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INSCRIBED CENTERS, REFLEXIVITY, AND SOME APPLICATIONS

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Abstract

We first define an inscribed center of a bounded convex body in a normed linear space as the center of a largest open ball contained in it (when such a ball exists). We then show that completeness is a necessary condition for a normed linear space to admit inscribed centers. We show that every weakly compact convex body in a Banach space has at least one inscribed center, and that admitting inscribed centers is a necessary and sufficient condition for reflexivity. We finally apply the concept of inscribed center to prove a type of fixed point theorem and also deduce a proposition concerning so-called Klee caverns in Hilbert spaces.

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1. Definitions and terminology

Let X be a normed linear space and B a bounded convex body in X (that is, B has a non-empty interior in X). Let us define the (*nearest*) *distance* of each point $b \in B$ from the complement cB of B by

$$d(b, cB) = \inf_{x \in cB} \|b - x\|.$$

We define the *inscribed radius* of B by

$$\rho(B) = \sup_{b \in B} d(b, cB).$$

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We shall also say that the bounded convex body B has an *inscribed center* if there exists some $c_0 \in B$ such that $d(c_0, cB) = \rho(B)$. When such a c_0 exists we call the open ball $B^0(c_0, \rho(B))$ an *inscribed ball* of B. If each bounded convex body in X has at least one inscribed center, we call X a *normed linear space admitting inscribed centers.*

2. Results

First of all, we point out that even in Euclidean spaces a bounded convex body may have lots of inscribed centers. For instance, in the Euclidean plane each point $(x, 0)$ with $-1 \leqslant x \leqslant 1$ is an insc ribed center for the rectangle with vertices $(-2, 1)$, $(-2, -1)$, $(2, -1)$, and $(2, 1)$. Also, as an example to show that in general the notion of "Chebyshev center" for a bounded convex body (see [2] for the definition) differs from that of "inscribed center", let us choose B to be the right triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 4)$ in the Euclidean plane. Then clearly $(1, 2)$ is the Chebyshev center of B, whereas B has $(\sqrt{3} - 1, \sqrt{3} - 1)$ as its inscribed center.

The following theorem shows that completeness is a necessary condition for a normed linear space to admit inscribed centers.

THEOREM 1. *Let X be an incomplete normed linear space. Then X contains a closed convex body with no inscribed center.*

PROOF. We let D denote the virtual ball of radius 1 in X, as constructed in the proof of Theorem 1 in [2], that is

$$D = \{x \in X : r(x) \leqslant 1\},$$

where $r(x) = \lim_n \|x - a_n\|$, and where $\{a_n\}$ is a fixed non-convergent Cauchy sequence in X with $\lim_n \|a_n\| = 1$. Since D equals the intersection with X of the ball $B(\lim_n a_n, 1)$ in the completion of X, D is dense in $B(\lim_n a_n, 1)$. Therefore $\rho(D) = \rho(B(\lim_n a_n, 1))$, and the only possible inscribed center for D would be $\lim_n a_n$ (the inscribed center of $B(\lim_n a_n, 1)$), which is not in X. Hence X admits no inscribed center for D.

The above theorem confines our attention to Banach spaces for this study of admitting inscribed centers. The following theorem has the corollary that reflexivity is a sufficient condition for a Banach space to admit inscribed centers.

THEOREM 2. *Let B be a weakly compact convex body in a Banach space X. Then B has at least one inscribed center in X.*

PROOF. For sufficiently large n let us define $C_n \subset X$ as follows:

$$C_n = c\left[cB + \left(\rho(B) - \frac{1}{n}\right)U(X)\right],$$

where $U(X)$ denotes the closed unit ball of X. Since cB (the complement of B in X) is open, it follows that each C_n is closed, and since

$$(1) \qquad x \in C_n \quad \Leftrightarrow \quad x + \left(\rho(B) - \frac{1}{n}\right)U(X) \subset B,$$

it follows that for each n we have $C_n \subset B$. On the other hand, for each n, we have

$$cB + \left(\rho(B) - \frac{1}{n}\right)U(B) \subset cB + \left(\rho(B) - \frac{1}{n+1}\right)U(X),$$

from which we deduce that $C_{n+1} \subset C_n$. That is, $\{C_n\}$ is a decreasing sequence. We now show that each C_n is convex. To this end, let b_1, $b_2 \in C_n$ and let $0 < t < 1$. By (1), for each $u \in U(X)$, we have

$$b_1 + \left(\rho(B) - \frac{1}{n}\right)u, \quad b_2\left(\rho(B) - \frac{1}{n}\right)u \in B.$$

