple B_j to $2p_j$. Then the defining tuple has the form

$$(3.3) \quad (g_1^{(p_1)}, \dots, g_l^{(p_l)}, (g'_{l+1})^{(p_{l+1})}, (g''_{l+1})^{(p_{l+1})}, \dots, (g'_m)^{(p_m)}, (g''_m)^{(p_m)})$$

with $g_1^2 t_1 = \dots = g_l^2 t_l = g'_{l+1} g''_{l+1} t_{l+1} = \dots = g'_m g''_m t_m$.

Additionally, if φ is a transpose involution, then each S_i is symmetric (skew-symmetric) at the same time as X_i , for any $i = 1, \dots, m$. If φ is a symplectic involution, then each S_i is symmetric (skew-symmetric) if and only if the respective X_i is skew-symmetric (symmetric), $i = 1, \dots, k$.

Conversely, if we have a grading by a group G on a matrix algebra R defined by a tuple as in (3.3), for the component C with elementary grading, and by an elementary abelian 2-subgroup T as the support of the component D with fine grading, and all of the above conditions are satisfied, then (3.2) defines a graded involution on R .

4 Change of Gradings

Our treatment of the remaining cases of non-simple involution simple algebras is based on a general result due to the first author. Let G be an abelian group and V a vector space. Suppose we have two G -gradings on V :

$$(4.1) \quad V = \bigoplus_{g \in G} \tilde{V}_g, \quad \alpha: \widehat{G} \rightarrow \text{Aut } V,$$

$$(4.2) \quad V = \bigoplus_{g \in G} \tilde{V}_g, \quad \beta: \widehat{G} \rightarrow \text{Aut } V,$$

where $\alpha, \beta: \widehat{G} \rightarrow \text{Aut } V$ are homomorphisms of the dual group \widehat{G} corresponding to the above gradings in the following way. Given $\chi \in \widehat{G}$, we define $\alpha(\chi)$ on an element v of \tilde{V}_g , for each g , by $\alpha(\chi)(v) = \chi(g)v$. Similarly for (4.2). Suppose $\Lambda \subset \widehat{G}$ is a subgroup such that $\alpha(\lambda) = \beta(\lambda)$, for each $\lambda \in \Lambda$. Let us denote by H the orthogonal complement $\Lambda^\perp = \{g \in G \mid \lambda(g) = 1, \text{ for all } \lambda \in \Lambda\}$. Assume further that the subgroups $\alpha(\widehat{G})$ and $\beta(\widehat{G})$ commute elementwise.

Let us consider a homomorphism $\gamma: \widehat{G} \rightarrow \text{Aut } V$ given by $\gamma(\chi) = \alpha^{-1}(\chi)\beta(\chi)$. In this case we can define an H -grading of V as follows:

$$V^{(h)} = \{v \mid \gamma(\chi)(v) = \chi(h)v\}.$$

It is obvious that $V = \bigoplus_{h \in H} V^{(h)}$. Indeed, since $H = \Lambda^\perp$ we can identify \widehat{G}/Λ with H if we set $(\chi\Lambda)(h) = \chi(h)$. Now we can define an action of \widehat{G}/Λ on V by setting $(\chi\Lambda)(v) = \gamma(\chi)(v)$ where $\chi \in \widehat{G}$ and $v \in V$. Since $\gamma(\lambda) = 1$ for any $\lambda \in \Lambda$, this formula is well defined and gives an automorphism of V . By the assumed commutativity of $\alpha(\widehat{G})$ and $\beta(\widehat{G})$, it follows that these operators commute with those of the action of \widehat{G}/Λ . Therefore, the gradings defined by α, β and γ are all compatible. Because $V = \bigoplus_{h \in H} V^{(h)}$ is defined by the action of \widehat{H} , this is an H -grading.

Theorem 4.1 (Exchange Theorem) *The three gradings defined above are connected by the following equations*

$$(4.3) \quad \tilde{V}_g = \bigoplus_{h \in H} (\tilde{V}_{gh} \cap V^{(h)}), \quad \tilde{V}_g = \bigoplus_{h \in H} (\tilde{V}_{gh} \cap V^{(h^{-1})}).$$

If V is an algebra and (4.1), (4.2) are algebra gradings, then (4.3) are relations among algebra gradings.

