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ON NUNOKAWA'S CONJECTURE FOR MULTIVALENT FUNCTIONS

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The object of this paper is to prove a conjecture given recently by Nunokawa that if $f(z) \in \mathcal{A}(p)$ satisfies $\operatorname{Re}\{1 + zf''(z)/f'(z)\} < p + 1/2$ in the unit disk $\mathcal{U}$, then $f(z)$ is p-valently starlike in $\mathcal{U}$.

1. INTRODUCTION

Let $\mathcal{A}(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (n \in \mathcal{N} = \{1, 2, 3, \ldots\}) \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U} = \{z\colon |z| < 1\}$. A function $f(z)$ belonging to the class $\mathcal{A}(p)$ is said to be *p-valently starlike* in the unit disk $\mathcal{U}$ if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \tag{1.2}$$

for all $z \in \mathcal{U}$.

For functions $f(z) \in \mathcal{A}(1)$ when $p = 1$, Singh and Singh [4] have proved

THEOREM A. *If $f(z) \in \mathcal{A}(1)$ satisfies*

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \frac{3}{2} \qquad (z \in \mathcal{U}), \tag{1.3}$$

then $f(z)$ is starlike in $\mathcal{U}$.

Recently, Nunokawa [3] has shown that

THEOREM B. *If $f(z) \in \mathcal{A}(p)$ satisfies*

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < p + \frac{1}{4} \quad (z \in \mathcal{U}), \tag{1.4}$$

then $f(z)$ is p-valently starlike in $\mathcal{U}$.

In view of Theorem A and Theorem B, Nunokawa [3] made the

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Conjecture. *If $f(z) \in \mathcal{A}(p)$ satisfies*

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < p + \frac{1}{2} \quad (z \in \mathcal{U}), \tag{1.5}$$

then $f(z)$ is p-valently starlike in $\mathcal{U}$.

We note that this is true for $p = 1$ by Theorem A. In this paper, we prove this conjecture for general p.

2. Main result

For an analytic function $f(z)$ in $\mathcal{U}$, if $g(z)$ is univalent in $\mathcal{U}$, $f(0) = g(0)$, and $f(\mathcal{U}) \subseteq g(\mathcal{U})$, then $f(z)$ is said to be *subordinate* to $g(z)$. We denote this subordination by $f(z) \prec g(z)$.

In order to give our main result, we have to recall here the following lemma due to Jack [**1**] (see also Miller and Mocanu [**2**]).

Lemma 1. *Let $w(z)$ be regular in $\mathcal{U}$ and such that $w(0) = 0$. Then if $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in \mathcal{U}$, we have*

$$z_0 w'(z_0) = k w(z_0), \tag{2.1}$$

where $k \geqslant 1$ is a real number.

Applying the above lemma, we derive

Theorem 1. *If $f(z) \in \mathcal{A}(p)$ satisfies the condition (1.5), then*

$$0 < \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \frac{2p(p+1)}{2p+1} \quad (z \in \mathcal{U}). \tag{2.2}$$

Therefore, $f(z)$ is p-valently starlike in $\mathcal{U}$. The result is sharp.

Proof: We define the function $w(z)$ by

$$\frac{zf'(z)}{f(z)} = \frac{p(p+1)(1-w(z))}{(p+1) - pw(z)}. \tag{2.3}$$

Then $w(z)$ is regular in $\mathcal{U}$ and $w(0) = 0$. It follows from (2.3) that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{p(p+1)(1-w(z))}{(p+1) - pw(z)} + \frac{pzw'(z)}{(p+1) - pw(z)} + \frac{zw'(z)}{1 - w(z)}. \tag{2.4}$$

Suppose that there exists a point $z_0 \in \mathcal{U}$ such that

$$\max_{|z| \leqslant |z_0|} |w(z)| = |w(z_0)| = 1.$$

Then, using Lemma 1, we have

$$z_0 w'(z_0) = k w(z_0) \quad (k \text{ is real and } k \geqslant 1),$$

and $w(z_0) = e^{i\theta}$. Thus we obtain that

$$\begin{aligned}
&\operatorname{Re}\left\{1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right\} \\
&= \operatorname{Re}\left\{\frac{p(p+1)(1-w(z_0))}{(p+1)-pw(z_0)} + \frac{pkw(z_0)}{(p+1)-pw(z_0)} - \frac{kw(z_0)}{1-w(z_0)}\right\} \\
&= \frac{p(p+1)(2p+1)(1-\cos\theta) + pk((p+1)\cos\theta - p)}{(2p^2+2p+1) - 2p(p+1)\cos\theta} + \frac{k}{2} \\
&= p + \frac{1}{2} + \frac{(2p+1)(k-1)}{(2p^2+2p+1) - 2p(p+1)\cos\theta} \\
&\geqslant p + \frac{1}{2}
\end{aligned}$$

