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Solution branches for mappings in cones, and applications

E.N. Dancer

We prove the existence of global solution branches for positive mappings. This improves an earlier result of the author. We also prove a related result for mappings in wedges. We then use these two results to prove the existence of solutions for boundary-value problems for systems of ordinary differential equations.

In this paper, we improve some of the results in the author's previous paper [3] and apply these results to boundary-value problems for systems of ordinary differential equations. In [10], Turner obtained related results for systems of partial differential equations. His methods, when applied to systems of ordinary differential equations, produce results somewhat weaker than ours.

In §1, we strengthen the main result (Theorem 2) in [3]. (A result similar to, but slightly weaker than, Theorem 2 in [3] was also obtained in [10].) The proof here is quite different from those in [3] and [10]. In §2, we prove a similar result for mappings in wedges and, in §3, we apply the results of §1 and §2 to boundary-value problems for systems of ordinary differential equations.

1. Mappings in cones

Our notation will follow that in [3]. It is assumed that the reader is familiar with [3]. Let K be a cone in E with $\bar{E}_K = E$ and suppose

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that $r, t > 0$ (where r and t may be ∞). *We let* $[a, \infty]$ *denote* $\{x \in R : x \geq a\}$ *with the usual topology*. This will simplify the statements of some of our theorems. Define $K_r = \{x \in K : \|x\| < r\}$. *Assume that*

(i) $A : \bar{K}_r \times [0, t] \to K$ *is completely continuous,*

(ii) $A(0, \lambda) = 0$ *for* $\lambda \in [0, t]$ *, and*

(iii) *there exist completely continuous positive mappings* $B(0)$ *and* D *from* E *into itself such that* $r(B(0)) < 1$ *and* $\|A(x, \lambda)-B(0)x-\lambda Dx\|$ *is* $o(\|x\|)$ *as* $\|x\| \to 0$ *(and* $x \in K$ *) uniformly in* λ *on compact subsets of* $[0, t]$.

Define $B(\lambda) : E \to E$ by $B(\lambda)x = B(0)x + \lambda Dx$ and let

$$C_K(B(\ )) = \{\lambda \in [0, \infty] : \text{there exists an } x \in K \text{ with } \|x\| = 1 \text{ and } x = B(\lambda)x\} .$$

Since $I - B(0)$ is invertible, Theorem V.1.8 in [*6*] implies that $C_K(B(\ ))$ is discrete. We shall use the notation in [*7*] for the degree of a mapping defined on a closed convex subset of a Banach space.

LEMMA 1. *If* $\lambda \geq 0$ *and* $r(B(\lambda)) < 1$ *, then* $i_K(B(\lambda), K_1) = 1$ *while, if* $\lambda \in [0, \infty]\backslash C_K(B(\ ))$ *and* $r(B(\lambda)) > 1$ *, then* $i_K(B(\lambda), K_1) = 0$.

Proof. Since $r(B(\lambda)) = r_K(B(\lambda))$ (*cf*. the remarks in §1 of [*3*]), this follows from Lemma 1 in [*3*] and Lemma 2 in [*1*].

Define $\tau(B(\ ))$ to be $\sup\{\lambda \in [0, \infty] : r(B(\lambda)) < 1\}$. When there is no possibility of confusion, we shall write τ instead of $\tau(B(\ ))$. If $\tau < \infty$, $C_K(B(\ )) \subseteq [\tau, \infty]$.

LEMMA 2. *If* $0 \leq \lambda < \tau$ *,* $r(B(\lambda)) < 1$. *If* $\tau < \infty$*,* $r(B(\tau)) \geq 1$ *,* $\tau \in C_K(B(\ ))$ *and* $r(B(\lambda)) > 1$ *for* $\lambda > \tau$.

Proof. If $r(B(\tau))$ is less than 1 , then, by Lemma 1 and the homotopy invariance of the degree, $i_K(B(\lambda), K_1) = i_K(B(\tau), K_1) = 1$ for λ near τ . Lemma 1 then implies that $r(B(\lambda)) < 1$ for λ near τ . Since this contradicts the definition of τ , $r(B(\tau)) \geq 1$. By a similar homotopy argument, $\tau \in C_K(B(\ ))$.