Proof Let us prove the first equality. Since all gradings are compatible, we have $\tilde{V}_g = \bigoplus_{h \in H} (\tilde{V}_g \cap V^{(h)})$. Thus it is enough to prove for any $g \in G, h \in H$ that $\tilde{V}_{gh} \cap V^{(h)} = \tilde{V}_g \cap V^{(h)}$. If $v \in \tilde{V}_{gh} \cap V^{(h)}$, then $\beta(\chi)(v) = \chi(gh)v$ and $\gamma(\chi)(v) = \chi(h)v$. Hence also $\gamma(\chi)^{-1}(v) = \chi(h)^{-1}v$. Now

$$\alpha(\chi)(v) = \alpha(\chi)\beta(\chi)^{-1}\beta(\chi)(v) = \gamma(\chi)^{-1}\beta(\chi)(v) = \chi(h)^{-1}\chi(gh)v,$$

proving $\tilde{V}_{gh} \cap V^{(h)} \subset \tilde{V}_g \cap V^{(h)}$.

If $b \in \tilde{V}_g \cap V^{(h)}$, then $\alpha(\chi)(b) = \chi(g)b, \gamma(\chi)(b) = \chi(h)b$. Therefore

$$\beta(\chi)(b) = \alpha(\chi)\alpha(\chi)^{-1}\beta(\chi)(a) = \alpha(\chi)\gamma(\chi)(a) = \chi(g)\chi(h)a = \chi(gh)a.$$

It follows that $\tilde{V}_g \cap V^{(h)} \subset \tilde{V}_{gh} \cap V^{(h)}$. Finally, $\tilde{V}_g \cap V^{(h)} = \tilde{V}_{gh} \cap V^{(h)}$ for any $g \in G$ and thus we have the first equality in (4.3). The second is similar. It is easy to check that if V is an algebra and (4.1), (4.2) are algebra gradings, then (4.3) provides us with the relations between algebra gradings as well. ■

Theorem 4.2 *Let R be a non-simple, involution simple, associative algebra over a field F of characteristic $\neq 2$. Then there exists a simple associative algebra A such that $R \cong A \oplus A^{\text{op}}$, with involution $*$ given by $(x, y)^* = (y, x)$.*

Proof See [8, Proposition 2.2.12]. ■

Proposition 4.3 *Let R be a G -graded algebra, G a not necessarily commutative group. Suppose R has an involution $*$ compatible with this grading and also that R is $*$ -simple. Then given any $g, h \in \text{Supp } R$, we have $gh = hg$.*

Proof Let $g, h \in \text{Supp } R$. Suppose $R_g R_h \neq 0$. Then $(R_g R_h)^* \subset R_h^* R_g^* = R_h R_g \subset R_{hg}$. Also $(R_g R_h)^* \subset R_{gh}^* = R_{gh}$. So R_{gh} and R_{hg} contain the non-zero subspace $(R_g R_h)^*$. It follows that $R_{gh} = R_{hg}$, that is $hg = gh$. Now take any $g, h \in \text{Supp } R$ and consider the two-sided ideal $I = R_g + RR_g + R_g R + RR_g R$. Since $R_g \neq 0$, we have that $I \neq 0$. Also $I^* = I$. Therefore, R being $*$ -simple, we have that $I = R$. In particular, $R_h \subset R_g + RR_g + R_g R + RR_g R$. The homogeneous components on the right-hand side of the latter containment are of one of the forms: g, kg, gl, pgq , for some $k, l, p, q \in G$. So, h is one of these forms. It follows that one of the spaces R_g (if $g = h$), of $R_k R_g$, or $R_g R_l$, or $R_p R_g R_q$ is different from zero, with either $h = g$, or $h = kg$, or $h = gl$, or $h = pgq$. The case $h = g$ being trivial, if $R_k R_g \neq 0$ with $h = kg$, then $kg = gk$ by what was proven before, and then $hg = (kg)g = g(kg) = gh$, as needed. We argue similarly if $R_g R_l \neq 0$ with $gl \neq 0$. Now if $R_p R_g R_q \neq 0$ with $h = pgq$, then $R_p R_g \neq 0$ and $R_g R_q \neq 0$ so that $pg = gp$ and $gq = qg$. Again, $hg = (pgq)g = (pg)(qg) = gpgq = gh$, as required. ■

This proposition allows us to consider an involution simple algebra R graded by a group G to be graded by the abelian subgroup H of G generated by $\text{Supp } R$.

