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Geometric Meaning of Isoparametric Hypersurfaces in a Real Space Form

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Abstract. We shall provide a characterization of all isoparametric hypersurfaces M's in a real space form $\tilde{M}(c)$ by observing the extrinsic shape of geodesics of M in the ambient manifold $\tilde{M}(c)$.

0 Introduction

In differential geometry it is interesting to know the shape of a Riemannian submanifold by observing the extrinsic shape of geodesics of the submanifold. For example: A hypersurface M^n isometrically immersed into a real space form $\widetilde{M}^{n+1}(c)$ of constant curvature c (that is, $\widetilde{M}^{n+1}(c) = \mathbb{R}^{n+1}$, $S^{n+1}(c)$ or $H^{n+1}(c)$ according as the curvature c is zero, positive, or negative) is totally umbilic in $\widetilde{M}^{n+1}(c)$ if and only if every geodesic of M, considered as a curve in the ambient space $\widetilde{M}^{n+1}(c)$, is a circle.

Here we recall the definition of circles in Riemannian geometry. A smooth curve $\gamma\colon \mathbb{R} \longrightarrow M$ in a complete Riemannian manifold M is called a *circle* of curvature $\kappa(\geqq 0)$ if it is parametrized by its arclength s and it satisfies the following equation:

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma(s) = -\kappa^2\dot\gamma(s),$$

where κ is constant and $\nabla_{\dot\gamma}$ denotes the covariant differentiation along γ with respect to the Riemannian connection ∇ of M. Since $\|\nabla_{\dot\gamma}\dot\gamma\| = \kappa$, this equation is equivalent to the equation of geodesics, when $\kappa = 0$. So we treat a geodesic as a circle of null curvature.

In general, a circle in a Riemannian manifold is not closed. Of course, any circles of positive curvature in Euclidean m-space $\mathbb{R}^m$ are closed. And also any circles in Euclidean m-sphere $S^m(c)$ are closed. But in the case of a real hyperbolic m-space $H^m(c)$, there exist many open circles. In fact, in $H^m(c)$ a circle with curvature κ is closed if and only if $\kappa > \sqrt{|c|}$ (for details, see [2]).

In this paper we are interested in a hypersurface M^n of a real space form $\widetilde{M}^{n+1}(c)$ satisfying that there exists an *orthonormal* basis $\{v_1, \dots, v_n\}$ at each point p of the hypersurface M^n such that all geodesics of M^n through p in the direction v_i, $(1 \leq i \leq n)$, lie on circles in the ambient space $\widetilde{M}^{n+1}(c)$. The classification problem of such hypersurfaces is concerned with studies about isoparametric hypersurfaces M^n's in a real space form $\widetilde{M}^{n+1}(c)$ (that is, all principal curvatures of M^n in $\widetilde{M}^{n+1}(c)$ are constant).

Theory of isoparametric submanifolds is one of the most interesting objects in differential geometry. In particular, É. Cartan studied extensively isoparametric hypersurfaces in a

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standard sphere. The classification problem of isoparametric hypersurfaces in a sphere is still open (see Problem 34 in [3]).

The main purpose of this paper is to provide a characterization of all isoparametric hypersurfaces by observing the extrinsic shape of geodesics of hypersurfaces in a real space form (Theorem 1 and Theorem 5).

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1 Results

Theorem 1 *Let M^n be a connected hypersurface of a real space form $\widetilde{M}^{n+1}(c)$ of constant curvature c. Then M^n is isoparametric in $\widetilde{M}^{n+1}(c)$ if and only if for each point p in M there exists an orthonormal basis $\{v_1, \dots, v_m\}$ of the orthogonal complement of $\ker A$ in $T_p(M)$ ($m = \operatorname{rank} A$) such that all geodesics of M through p in the direction v_i, ($1 \le i \le m$), lie on circles of nonzero curvature in the ambient space $\widetilde{M}^{n+1}(c)$.*

Proof Let M be an isoparametric hypersurface of a real space form $\widetilde{M}(c)$ with constant principal curvatures $\kappa_1, \dots, \kappa_g$. Then the tangent bundle TM is decomposed as: $TM = T_{\kappa_1} \oplus \cdots \oplus T_{\kappa_g}$, where $T_{\kappa_i} = \{X \in TM : AX = \kappa_i X\}$ $(i = 1, \dots, g)$. We here recall the fact that each distribution T_{κ_i} is integrable and moreover, every leaf of T_{κ_i} is totally geodesic in the hypersurface M and totally umbilic in the ambient space $\widetilde{M}(c)$ (see [1]), which implies that every geodesic of such leaves is a geodesic in M and a circle in $\widetilde{M}(c)$.