By Lemmas 1 and 2 in [3], $r(B(\lambda)) \geq r(B(\nu))$ if $\lambda \geq \nu \geq 0$. Since $\lambda \in C_K(B(\ ))$ if $r(B(\lambda)) = 1$ and since $C_K(B(\ ))$ is discrete, the remaining assertions of the lemma follow.

It can be shown that $r(B(\tau)) = 1$ if $\tau < \infty$. Define

$$D_K(A) = \{(x, \lambda) \in \overline{K}_r \times [0, t] : x = A(x, \lambda),\ x \neq 0\} \cup \{(0, \lambda) : \lambda \in C_K(B(\ )) \cap [0, t]\} .$$

By similar arguments to those in [3], $D_K(A)$ intersects any closed bounded subset of $\overline{K}_r \times [0, t]$ in a compact set.

THEOREM 1. *If* $\tau < t$, *then one of the following possibilities holds for the component* T *of* $D_K(A)$ *containing* $(0, \tau)$:

(i) T *intersects* $(K\backslash\{0\}) \times \{0\}$;

(ii) $\sup\{\|x\| : (x, \lambda) \in T\} = r$; *or*

(iii) $\sup\{\lambda : (x, \lambda) \in T\} = t$.

Moreover, if there exist a linear operator V *on* E *an* $\alpha \in (0, t)$ *and a* $y \in K\backslash\{0\}$ *such that* $Vy \geq y$ *and* $A(x, \alpha) \geq Vx \geq 0$ *if* $x \in \overline{K}_r$, *then (i) or (ii) holds for the component of* $\{(x, \lambda) \in D_K(A) : \lambda \leq \alpha\}$ *containing* $(0, \tau)$.

Proof. It obviously suffices to prove the result when $r, t < \infty$. Thus $D_K(A)$ is compact. Consider the first assertion. Because the proof is similar to that of Theorem 1 in [5] (which is based on Theorem 1.16 in [9]), we shall only outline it. Suppose that the result is false. Then $T \subseteq K_r \times (0, t)$. By repeating the argument in the proof of Lemma 1.2 in [9], we find that there exists an open set U in $K_r \times (0, t)$ such that $T \subseteq U$ and $\partial U \cap D_K(A) = \emptyset$. (Here ∂U denotes the boundary *in* $\overline{K}_r \times [0, t]$.) By an earlier remark, $C_K(B(\ ))$ is a finite set $\{\lambda_i : i = 1, \ldots, n\}$ (where $\lambda_1 = \tau$). It follows from Lemma 1 that, if $0 < \varepsilon < \inf\limits_{j \neq i} |\lambda_i - \lambda_j|$ and δ is sufficiently small, then

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$$i_K\big(A(\ ,\ \lambda_i+\varepsilon),\ K_\delta\big) - i_K\big(A(\ ,\ \lambda_i-\varepsilon),\ K_\delta\big) = \begin{cases} 0\ , & i \neq 1 \\ -1\ , & i = 1 \end{cases} .$$

The proof of the first assertion can now be completed by using a similar homotopy argument to that in [5].

Now consider the second assertion. Define $\tilde{A} : \overline{K}_r \times [0,\ t] \to K$ by $\tilde{A}(x,\ \lambda) = A(x,\ \lambda)$ if $\lambda \leq \alpha$ and $\tilde{A}(x,\ \lambda) = A(x,\ \alpha) + (\lambda-\alpha)Dx$ if $\lambda \geq \alpha$. It obviously suffices to prove the result when A is replaced by $\tilde{A}$. For n a positive integer, define $\tilde{A}_n : \overline{K}_r \times [0,\ t] \to K$ by $\tilde{A}_n(x,\ \lambda) = \tilde{A}(x,\ \lambda) + n^{-1}\|x\|^2 y$. Note that $\tilde{A}_n$ and A have the same linearization at zero. Since $\tilde{A}(x,\ \lambda) \geq Vx$ if $x \in \overline{K}_r$ and $\lambda \in [\alpha,\ t]$, a similar argument to that in the proof of the second assertion of Theorem 1 in [3] shows that $\lambda \leq \alpha$ if $(x,\ \lambda) \in D_K(\tilde{A}_n)$ and $x \neq 0$. Hence the component of $D_K(\tilde{A}_n)$ containing $(0,\ \tau)$ is contained in $\overline{K}_r \times [0,\ \alpha]$. The proof can now be completed by a similar argument to that at the end of the proof of Theorem 2 in [3].

