$$\therefore \mathrm{K} \equiv 4rlm(p+q) - 32pqnrs - 8pqmnr - 2lmr(p+q) - 16pqrsn^2$$
$$- 8pqmn(r+s) - 8rsln(p+q) - 2lmr(p+q) - 2lms(p+q).$$
$$\equiv 2s[-16pqnr - 8pqrn^2 - 4pqmn - (p+q)(rln + lm)]$$
$$\equiv s[-4l^2r - 2l^2rn - l^2m - 2l(p+q)(rn+m)]$$
$$\equiv -ls[4lr + 2lrn + lm + 2(p+q)(rn+m)].$$

Now if we eliminate s between the equations (3) we get

$$4lr + 2lrn + lm + 2(p+q)(rn+m) = 0\,;\ \therefore \mathrm{K} = 0.$$

It may easily be proved that the eleven-point conic passes through the following eleven points :—the vertices of the diagonal triangle of ABCD, the middle points of the six sides of ABCD and two definite points at infinity, namely, the centres of the two parabolas that pass through A, B, C, D ; and by means of the equation to the eleven-point conic given above, various particular cases may be considered.

[*Note.*—Since writing the above, I have found references to the following papers by Professor Beltrami on the eleven-point conic :— *Intorno alle coniche di nove punti e ad alcune questioni che ne dipendono*, Bologna, Mem. Accad., II., 1862, pp. 361-365, and Bologna, Rendiconto 1862-63, pp. 82-85; *Sulle coniche di nove punti*, Napoli, Giorn. di Mat., I., 1863, pp. 109-118; *Estenzione allo spazio di tre dimensioni dei teoremi, relativi alle coniche di nove punti*, Napoli, Giorn. di Mat., I., 1863, pp. 208-217, 354-360.]

Deduction of the Thermodynamical Relations.

By WILLIAM PEDDIE, D.Sc.

In his *Theory of Heat*, Clerk-Maxwell gives a very elegant and simple geometrical proof of the four thermodynamical relations, and points out that his construction shows that the truth of any one is a necessary consequence of the truth of any other. The usual analytical proof of these relations can be made as simple as Maxwell's geometrical proof, and the fact that any one is a necessary consequence of any other becomes evident when it is considered that they are all deduced by a common process from identical transformations of one equation. The following method of proof seems to me to bring out more directly their necessary interdependence.

Let u, v, x, y, be four variables any two of which may be taken as independent. If we take x and y as the independent variables we get

$$dv = \frac{\delta v}{\delta x} dx + \frac{\delta v}{\delta y} dy$$

where dx and dy are subject to no restriction other than extreme smallness of magnitude. If we now impose the condition u = a constant, this equation takes the form

$$\left(\frac{\delta v}{\delta x}\right)_u - \left(\frac{\delta v}{\delta x}\right)_y = \left(\frac{\delta v}{\delta y}\right)_x \left(\frac{\delta y}{\delta x}\right)_u,$$

where the suffixes denote the quantity which is kept constant.

Now take u and x as the independent variables and we get

$$\left(\frac{\delta v}{\delta x}\right)_y - \left(\frac{\delta v}{\delta x}\right)_u = \left(\frac{\delta v}{\delta u}\right)_x \left(\frac{\delta u}{\delta x}\right)_y$$

whence $$\left(\frac{\delta v}{\delta y}\right)_x \left(\frac{\delta y}{\delta x}\right)_u = -\left(\frac{\delta v}{\delta u}\right)_x \left(\frac{\delta u}{\delta x}\right)_y \quad \dots \quad \dots \quad (1)$$

From (1) we get

$$\left(\frac{\delta v}{\delta y}\right)_x = \phi(x, y)\left(\frac{\delta u}{\delta x}\right)_y \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (2)$$

$$\left(\frac{\delta v}{\delta u}\right)_x = -\phi(x, y)\left(\frac{\delta y}{\delta x}\right)_u \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (3)$$

where ϕ is a function whose dimensions are $[vxu^{-1}y^{-1}]$ and are therefore zero (the simplest case) when $[vx] = [uy]$. In particular, when this condition holds, ϕ may be a constant (again the simplest case) which we may call positive or negative unity since its absolute magnitude depends upon the (arbitrary) units in which v, u, x, and y, are measured.

Equations (2) and (3) then become

$$\left(\frac{\delta v}{\delta y}\right)_x = \pm \left(\frac{\delta u}{\delta x}\right)_y \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (4)$$

$$\left(\frac{\delta v}{\delta u}\right)_x = \mp \left(\frac{\delta y}{\delta x}\right)_u \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (5)$$

From these equations we see that interchange of two dependent, or of two independent, variables does not change the $\pm$ sign; while the sign must be reversed when we interchange a dependent and an independent variable. Hence, remembering this rule, we may consider (4) above as the type of all such equations.

Now if
$$\left(\frac{\delta v}{\delta y}\right)_x = \pm\left(\frac{\delta u}{\delta x}\right)_y$$

we have some function, w, of x and y such that

$$\left(\frac{\delta w}{\delta x}\right)_y = \pm v\,;\ \left(\frac{\delta w}{\delta y}\right)_x = \pm u$$

or
$$dw = \pm vdx \pm udy.$$

In the case of thermodynamics, the quantities e (energy), ϕ (entropy), t (temperature, p (pressure), v (volume), satisfy the conditions $[t\phi]=[pv]$ and $de=td\phi-pdv$. Hence, putting, in (4), $v=t$, $y=v$, $x=\phi$, $u=p$, we get

$$\left(\frac{\delta t}{\delta v}\right)_\phi = -\left(\frac{\delta p}{\delta \phi}\right)_v .$$

Interchanging t and ϕ,

$$\left(\frac{\delta \phi}{\delta v}\right)_t = \left(\frac{\delta p}{\delta t}\right)_v$$

Interchanging now p and v,

$$\left(\frac{\delta \phi}{\delta p}\right)_t = -\left(\frac{\delta v}{\delta t}\right)_p$$

Interchanging now t and ϕ,

$$\left(\frac{\delta t}{\delta p}\right)_\phi = \left(\frac{\delta v}{\delta \phi}\right)_p .$$

The investigation shows that the relations between the quantities ϕ, p, v, t, are the simplest which can satisfy the general relation (1) connecting four quantities of which any two may be regarded as independent.