Hence, for each point p of M, taking an orthonormal basis $\{v_1, \dots, v_m\}$ of the orthogonal complement of $\ker A$ in $T_p(M)$ in such a way that each v_i $(1 \le i \le m)$ is a principal curvature vector, we find that the vectors $v_1, \dots, v_m$ satisfy the statement of Theorem 1.

Conversely, let M be a hypersurface satisfying the condition that for each point p in M there exists an orthonormal basis $\{v_1, \dots, v_m\}$ of the orthogonal complement of $\ker A$ in $T_p(M)$ such that all geodesics of M through p in the direction v_i $(1 \le i \le m)$, lie on circles of nonzero curvature in the ambient space $\widetilde{M}^{n+1}(c)$. We consider the open dense subset $\mathcal{U} = \{p \in M \mid$ the multiplicity of each principal curvature of M in $\widetilde{M}(c)$ is constant on some neighborhood $\mathcal{V}_p (\subseteq \mathcal{U})$ of $p\}$ of M. We here note that all principal curvatures are differentiable on $\mathcal{U}$ and in a neighborhood of any point p in $\mathcal{U}$ the principal curvature vectors can be chosen to be smooth. In the following, we shall study on a fixed neighborhood $\mathcal{V}_p$. We remark that the shape operator A has constant rank on $\mathcal{V}_p$.

Let $\gamma_i = \gamma_i(s)$ $(1 \le i \le m)$ be geodesics of M (with metric $\langle\ ,\ \rangle$) with $\gamma_i(0) = p$ and $\dot\gamma_i(0) = v_i$, where $\{v_1, \dots, v_m\}$ is an orthonormal basis of $(\ker A)^\perp$ in $T_p(M)$. We denote by $\widetilde\nabla$ and ∇ the Riemannian connections of $\widetilde{M}(c)$ and M, respectively. Then they satisfy

$$\widetilde\nabla_{\dot\gamma_i} \widetilde\nabla_{\dot\gamma_i} \dot\gamma_i = -k_i^2 \dot\gamma_i \tag{1.1}$$

for some positive constants k_i. Here, without loss of generality we can set $k_1 \le k_2 \le \cdots \le k_m$. We recall the Gauss formula $\widetilde\nabla_X Y = \nabla_X Y + \langle AX, Y\rangle N$ and the Weingarten formula $\widetilde\nabla_X N = -AX$, where N is a unit normal vector field on M and A is the shape operator of

M in $\widetilde{M}(c)$. From these two formulas we get

$$\widetilde{\nabla}_{\dot{\gamma}_i}\widetilde{\nabla}_{\dot{\gamma}_i}\dot{\gamma}_i = -\langle A\dot{\gamma}_i, \dot{\gamma}_i\rangle A\dot{\gamma}_i + \langle(\nabla_{\dot{\gamma}_i}A)\dot{\gamma}_i, \dot{\gamma}_i\rangle N. \tag{1.2}$$

Comparing the tangential component of (1.1) and (1.2), we obtain

$$\langle A\dot{\gamma}_i, \dot{\gamma}_i\rangle A\dot{\gamma}_i = k_i^2\dot{\gamma}_i$$

so that at $s = 0$

$$\langle Av_i, v_i\rangle Av_i = k_i^2 v_i.$$

Hence

$$Av_i = k_i v_i \quad or \quad Av_i = -k_i v_i \quad (1 \le i \le m),$$

which means that the tangent space $T_p(M)$ is decomposed as:

$$T_p(M) = \ker A \oplus \{v \in T_p(M) : Av = -k_{i_1}v\} \oplus \{v \in T_p(M) : Av = k_{i_1}v\}$$
$$\oplus \cdots \oplus \{v \in T_p(M) : Av = -k_{i_g}v\} \oplus \{v \in T_p(M) : Av = k_{i_g}v\},$$

where $0 < k_{i_1} < k_{i_2} < \cdots < k_{i_g}$ and g is the number of positive distinct k_j $(j = 1, \ldots, m)$. Hence our discussion yields that every k_{i_j} is differentiable on $\mathcal{V}_p$. Next, we shall show the constancy of k_{i_j}. It suffices to check the case that $Av_{i_j} = k_{i_j}v_{i_j}$. First we note that $v_{i_j}k_{i_j} = 0$ (see the normal component of Equation (1.2)). For any v_l $(1 \le l \ne i_j \le n)$, since A is symmetric, we see

$$\langle(\nabla_{v_{i_j}}A)v_l, v_{i_j}\rangle = \langle v_l, (\nabla_{v_{i_j}}A)v_{i_j}\rangle. \tag{1.3}$$

Here $\{v_{m+1}, \ldots, v_n\}$ is an orthonormal basis of $\ker A$. In order to compute Equation (1.3) easily, we extend an orthonormal basis $\{v_1, \ldots, v_n\}$ to principal curvature unit vector fields on some neighborhood $\mathcal{W}_p(\subset \mathcal{V}_p)$, say $\{V_1, \ldots, V_n\}$. Moreover we can choose $\nabla_{V_{i_j}}V_{i_j} = 0$ at the point p, where $(V_{i_j})_p = v_{i_j}$. Such a principal curvature vector field V_{i_j} can be obtained as follows:

First we define a smooth vector field W_{i_j} on some sufficient small neighborhood $\mathcal{W}_p(\subset \mathcal{V}_p)$ by using parallel displacement for the vector v_{i_j} along each geodesic with origin p. We remark that in general W_{i_j} is not principal on $\mathcal{W}_p$, but $AW_{i_j} = k_{i_j}W_{i_j}$ on the geodesic $\gamma = \gamma(s)$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = v_{i_j}$. We here define the vector field U_{i_j} on $\mathcal{W}_p$ as: $U_{i_j} = \prod_{\alpha \ne k_{i_j}}(A - \alpha I)W_{i_j}$, where α runs over the set of all distinct principal curvatures of M except for the principal curvature k_{i_j}. Then we find that $AU_{i_j} = k_{i_j}U_{i_j}(\ne 0)$ on $\mathcal{W}_p$. We define V_{i_j} by normalizing U_{i_j}. Our construction shows that the integral curve of V_{i_j} through the point p is a geodesic on M, so that in particular $(\nabla_{V_{i_j}}V_{i_j})_p = 0$.

Thanks to the Codazzi equation $\langle(\nabla_X A)Y, Z\rangle = \langle(\nabla_Y A)X, Z\rangle$, at the point p we find

$$\begin{aligned}
(\text{the left-hand side of } (1.3)) &= \langle(\nabla_{v_l}A)v_{i_j}, v_{i_j}\rangle \\
&= \langle(\nabla_{V_l}A)V_{i_j}, V_{i_j}\rangle \\
&= \langle\nabla_{V_l}(k_{i_j}V_{i_j}) - A\nabla_{V_l}V_{i_j}, V_{i_j}\rangle \\
&= \langle(V_l k_{i_j})V_{i_j} + (k_{i_j}I - A)\nabla_{V_l}V_{i_j}, V_{i_j}\rangle \\
&= v_l k_{i_j}.
\end{aligned}$$

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Similarly we get

$$\begin{aligned}(\text{the right-hand side of } (1.3)) &= \langle V_l, (\nabla_{V_{i_j}} A) V_{i_j} \rangle \\ &= \langle V_l, \nabla_{V_{i_j}} (k_{i_j} V_{i_j}) - A \nabla_{V_{i_j}} V_{i_j} \rangle \\ &= \langle v_l, (v_{i_j} k_{i_j}) v_{i_j} \rangle = 0.\end{aligned}$$

Therefore we can see that the differential dk_{i_j} of k_{i_j} vanishes at the point p, which shows that every $k_{i_j} (> 0)$ is constant on $\mathcal{W}_p$, since p is an arbitrary point of $\mathcal{W}_p$.