The proof of the second assertion shows that $\tau \leq \alpha$. Lemmas 1 and 2 in [3] imply that $\tau < \infty$ if $r(D) > 0$. However, it is easy to construct examples in which $r(D) = 0$ but $\tau < \infty$.

For simplicity, we have not proved our result under the weakest assumptions. With a little more care in the proof, it can be shown that the result remains true if, instead of assuming the mapping $\lambda \to B(\lambda)$ is affine, we assume that $B(\lambda) \geq B(\nu)$ if $\lambda \geq \nu \geq 0$. Note that, in this case, $C_K\big(B(\ )\big)$ need not be discrete. Moreover, as in [3], we need only assume that $B(\lambda)$ is completely continuous *on* K. Thus we can give an alternative proof of the main result in [3].

2. Bifurcation in wedges

In this section, we improve the results for mappings in Borisovič [2]. Assume that K is a cone in a real Banach space E with $\overline{E}_K = E$, and F is a real Banach space. Let $G = E \oplus F$, and $W = K \oplus F$. We shall write elements of G as $(x,\ y)$ (or as $\underline{z}$), $(x,\ y,\ \lambda)$ or $(\underline{z},\ \lambda)$ will

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denote an element of $G \times R$, $\| \ \|$ will denote the usual product norm on G and $W_r = \{\underline{z} \in W : \|\underline{z}\| < r\}$. Suppose $r, t > 0$. Assume that

(i) $A : \overline{W}_r \times [0, t] \to W$ is completely continuous,

(ii) $A(\underline{0}, \lambda) = 0$ for $\lambda \in [0, t]$, and

(iii) there exist completely continuous mappings $T_1 : E \to E$, $T_2 : E \to E$, $P : E \to F$ and $Q : F \to F$ such that $T_1(K) \subseteq K$, $T_2(K) \subseteq K$, $r(T_1) < 1$ and $\|A(\underline{z}, \lambda) - B(\lambda)(\underline{z})\|$ is $o(\|\underline{z}\|)$ as $\|\underline{z}\| \to 0$ (and $\underline{z} \in W$) uniformly in λ on compact subsets of $[0, t]$.

Here $B(\lambda)$ is defined by $B(\lambda)(x, y) = \left(T_1x+\lambda T_2x, \lambda Px+\lambda Qy\right)$. If $\lambda \geq 0$, then $B(\lambda)(W) \subseteq W$. Our assumed form for $B(\lambda)$ may seem restrictive but an argument in [2] shows that our other assumptions ensure that the first component of $B(\lambda)(0, y)$ is zero for all y in F.

Let

$$C_W\big(B(\lambda)\big) = \{\lambda \in [0, \infty] : \text{there exists a } \underline{z} \in W \text{ such that } \|\underline{z}\| = 1 \text{ and } B(\lambda)\underline{z} = \underline{z}\},$$

and let S denote the set of non-negative characteristic values of Q. It is easy to see that $C_W\big(B(\ )\big) = C_K\big(T_1+\lambda T_2\big) \cup S$. Let $0 < \gamma_1 < \gamma_2 < \ldots$ denote the distinct positive characteristic values of Q of odd (algebraic) multiplicity. Define a function n_W on $C_W\big(B(\ )\big)$ by

(i) $n_W(\nu) = 0$ if $\nu > \tau\big(T_1+\lambda T_2\big)$ or if $\nu < \tau\big(T_1+\lambda T_2\big)$ and ν is a characteristic value of Q of even multiplicity;

(ii) $n_W(\gamma_i) = 2(-1)^i$ if $\gamma_i < \tau\big(T_1+\lambda T_2\big)$; and

(iii) $n_W\big(\tau\big(T_1+\lambda T_2\big)\big) = (-1)^{j+1}$, where γ_j is the largest positive characteristic value of Q of odd multiplicity with $\gamma_j < \tau\big(T_1+\lambda T_2\big)$.