Therefore, by the convexity of B, for each $u \in U(X)$, we have

$$tb_1 + (1-t)b_2 + \left(\rho(B) - \frac{1}{n}\right)u$$
$$= t\left[b_1 + \left(\rho(B) - \frac{1}{n}\right)u\right] + (1-t)\left[b_2 + \left(\rho(B) - \frac{1}{n}\right)u\right] \in B.$$

Hence $tb_1 + (1-t)b_2 + (\rho(B) - 1/n)U(X) \subset B$, and, by (1), $tb_1 + (1-t)b_2 \in C_n$.

Now each C_n (being closed and convex) is weakly closed [3, Theorem 13, page 422]. Also, by the weak compactness of B and by the fact that $C_n \subset B$, it follows that each C_n is weakly compact. Since $\{C_n\}$ is a decreasing sequence of weakly compact subsets of B, we deduce from a theorem of Smullian (cf. [3, Theorem 2, page 433]) that $C = \bigcap_n C_n$ is non-empty. If now $c_0 \in C$, it follows from (1) that for each n we have

$$c_0 + \left(\rho(B) - \frac{1}{n}\right)U(X) \subset B.$$

Therefore each ball $B(c_0, \rho(B) - 1/n)$ is contained in B. Therefore

$$B^0(c_0, \rho(B)) = \bigcup_n B\left(c_0, \rho(B) - \frac{1}{n}\right) \subset B.$$

We deduce that $d(c_0, cB) \geqslant \rho(B)$. Hence $d(c_0, cB) = \rho(B)$, and the result follows.

COROLLARY 1. *The set consisting of all inscribed centers of a bounded convex body B in a normed linear space X is closed convex, and nowhere dense in X.*

PROOF. We may assume without loss of generality that B is closed. Then it is enough to observe that the closed convex subsets $C_n \subset B$ (and hence $C = \bigcap_n C_n$) in the proof of Theorem 2 may be constructed, even if X is an arbitrary normed linear space. Hence, with the convention that the empty set is convex, the set $C \subset B$ is closed and convex. The nowhere density of C is obvious; for otherwise C would contain a ball $B(c, \delta)$, and then we would have $B(c, \rho(B) + \delta) \subset B$, which is absurd.

Our next theorem shows that admitting inscribed centers characterises reflexivity. To prove this we shall use Theorem 2 above, a well known theorem of R. C. James in [4], and also the following lemma.

LEMMA 1. *Let X be a normed linear space and f a continuous linear functional of norm 1 on X. Then the inscribed radius of the (upper) half unit ball $B = \mathrm{U}(X) \cap f^{-1}([0, \infty))$ equals $\frac{1}{2}$.*

PROOF. We first show that $\rho(B) \geqslant \frac{1}{2}$. To do so, let $\varepsilon > 0$ be arbitrary, and choose $z \in B$ such that $\|z\| = 1$ and such that $1 - \varepsilon < f(z) < 1$. Then

$$(1) \qquad B\left(\frac{z}{2}, \frac{1}{2} - \frac{\varepsilon}{2}\right) \subset B\left(\frac{z}{2}, \frac{1}{2}\right) \subset U(X).$$