Now let $\{\lambda_i\}$ be the n principal curvature functions on M numbered in descending order. Then each λ_i is continuous on M. The above argument guarantees that the set where $\{q \in M : \lambda_i(q) = \lambda_i(p)\}$ for the fixed point $p (\in \mathcal{U})$ is both open and closed in M, so that every principal curvature is constant on M. Thus M is an isoparametric hypersurface. ■

As immediate consequences of Theorem 1 we establish the following

Theorem 2 *Let M^n be a connected hypersurface of a real space form $\widetilde{M}^{n+1}(c)$ of constant curvature c. Then M^n is isoparametric with nonzero constant principal curvatures in $\widetilde{M}^{n+1}(c)$ if and only if for each point p of M, there exists an orthonormal basis $\{v_1, \ldots, v_n\}$ of $T_p(M)$ such that all geodesics of M through p in the direction v_i, ($1 \le i \le n$), lie on circles of nonzero curvature in the ambient space $\widetilde{M}^{n+1}(c)$.*

Theorem 3 *Let M^n be a connected hypersurface of a real space form $\widetilde{M}^{n+1}(c)$ of constant curvature c. Then M^n is totally umbilic* (but not totally geodesic) *in $\widetilde{M}^{n+1}(c)$ or locally congruent to a product of spheres $S^r(2c) \times S^{n-r}(2c)$ ($1 \le r \le n-1$) which is naturally imbedded into $S^{n+1}(c)$ if and only if there exists an orthonormal basis $\{v_1, \ldots, v_n\}$ at each point p of M such that all geodesics of M through p in the direction v_i, ($1 \le i \le n$), lie on circles with the same nonzero curvature in the ambient space $\widetilde{M}^{n+1}(c)$.*

Proof of Theorem 3 By virtue of the proof of Theorem 1 we know that the hypersurface M^n in $\widetilde{M}^{n+1}(c)$ satisfying the condition that there exists an orthonormal basis $\{v_1, \ldots, v_n\}$ at each point p of M such that all geodesics of M through p in the direction v_i, $(1 \le i \le n)$, lie on circles with the same nonzero curvature, say, k in the ambient space $\widetilde{M}^{n+1}(c)$ has at most two nonzero constant principal curvatures k, $-k$. Then we get the conclusion (see [1]). It is well-known that the hypersurface $S^r(c_1) \times S^{n-r}(c_2)$ $(1 \le r \le n-1, \frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c})$ in $S^{n+1}(c)$ has two constant principal curvatures $\frac{c_1}{\sqrt{c_1+c_2}}$ with multiplicity r and $-\frac{c_2}{\sqrt{c_1+c_2}}$ with multiplicity $n-r$. ■

In connection with Theorem 1 we recall the following example.

Example 4 A hypersurface M^n in a real space form $\widetilde{M}^{n+1}(c)$ is called a *Dupin hypersurface* (*cf.* [1]) if each of its principal curvatures has constant multiplicity and is constant along the leaves of its principal foliation. So each leaf of its principal foliation is totally umbilic in $\widetilde{M}^{n+1}(c)$, but generally is not totally geodesic in M^n by Theorem 1.

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Finally we rewrite Theorem 1 as follows:

Theorem 5 *Let M^n be a connected hypersurface of a real space form $\widetilde{M}^{n+1}(c)$ of constant curvature c. Then M^n is isoparametric in $\widetilde{M}^{n+1}(c)$ if and only if for each point p of M, there exists an orthonormal basis $\{v_1, \dots, v_n\}$ of $T_p(M)$ of principal curvature vectors such that all geodesics of M through p in the direction v_i, ($1 \leq i \leq n$), lie on circles in the ambient space $\widetilde{M}^{n+1}(c)$.*

Proof of Theorem 5 If $\langle Av_i, v_i \rangle = 0$, then $Av_i = 0$, because v_i is a principal curvature vector. Hence the proof of Theorem 1 yields that all principal curvatures of M are constant. ∎

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