Now let G be a finite abelian group and R an involution simple algebra. Then the following define gradings on $R = A \oplus A^{\text{op}}$ provided we know some G -gradings or involution G -gradings on A . If $A = \bigoplus_{g \in G} A_g$ is a grading of A , then we can set

$$R_g = A_g \oplus A_g^{\text{op}},$$

for any $g \in G$. This is indeed a grading of R since for any $g, g' \in G$ and $x, y \in A_g (= A_g^{op})$, $x', y' \in A_{g'}$ we have $(x, y)(x', y') = (xx', y'y) \in R_{gg'}$, thanks to the commutativity of G . Also $(x, y)^* = (y, x) \in R_g$, so this is an involution grading of R . We will call such gradings of the first type (type I).

Now suppose A has an involution on its own $x \rightarrow x^\dagger$, and an involution grading

$$(4.4) \quad A = \bigoplus_{g \in G} A_g.$$

So we have $A_g^\dagger \subset A_g$ for any $g \in G$. In this case we can define a G -grading of R as follows. We fix an element $h \in G$, $o(h) = 2$, and set

$$(4.5) \quad R_g = \{(x, x^\dagger) \mid x \in A_g\} \oplus \{(x, -x^\dagger) \mid x \in A_{gh}\}.$$

Let us check that (4.5) defines a G -grading of R . Indeed, if $x \in A_g, y \in A_{g'}$, then $(x, x^\dagger)(y, y^\dagger) = (xy, y^\dagger x^\dagger) = (xy, (xy)^\dagger)$. Next if $x \in A_{gh}, y \in A_{g'}$, then $(x, -x^\dagger)(y, y^\dagger) = (xy, -(xy)^\dagger) \in R_{gg'}$. Also for $x \in A_g, y \in A_{g'h}$ we obtain $(x, x^\dagger)(y, -y^\dagger) = (xy, -(xy)^\dagger) \in R_{gg'}$. Finally $x \in A_{gh}, y \in A_{g'h}$ give that $(x, -x^\dagger)(y, -y^\dagger) = (xy, (xy)^\dagger) \in R_{gg'} = R_{(gh)(g'h)}$.

This grading is compatible with the exchange involution $*$ of R . In fact if $x \in A_g$, $(x, x^\dagger) \in R_g$ and $x^\dagger \in A_g$ because (4.4) was an involution grading of A . It follows that $(x, x^\dagger)^* = (x^\dagger, x) = (x^\dagger, (x^\dagger)^\dagger) \in R_g$. Also, for $x \in A_{gh}$, $(x, -x^\dagger) \in R_g$. Then $(x, -x^\dagger)^* = (-x^\dagger, x) = -(x^\dagger, -(x^\dagger)^\dagger) \in R_g$ since $x^\dagger \in A_{gh}$.

Any grading defined by (4.5) is called of the second type (type II grading).

Recall that $\text{Aut}^* R$ denotes the group of automorphisms and antiautomorphisms of the algebra R . Our next result here is the following.

Theorem 4.4 *Let R be a non-simple involution simple algebra over an algebraically closed field of characteristic zero or coprime to the order of G . Then any involution grading of R by G is either of the first or the second type.*

Proof Let $R = M_n \oplus M_n^{op}$ and let \widehat{G} be the dual group of G . It is well known (see [5]) that a subspace V of R is G -graded if and only if V is \widehat{G} -invariant where the action of \widehat{G} on R by automorphisms is given by

$$\chi \circ \sum_{g \in G} x_g = \sum_{g \in G} \chi(g)x_g \quad (x_g \in R_g).$$

Now, inside the group of automorphisms $\text{Aut} R$ one can select the subgroup $\text{Aut}^o R$ of all automorphisms preserving the direct summands of R . For convenience, we write I for M_n and J for M_n^{op} . Then any element of R is of the form (x, y) where $x, y \in M_n$. If $\varphi \in \text{Aut}^o R$, then there exist two linear mappings $\varphi_0, \varphi_1: M_n \rightarrow M_n$ such that $\varphi \circ (x, y) = (\varphi_0(x), \varphi_1(y))$. Now φ commutes with the involution $*$ of R . Hence

$$\begin{aligned} (\varphi_1(y), \varphi_0(x)) &= (\varphi_0(x), \varphi_1(y))^* = (\varphi \circ (x, y))^* \\ &= \varphi \circ (x, y)^* = \varphi \circ (y, x) = (\varphi_0(y), \varphi_1(x)). \end{aligned}$$