(Take j to be zero if no such γ_j exists.)

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LEMMA 3. *Suppose that* $\nu \in C_W(B(\ ))$ *and* $\varepsilon > 0$ *such that* $[\nu-\varepsilon, \nu+\varepsilon] \cap C_W(B(\ )) = \{\nu\}$. *Then*

$$i_W(B(\nu+\varepsilon), W_1) - i_W(B(\nu-\varepsilon), W_1) = n_W(\nu) .$$

Proof. Suppose that $\lambda \in [0, \infty]\backslash C_W(B(\ ))$. By a similar argument to that in [2],

$$\begin{aligned} i_W(B(\lambda), W_1) &= i_K(T_1+\lambda T_2, K_1) \times i_F(\lambda Q, F_1) \\ &= \begin{cases} 0 & \text{if } \lambda < \tau(T_1+\lambda T_2) \\ (-1)^\beta & \text{if } \lambda > \tau(T_1+\lambda T_2) \end{cases} . \end{aligned}$$

Here β is the sum of the multiplicities of the elements of $S \cap [0, \lambda)$. The last equality follows from Theorem 2.4.6 in [7] and our Lemma 1. Lemma 3 follows from this equality and the definition of n_W .

Define

$$D_W(A) = \{(\underline{z}, \lambda) \in \overline{W}_r \times [0, t] : \underline{z} = A(\underline{z}, \lambda), \underline{z} \neq \underline{0}\} \cup$$
$$\cup \{(\underline{0}, \lambda) : \lambda \in C_W(B(\ )) \cap [0, t]\} .$$

THEOREM 2. *If* H *is a component of* $D_W(A)$, *then*

(i) H *intersects* $(W\backslash\{(\underline{0})\}) \times \{0\}$, *or*

(ii) $\sup\{\|\underline{x}\| : (\underline{x}, \lambda) \in H\} = r$, *or*

(iii) $\sup\{\lambda : (\underline{x}, \lambda) \in H\} = t$, *or*

(iv) $\sum n_W(\lambda) = 0$, *where the summation is over* $\{\lambda : (\underline{0}, \lambda) \in H\}$.

In particular, if $\tau(T_1+\lambda T_2) < t$, *(i), (ii) or (iii) holds for the component containing* $(\underline{0}, \tau(T_1+\lambda T_2))$. *Moreover, if there exist a linear operator* V *on* E , *an* $\alpha \in (0, t)$ *and a* $\underline{y} \in W$ *such that* $-\underline{y} \notin W$, $V\underline{y} - \underline{y} \in W$, $V(W) \subseteq W$ *and* $A(\underline{z}, \alpha) - V\underline{z} \in W$ *for* $\underline{z} \in \overline{W}_r$, *then (i) or (ii) holds for the component of* $\{(\underline{z}, \lambda) \in D_W(A) : \lambda \leq \alpha\}$ *containing* $(\underline{0}, \tau(T_1+\lambda T_2))$.

Proof. The proof of the first assertion is similar to the proof of the first assertion of Theorem 1 except that Lemma 3 is used instead of Lemma 1 and $i_W(A, W_\delta)$ is used instead of $i_K(A, K_\delta)$. The second assertion follows from the first and the definition of n_W . The proof of the third assertion is similar to the proof of the second assertion of Theorem 1.

With more care in the proof, it could be shown that, if the assumptions of the last assertion of Theorem 2 hold, then *(i)*, *(ii)* or *(iv)* holds for each component of $\{(\underline{z}, \lambda) \in D_W(A) : \lambda \le \alpha\}$.

The most useful parts of Theorem 2 are those which involve the component containing $(\underline{0}, \tau(T_1+\lambda T_2))$. However, the statements involving the components which intersect $\{(\underline{0}, \gamma_i) : \gamma_i < \tau\}$ do give more information about the solutions than one can obtain from Theorem 1.16 in [9]. With more care in the proof, one could prove rather more general versions of Theorem 2. However, the above result suffices for most applications. It is also possible to prove an analogue of Theorem 3 in [3].