which contradicts our condition (1.5). Hence, we conclude that $|w(z)| < 1$ for all $z \in \mathcal{U}$. Further, noting that the function $g(z)$ defined by

$$g(z) = \frac{p(p+1)(1-z)}{(p+1)-pz} \tag{2.6}$$

is univalent in $\mathcal{U}$ and $g(0) = p$, we have that

$$\frac{zf'(z)}{f(z)} \prec g(z) \quad (z \in \mathcal{U}),$$

that is, that

$$\begin{aligned}
\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} &\geqslant \min_{|z|=r} \operatorname{Re} g(z) \\
&= \frac{p(p+1)(1-r)}{(p+1)-pr} \\
&> 0 \qquad (|z| = r < 1)
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} &\leqslant \max_{|z|=r} \operatorname{Re}\{g(z)\} \\
&= \frac{p(p+1)(1+r)}{(p+1)+pr} \\
&< \frac{2p(p+1)}{2p+1} \quad (|z| = r < 1).
\end{aligned} \tag{2.8}$$

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Further, we see that the result is sharp for the function

$$f(z) = z^p\{(p+1) - pz\}.$$

This completes the proof. $\square$

Making $p = 1$ in Theorem 1, we have

Corollary 1. *If $f(z) \in \mathcal{A}(1)$ satisfies*

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \frac{3}{2} \quad (z \in \mathcal{U}),$$

then

$$0 < \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \frac{4}{3} \quad (z \in \mathcal{U}).$$

The result is sharp.

Next, we prove

Theorem 2. *If $f(z) \in \mathcal{A}(p)$ satisfies*

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{p}{4} - 1 \quad (z \in \mathcal{U}), \tag{2.9}$$

then

$$\operatorname{Re}\sqrt{\frac{zf'(z)}{f(z)}} > \frac{p^{1/2}}{2} \quad (z \in \mathcal{U}). \tag{2.10}$$

The result is sharp.

Proof: Defining the function $w(z)$ by

$$\sqrt{\frac{zf'(z)}{f(z)}} = \frac{p^{1/2}}{1 + w(z)}, \tag{2.11}$$

we have

$$1 + \frac{zf''(z)}{f'(z)} = \frac{p}{(1 + w(z))^2} - \frac{2zw'(z)}{1 + w(z)}. \tag{2.12}$$

Assuming that there exists a point $z_0 \in \mathcal{U}$ such that

$$\max_{|z| \leqslant |z_0|} |w(z)| = |w(z_0)| = 1,$$

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Lemma 1 implies that

$$\text{(2.13)} \qquad \operatorname{Re}\left\{1+\frac{z_0 f''(z_0)}{f'(z_0)}\right\} = \operatorname{Re}\left\{\frac{p}{(1+w(z_0))^2} - \frac{2kw(z_0)}{1+w(z_0)}\right\}$$
$$= \frac{p\cos\theta}{2(1+\cos\theta)} - k$$
$$\leqslant \frac{p}{4} - 1,$$

where $w(z_0) = e^{i\theta}$. This proves that

$$\text{(2.14)} \qquad \operatorname{Re}\sqrt{\frac{zf'(z)}{f(z)}} = \operatorname{Re}\left\{\frac{p}{1+w(z)}\right\} > \frac{p}{2} \quad (z \in \mathcal{U}).$$

Noting that $g(z) = p^{1/2}/(1+z)$ is univalent in $\mathcal{U}$ and $g(0) = p^{1/2}$, so that

$$\text{(2.15)} \qquad \sqrt{\frac{zf'(z)}{f(z)}} \prec g(z) = \frac{p^{1/2}}{1+z},$$

we see that the result is sharp with the extremal function

$$f(z) = \left(\frac{z}{1+z}e^{z/(1+z)}\right)^p.$$

□

Setting $p = 1$ in Theorem 2, we have

Corollary 2. *If $f(z) \in \mathcal{A}(1)$ satisfies*

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > -\frac{3}{4} \quad (z \in \mathcal{U}),$$

then

$$\operatorname{Re}\sqrt{\frac{zf'(z)}{f(z)}} > \frac{1}{2} \quad (z \in \mathcal{U}).$$

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