Also we have

$$(2) \qquad B\left(\frac{z}{2}, \frac{1}{2} - \frac{\varepsilon}{2}\right) \subset f^{-1}([0, \infty)),$$

for if $y \in B(\frac{z}{2}, \frac{1}{2} - \frac{\varepsilon}{2})$, then $f(\frac{z}{2}) - f(y) \leqslant \|\frac{z}{2} - y\| \leqslant \frac{1}{2} - \frac{\varepsilon}{2}$, and hence $f(y) \geqslant f(\frac{z}{2}) - \frac{1}{2} + \frac{\varepsilon}{2} > 0$. From (1) and (2) we get

$$B\left(\frac{z}{2}, \frac{1}{2} - \frac{\varepsilon}{2}\right) \subset B.$$

Now since $\varepsilon > 0$ was arbitrary, we deduce that $\rho(B) \geqslant \frac{1}{2}$. We end the proof of the lemma by showing that $\rho(B) > \frac{1}{2}$ is impossible. If $\rho(B) > \frac{1}{2}$, then there would exist $b \in B$ such that $d(b, cB) > \frac{1}{2}$, and hence for some $\alpha > 0$ we would have

$$(3) \qquad \tfrac{1}{2} + \alpha < d(b, cB) = \inf_{z \in cB} \|b - z\| \leqslant \|b\|.$$

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On the other hand (3) implies that

$$B(b, \tfrac{1}{2} + \alpha) \subset U(X).$$

Now, observing that

$$b' = \left(1 + \frac{1}{2\|b\|}\right) b \in B(b, \tfrac{1}{2} + \alpha) \subset U(X),$$

we get from (3) that

$$1 \geqslant \|b''\| = \|b\| + \tfrac{1}{2} > \tfrac{1}{2} + \alpha + \tfrac{1}{2} = 1 + \alpha.$$

This contradiction shows that we must have $\rho(B) = \frac{1}{2}$.

THEOREM 3. *For a Banach space X the following conditions are equivalent*:
(i) *X is reflexive*;
(ii) *X admits inscribed centers.*

PROOF. (i) ⇒ (ii). If X is reflexive, and if $B \subset X$ is a bounded convex body, then its norm closure $\bar{B}$ is weakly closed. By reflexivity of X, $\bar{B}$ is weakly compact [3, Corollary 8, page 425]. Hence by Theorem 2, $\bar{B}$, and therefore B, has an inscribed center in X.

(ii) ⇒ (i). To prove this we only need to show that every continuous linear functional f on X attains its supremum on $U(X)$ (cf. Theorem 5 in [4]). Let f be an arbitrary continuous linear functional on X. We assume without loss of generality that $\|f\| = 1$. Let $B = U(X) \cap f^{-1}([0, \infty))$ be the (upper) half unit ball determined by f. By Lemma 1, $\rho(B) = \frac{1}{2}$, and by hypothesis there exists $c \in B$ such that

$$d(c, cB) = \rho(B) = \tfrac{1}{2}.$$

We now claim that $\|c\| \leqslant \frac{1}{2}$. For otherwise (as in the proof of Lemma 1), the inequality $\frac{1}{2} < \|c\|$ together with the relationships

$$c'' = \left(1 + \frac{1}{2\|c\|}\right) c \in B(c, \tfrac{1}{2}) \subset U(X)$$

imply the following contradiction:

$$1 \geqslant \|c''\| = \|c\| + \tfrac{1}{2} > 1.$$

Hence $\|d\| \leqslant \frac{1}{2}$. On the other hand

$$\tfrac{1}{2} = \rho(B) = d(c, cB) = \inf_{z \in cB} \|c - z\| \leqslant \inf\{\|c - z\| : z \in f^{-1}(\{0\}) \cap U(X)\}$$
$$= f(c) \leqslant \|c\| \leqslant \tfrac{1}{2}.$$

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Therefore $f(c) = \|c\| = \frac{1}{2}$, and $f(2c) = \|2c\| = 1 = \|f\|$. We deduce that f attains its supremum on $U(X)$, and this completes the proof of the theorem.

COROLLARY 2. *In every non-reflexive Banach space X there exists a partition of the unit ball $U(X)$ into two half balls, neither of which contains a ball of radius $\frac{1}{2}$. These half balls are $B_1 = U(X) \cap f^{-1}([0, \infty))$ and $B_2 = -B_1$, where f is a continuous linear functional on X which does not attain its supremum on $U(X)$.*

EXAMPLE. In c_0, the Banach space of all real sequences (x_n) converging to 0, the subsets $B_1 = \{(x_n): \|(x_n)\| \leqslant 1; 0 \leqslant \sum_{n=1}^{\infty} x_n/2^n\}$ and $B_2 = -B_1$ are two half balls which do not contain a largest ball (of radius $\frac{1}{2}$). This follows since the continuous linear functional f defined by $f(x_n) = \sum_{n=1}^{\infty} x_n/2^n$ on c_0, does not attain its supremum on $U(c_0)$ (see [5, Example 18.8, page 173]).