Note that the set $K \oplus F$ is a wedge in the sense of [3]. In [3], we mentioned rather less precise results for general wedges. These are proved by using degree arguments similar to those used here.

3. Applications to systems of differential equations

In this section, we use Theorems 1 and 2 to obtain some new results for boundary-value problems for systems of ordinary differential equations. Let $C[0, 1]$ denote the space of continuous real-valued functions on $[0, 1]$ and $C^1[0, 1]$ the space of real-valued continuously differentiable functions on $[0, 1]$. Assume that, for $i = 1, 2$, $p_i \in C^1[0, 1]$, $q_i \in C[0, 1]$ and $p_i(t) > 0$ for $t \in [0, 1]$. Define L_i by

$$L_i y(t) = -\big(p_i(t)y'(t)\big)' + q_i(t)y(t) .$$

Finally, assume that $f_i : R^4 \to R$ are continuously differentiable for $i = 1, 2$, $f_i(0, 0, 0, 0) = 0$ for $i = 1, 2$, and the partial derivatives

$D_j f_i(0, 0, 0, 0) = 0$ for $i = 1, 2$ and $j = 3, 4$. We wish to find twice continuously differentiable solutions of the boundary-value problem

$$(1)\qquad \begin{aligned} L_1 x(t) - by(t) &= \lambda f_1\big(x(t), y(t), x'(t), y'(t)\big) , \\ L_2 y(t) - dx(t) &= \lambda f_2\big(x(t), y(t), x'(t), y'(t)\big) , \end{aligned}$$

$x(0) = x(1) = y(0) = y(1) = 0$.

We first consider the application of Theorem 1 to this problem. Let T denote the solutions (x, y, λ) of (1) for which $x(t) \ge 0$ and $y(t) \ge 0$ for $t \in [0, 1]$, $\lambda \ge 0$ and $(x, y) \ne (0, 0)$. Define $\tilde{D}_K = T \cup \big(\{(0, 0)\}\times C\big)$, where C is the set of non-negative eigenvalues of the linear problem

$$L_1 x(t) - by(t) = \lambda\big(\alpha_{11}x(t)+\alpha_{12}y(t)\big) ,$$

$$L_2 y(t) - dx(t) = \lambda\big(\alpha_{21}x(t)+\alpha_{22}y(t)\big) ,$$

$x(0) = x(1) = y(0) = y(1) = 0$. Here α_{ij} denotes $D_j f_i(0, 0, 0, 0)$ for $i, j = 1, 2$. If C is non-empty, it has a least element μ_1.

THEOREM 3. *Suppose that all the eigenvalues of* L_1 *and* L_2 *are positive,* $b \ge 0$, $d \ge 0$, bd *is sufficiently small,* $f_1(0, x_2, 0, x_4) \ge 0$ *if* $x_2 \ge 0$ *and* $x_4 \in R$, $f_2(x_1, 0, x_3, 0) \ge 0$ *if* $x_1 \ge 0$ *and* $x_3 \in R$, $\alpha_{ij} \ge 0$ *for* $i, j = 1, 2$, *and* $\alpha_{11} > 0$ *or* $\alpha_{22} > 0$. *Then* C *is non-empty and there exists a connected subset* S *of* $\tilde{D}_K$ *such that* $(0, 0, \mu_1) \in S$ *and* S *is unbounded in* $C^1[0, 1] \times C^1[0, 1] \times R$.

Proof. We apply Theorem 1. Let $E = C^1[0, 1] \times C^1[0, 1]$ and let K be the cone $\{(x, y) \in E : x, y$ are non-negative on $[0, 1]\}$. If $c \ge 0$, define $B_c(\lambda) : E \to E$ by

$$B_c(\lambda)(x, y) = \Big\{\big(L_1+cI\big)^{-1}\big[by+cx+\lambda\alpha_{11}x+\lambda\alpha_{12}y\big], \\ \big(L_2+cI\big)^{-1}\big[dx+cy+\lambda\alpha_{21}x+\lambda\alpha_{22}y\big]\Big\} .$$

