Some problems in the enumeration
of labelled graphs

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A method of counting labelled 3-connected graphs is presented. The
enumeration is accomplished by employing a new characterisation of
3-connected graphs, based on an examination of the graphs which may result
when a line is removed from a 3-connected graph. The same general
principle is first applied to counting connected and 2-connected labelled
graphs. In all three cases, it is necessary to keep track of both the
numbers of points and of lines. Numerical results are supplied, showing
the numbers of connected, 2-connected and 3-connected labelled graphs
with p points for p up to 20 .

The characterisation result for 3-connected graphs also allows us to
count labelled 3-connected cubic graphs. Labelled cubic graphs and
labelled connected cubic graphs have already been counted by Read. As no
cubic graph is 4-connected, the picture is completed by counting labelled
2-connected cubic graphs. This is done by taking into consideration graphs
which are cubic except for one or two points of degree 2 . The same
method is used to re-derive some of Read's results for labelled cubic
graphs and labelled connected cubic graphs. Numerical results are listed
for graphs with up to 30 points.

An asymptotic formula for the number of labelled cubic graphs with a
given number of points was also found by Read. A generalisation of this
problem is solved here by beginning with a known result concerning the
asymptotic enumeration of matrices with given row and column sums. By this

means we find an asymptotic formula for the number of labelled graphs (or
pseudographs) with points of given degrees, as long as the degrees of the
points in the graphs (pseudographs) are bounded. The numbers of cycles of
given fixed length can also be specified. This yields, for example, an
asymptotic formula for the number of labelled r-regular graphs with girth
j , for fixed r and j .

The asymptotic formula for the number of labelled graphs with points
of given degrees is used to show that almost all labelled graphs with
points of given degrees lying between r and R are r-connected, where
r and R are fixed, $3 \leq r \leq R$. This implies, for instance, that almost
all labelled r-regular graphs are r-connected for any fixed $r \geq 3$. A
similar procedure is used to show that for fixed $k > 0$ and $r \geq 3$,
almost all labelled r-regular graphs with girth at least $\lceil k/(r-2) \rceil$ are
cyclically k-connected. This provides an asymptotic formula for the
number of labelled cyclically k-connected r-regular graphs with p
points, for fixed k and r .