Thus any element φ of $\text{Aut}^o R$ is completely defined by $\varphi_0: M_n \rightarrow M_n$, $\varphi \circ (x, y) = (\varphi_0(x), \varphi_0(y))$. Since φ should also preserve the product in R , we must have $\varphi \circ ((x, y)(x', y')) = (\varphi \circ (x, y)\varphi \circ (x', y'))$. Performing all calculations we find

$$(\varphi_0(xx'), \varphi_0(y'y)) = (\varphi_0(x)\varphi_0(x'), \varphi_0(y')\varphi_0(y)).$$

Hence φ_0 must be an automorphism of M_n . It is obvious that $\varphi \rightarrow \varphi_0$ is an isomorphism of $\text{Aut}^o R$ onto $\text{Aut } M_n$.

Now let us consider any automorphism ψ of R from outside of $\text{Aut}^o R$. Then there exist two linear mappings $\psi_0, \psi_1: M_n \rightarrow M_n$, such that $\psi \circ (x, y) = (\psi_0(y), \psi_1(x))$. Since φ must commute with the involution, we must have

$$(\psi_1(x), \psi_0(y)) = \psi \circ (x, y)^* = \psi \circ (y, x) = (\psi_0(x), \psi_1(y)).$$

Again, as before $\psi_0 = \psi_1$. Now ψ is an automorphism of R . So,

$$\begin{aligned} (\psi_0(y'y), \psi_0(xx')) &= \psi \circ (xx', y'y) = \psi \circ ((x, y)(x', y')) \\ &= (\psi \circ (x, y))(\psi \circ (x', y')) = (\psi_0(y), \psi_0(x))(\psi_0(y'), \psi_0(x')) \\ &= (\psi_0(y)\psi_0(y'), \psi_0(x')\psi_0(x)). \end{aligned}$$

Thus $\psi_0(y'y) = \psi_0(y)\psi_0(y')$ and $\psi_0(xx') = \psi_0(x')\psi_0(x)$. Therefore ψ_0 is an antiautomorphism of M_n . We also have $[\text{Aut } R: \text{Aut}^o R] = 2$. It follows that for any $\varphi \in \text{Aut } R$ there is $\varphi_0 \in \text{Aut}^* M_n$ (that is either an automorphism or an antiautomorphism) such that $\theta: \varphi \rightarrow \varphi_0$ is an isomorphism of $\text{Aut } R$ and $\text{Aut}^* M_n$.

Now \widehat{G} acts on R . Therefore, there is a homomorphism α of \widehat{G} into $\text{Aut } R$. By the above, the subgroup Λ such that $\alpha(\Lambda) \subset \text{Aut}^o R$ is of index 1 or 2 in \widehat{G} .

Suppose first that $\Lambda = \widehat{G}$. Then the mapping $\varphi \rightarrow \varphi_0$ is a homomorphism of \widehat{G} into $\text{Aut } M_n$ and hence we have a G -grading of M_n given by

$$(M_n)_g = \{x \mid \varphi_0(x) = \varphi(g)x, \text{ for all } \varphi \in \widehat{G}\}.$$

This formula is correct because φ_0 completely defines φ . The grading of R is given by $R_g = \{(x, y) \mid \varphi \circ (x, y) = \varphi(g)(x, y), \text{ for all } \varphi \in \widehat{G}\}$. It follows that $\varphi_0(x) = \varphi(g)x$ and $\varphi_0(y) = \varphi(g)y$. Therefore, $R_g = (M_n)_g \oplus (M_n)_g$; I is a G -graded subspace isomorphic to M_n . We have the subspace

$$R_g \cap I = \{(x, 0) \mid \varphi_0(x) = \varphi(g)x, \text{ for all } \varphi \in G\}.$$

So R is G -graded by the subspaces $R_g = I_g \oplus J_g$. It is easy to check that the mapping $(x, 0) \rightarrow (0, x)$ is the graded isomorphism of M_n and M_n^{op} .