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Since all the eigenvalues of $L_i + cI$ are positive (for $i = 1, 2$), $(L_i+cI)^{-1}$ maps non-negative functions to non-negative functions. Thus, if $\lambda \geq 0$, $B_c(\lambda)K \subseteq K$. Note that $C_K(B_c(\ ))$ (and thus $\tau(B_c(\ ))$) is independent of c. Lemma 1 and the homotopy invariance of the degree imply that, if $r(B_0(0)) < 1$, then $r(B_c(0)) < 1$ for all $c \geq 0$. If $(x, y) = \lambda B_0(0)(x, y)$, then $x = \lambda^2 bdL_1^{-1}L_2^{-1}x$. Thus, if bd is sufficiently small, any positive characteristic value of $B_0(0)$ is greater that 1, that is, $r(B_0(0)) < 1$. Since $\alpha_{11} > 0$ or $\alpha_{22} > 0$, Lemma 2 in [3] shows that the mapping D_c defined by

$$D_c(x, y) = \left((L_1+cI)^{-1}[\alpha_{11}x+\alpha_{12}y],\ (L_2+cI)^{-1}[\alpha_{21}x+\alpha_{22}y]\right)$$

has positive spectral radius. (A similar argument is used in §4 of [10].) By a remark after Theorem 1, it follows that $\tau(B_c(\ )) < \infty$.

Now $f_1(x_1, x_2, x_3, x_4) = x_3g_1(x_1, x_2, x_3, x_4) + f_1(x_1, x_2, 0, x_4)$, where $g_1(x_1, x_2, x_3, x_4) = \int_0^1 D_3f_1(x_1, x_2, tx_3, x_4)dt$. Similarly, $f_2(x_1, x_2, x_3, x_4) = x_4g_2(x_1, x_2, x_3, x_4) + f_2(x_1, x_2, x_3, 0)$. Choose n a positive integer with $n > \tau(B_0(\ ))$ and then choose $c > 0$ such that both $cx_1 + bx_2 + \lambda f_1(x_1, x_2, 0, x_4)$ and $cx_2 + dx_1 + \lambda f_2(x_1, x_2, x_3, 0)$ are non-negative if λ, x_1, x_2 are non-negative, $x_1 + x_2 + |x_3| + |x_4| \leq n$ and $\lambda \leq n$. Our assumptions on f_1 and f_2 ensure that this can be done.

Define $L_{i,u,v,\lambda} : \{z \in C^2[0, 1] : z(0) = z(1) = 0\} \to C[0, 1]$ by

$$L_{i,u,v,\lambda}(z) = L_iz - \lambda g_i(u, v, u', v')z$$

and $A : E \to E$ by

$$A(\lambda)(u,\ v) = \Big((L_{1,u,v,\lambda}+cI)^{-1}[cu+bv+\lambda f_1(u,\ v,\ 0,\ v')],$$
$$(L_{2,u,v,\lambda}+cI)^{-1}[du+cv+\lambda f_2(u,\ v,\ u',\ 0)]\Big)\ .$$

If $(x,\ y,\ \lambda) \in D_K(A)$, then $(x,\ y,\ \lambda) \in \tilde{D}_K$. It is easy to check that this mapping satisfies the assumptions of Theorem 1 with $r = t = n$ and linear term $B_c(\ )$. The conditions on the linear term are verified in the preceding paragraph. The remaining conditions are verified by similar arguments to those in [10]. The results in the previous paragraph also show that C is non-empty. Thus, we find that, for each $n > 0$, the component S of $\tilde{D}_K$ containing $(0,\ 0,\ \mu_1)$ intersects $\{(x,\ y,\ \lambda) \in E \times R : \|x\|_1+\|y\|_1 = n \text{ or } \lambda = n\}$. (Our assumptions ensure that there are no non-trivial solutions with $\lambda = 0$.) Since n can be arbitrarily large, this completes the proof.

With more care, one could prove more general results. A similar result still holds if $\alpha_{12}\alpha_{21} > 0$ and $\alpha_{11} = \alpha_{22} = 0$ or if $\alpha_{12}d > 0$ or $\alpha_{21}b > 0$. In the latter cases, we could relax the positivity assumptions on the f_i and still prove that μ_1 is a bifurcation point.