3. Applications

In this section we point out two applications of the concept of inscribed centers. The first application is to deduce the following fixed point theorem. In this theorem Inscr(B) denotes the set of all inscribed centers of a given bounded convex body.

THEOREM 4. *Let X be a normed linear space, and let $B \subset X$ be a bounded convex body with* Inscr$(B) \neq \varnothing$. *Let $K: B \to [1, \infty)$ be a given function, and assume that $T: B \to B$ is a map such that for each $x \in B$ and $y \in cB$ we have*

$$d(x, cB) \leqslant K(x)\|y - Tx\|. \tag{1}$$

Then T leaves Inscr(B) *invariant. In particular, if* Inscr(B) *is a singleton, then its only member is a fixed point for T.*

PROOF. We only need to prove the first assertion of the theorem. Let $z \in$ Inscr(B) be given. By (1), for each $y \in cB$ we have

$$d(z, cB) \leqslant K(z)\|y - Tz\|.$$

Taking the infimum over cB in the right side of this inequality and noting that $Tz \in B$, we get

$$\rho(B) = d(z, cB) \leqslant K(z) d(Tz, cB) \leqslant \rho(B).$$

Therefore $d(Tz, cB) = \rho(B)$ and $Tz \in$ Inscr(B). Hence the result follows.

We may recall that under the conditions of the above theorem the map T may not have a fixed point if Inscr(B) contains more than one point. As an example, let B be the rectangle with vertices $(-2,1)$, $(-2,-1)$, $(2,-1)$, and $(2,1)$ in the Euclidean plane. As we mentioned at the beginning of Section 2, $\operatorname{Inscr}(B) = \{(a,0)\colon -1 \leqslant a \leqslant 1\}$. If we consider the map $T\colon B \to B$ defined by $T(0,0) = (1,0)$, and $T(a,b) = (-a, b/2)$ for $(a,b) \neq (0,0)$, then (for $K = 1$) T satisfies the conditions of Theorem 4 (since for each $(a, b) \in B$, $d((a, b), cB) \leqslant d(T(a, b), cB)$), while clearly T has no fixed point in B.

As our next application of the concept of inscribed center, we point out the following proposition concerning so-called Klee caverns in Hilbert spaces. Recall that a subset K of a normed linear space X is called *Chebyshev* if K admits a unique nearest point to each point of X. Chebyshev subsets of Hilbert spaces whose complements are bounded and convex have been called *Klee caverns* by Asplund in [1]. Asplund showed that Klee caverns exist, provided that non-convex Chebyshev sets exist (see [1, page 239]).

PROPOSITION 1. *If a Hilbert space H contains a non-convex Chebyshev subset, then H contains a Klee cavern whose complement has a unique inscribed center.*

PROOF. We adopt the notations and the details stated in [1, pages 238–239]. Thus, let K be a non-convex Chebyshev subset of H and let G be the subset (with the unique farthest point property) of H obtained from K by Ficken's method of inversion (see [1, page 238]). Let y denote the unique Chebyshev center of G. Then the subset $C = \{x \in H\colon t(x) \geqslant t(y) + 1\}$, where $t(x) = \sup_{z \in G}\|x - z\|$, is a Klee cavern. If b is the metric projection onto C, then for each $x \notin C$ the following equality holds:

$$t(x) + \|x - b(x)\| = t(y) + 1.$$

The above equation with its constant right hand side reveals that as $t(x)$ decreases to reach its greatest lower bound over cC (the complement of C), $\|x - b(x)\|$ increases to reach its least upper bound. Since the only point at which $t(x)$ takes its minimum is y, it follows that y is the unique point in cC for which $\|x - b(x)\|$ takes its maximum. Therefore y is at the same time the Chebyshev center of G and the unique inscribed center of cC, and the proposition follows

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