Now suppose $\widehat{G} \neq \Lambda$. Then there is $\psi \in \widehat{G} \setminus \Lambda$. As we learned, there exists an antiautomorphism $\varphi_0: M_n \rightarrow M_n$ such that $\varphi \circ (x, y) = (\varphi_0(y), \varphi_0(x))$ for any $x \in M_n, y \in M_n^{op}$. Let us consider $\theta(\widehat{G}) \subset \text{Aut}^* M_n$. We have that $\theta(\widehat{G})$ is an abelian subgroup of $\text{Aut}^* M_n$ with $\theta(\Lambda) \subset \text{Aut } M_n$. In [7] the authors considered group gradings of the Lie algebra $M_n^{(-)}$: the action of $\theta(\widehat{G})$ defined a grading of $M_n^{(-)}$, the

action of $\theta(\Lambda)$ being inner, while that of φ_0 outer. It was proved in that paper that in that situation there is an inner automorphism ψ_0 of $M_n^{(-)}$ such that $\psi_0^2 = \varphi_0^2$ and ψ_0 commutes with $\theta(\widehat{G})$. This allows us to consider $\psi \in \text{Aut } R$ such that $\psi \circ (x, y) = (\psi_0(x), \psi_0(y))$. Since ψ_0 commutes with $\theta(\widehat{G})$, we have that ψ commutes with \widehat{G} , and $\psi^2 = \varphi^2$. Indeed, $\psi^2 \circ (x, y) = (\psi_0^2(x), \psi_0^2(y))$ while

$$\varphi^2 \circ (x, y) = \varphi \circ (\varphi_0(x), \varphi_0(y)) = (\varphi_0^2(x), \varphi_0^2(y)) = (\psi_0^2(x), \psi_0^2(y)) = \psi^2 \circ (x, y).$$

Now we would like to apply the Exchange Theorem. We consider two gradings of R , the first being our original one, $R = \bigoplus_{g \in G} R_g$, which corresponds to the action of \widehat{G} on R via the homomorphism $\alpha: \widehat{G} \rightarrow \text{Aut } R$. We also define $\beta: \widehat{G} \rightarrow \text{Aut } R$ by $\beta|_{\Lambda} = \alpha|_{\Lambda}$ and $\beta(\varphi) = \psi$. It is easy to check that β is indeed a homomorphism. Now according to the Exchange Theorem there exists a grading by a subgroup $H = \Lambda^\perp$, corresponding to the action of $\alpha(\varphi)^{-1}\beta(\varphi)$, that is $\varphi\psi^{-1}$, which is an element of order 2. Now if we denote $\omega = \varphi\psi^{-1}$, then $\omega \circ (x, y) = (\omega_0(y), \omega_0(x))$, $\omega_0^2 = 1$, and

$$\begin{aligned} \omega_0(x_1x_2) &= \varphi_0\phi_0^{-1}(x_1x_2) = \varphi_0(\phi_0^{-1}(x_1)\phi_0^{-1}(x_2)) \\ &= (\varphi_0\psi_0^{-1}(x_2))(\varphi_0\psi_0^{-1}(x_1)) = \omega_0(x_2)\omega_0(x_1), \end{aligned}$$

for any $x_1, x_2 \in M_n$. Therefore ω_0 is an involution of M_n which we denote by $\omega_0(x) = x^\dagger$. Since ω_0 commutes with ψ and $\theta(\Lambda)$, we have that the G -grading of M_n defined by this action is an involution grading with respect to \dagger . So $M_n = \bigoplus_{g \in G} (M_n)_g$ is a \dagger -grading by G . It defines a G -grading of R by $R = \bigoplus_{g \in G} \tilde{R}_g$ where $\tilde{R}_g = \{(x, y) \mid \beta(\chi) \circ (x, y) = \chi(g)(x, y)\}$, $\tilde{R}_g = (M_n)_g \oplus (M_n^{\text{op}})_g$.

Now by the Exchange Theorem $R_g = \tilde{R}_g \cap R^{(e)} + \tilde{R}_{gh} \cap R^{(h)}$. Here

$$\begin{aligned} R^{(e)} &= \{(x, y) \mid \omega \circ (x, y) = (x, y)\} = \{(x, y) \mid (y^\dagger, x^\dagger) = (x, y)\} \\ &= \{(x, x^\dagger) \mid x \in M_n\}. \end{aligned}$$

Also

$$R^{(h)} = \{(x, y) \mid \omega \circ (x, y) = -(x, y)\} = \{(x, -x^\dagger) \mid x \in M_n\}.$$

This allows us to write $\tilde{R}_g \cap R^{(e)} = \{(x, x^\dagger) \mid x \in (M_n)_g\}$ while $\tilde{R}_{gh} \cap R^{(h)} = \{(y, -y^\dagger) \mid y \in (M_n)_{gh}\}$. Finally, we have

$$R_g = \{(x, x^\dagger) \mid x \in (M_n)_g\} \oplus \{(y, -y^\dagger) \mid y \in (M_n)_{gh}\},$$

as required. ■

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