The second assertion of Theorem 1 can be used to obtain additional results. For example, it implies that, if there exists an $\alpha > 0$ such that $f_1(x_1,\ x_2,\ x_3,\ x_4) \geq \alpha x_1$ for $x_1,\ x_2 \geq 0$ and $x_3,\ x_4 \in R$ and if all the assumptions of Theorem 3 are satisfied, then the component of $\left\{(x,\ y,\ \lambda) \in \tilde{D}_K : \lambda \leq \left[\alpha r\left(L_1^{-1}\right)\right]^{-1}\right\}$ containing $(0,\ 0,\ \mu_1)$ is unbounded. Moreover, in this case, we could weaken the assumption on f_2 to $f_2(x_1,\ 0,\ x_3,\ 0) \geq -\alpha r\left(L_1^{-1}\right)dx_1$ for $x_1 \geq 0$ and $x_3 \in R$.

We now consider the applications of the results of §2 to (1). In this case, we place stronger assumptions on one of the equations and weaker ones on the other. Define T' to be the set of solutions $(x,\ y,\ \lambda)$ of (1) for which $\lambda \geq 0$, x is non-negative and $(x,\ y) \neq (0,\ 0)$ and let $\tilde{D}_W = T' \cup (\{(0,\ 0)\}\times C)$.

THEOREM 4. *Suppose that all the eigenvalues of* L_1 *are positive,*

L_2 is invertible, $b = 0$, $f_1(0, x_2, 0, x_4) \geq 0$ *for* $x_2, x_4 \in R$ *and* $\alpha_{11} > 0$. *Then there exists a connected subset* S *of* $\tilde{D}_W$ *such that* $\left(0, 0, \left[\alpha_{11} r\left(L_1^{-1}\right)\right]^{-1}\right) \in S$ *and* S *is unbounded in* $C^1[0, 1] \times C^1[0, 1] \times R$.

Proof. It suffices to show that, for each $n > 0$, the component of $\{(x, y, \lambda) \in \tilde{D}_W : \|x\|_1 + \|y\|_1 \leq n, \lambda \leq n\}$ containing $\left(0, 0, \left[\alpha_{11} r\left(L_1^{-1}\right)\right]^{-1}\right)$ contains a point (x, y, λ) with $\|x\|_1 + \|y\|_1 = n$ or $\lambda = n$. Let $G = C^1[0, 1] \times C^1[0, 1]$, $W = \{(x, y) \in G : x$ is non-negative$\}$. Choose $c > 0$ such that $cx_1 + \lambda f_1(x_1, x_2, 0, x_4) \geq 0$ if x_1 and λ are non-negative, $\lambda \leq n$ and $x_1 + |x_2| + |x_4| \leq n$. Define $\tilde{A}(\lambda) : G \to G$ by

$$\tilde{A}(\lambda)(u, v) = \Big(\left(L_{1,u,v,\lambda} + cI\right)^{-1}\left[cu + \lambda f_1(u, v, 0, v')\right], \quad \left(L_2\right)^{-1}\left[du + \lambda f_2(u, v, u', 0)\right]\Big) .$$

The proof of Theorem 4 is completed by applying Theorem 2 to $\tilde{A}$. Similar arguments to those in the proof of Theorem 3 ensure that $\tilde{A}$ verifies the assumptions of Theorem 2.

The condition that L_2 is invertible can be removed by an approximation argument. Moreover, we could apply the third assertion of Theorem 2 if there exists an $\alpha > 0$ such that $f_1(x_1, x_2, x_3, x_4) \geq \alpha x_1$ if $x_1 \geq 0$, $x_2, x_3, x_4 \in R$.

Our assumptions in Theorems 3 and 4 are much stronger than is really necessary. For example, f_1, f_2 could be allowed to depend on t and λ while p_1, p_2, q_1, q_2, b, d could be allowed to depend on t, x, x', y, y' and λ . Our methods could also be applied to systems of n equations and to systems of differential equations of higher order. In [*10*], Turner considers systems of elliptic partial differential equations. Our methods could be used to considerably strengthen his results. It is possible to obtain variants of our results by considering slightly different cones and wedges, for example, the cone $K = \{(x, y) \in E : x, -y$ are non-negative$\}$.

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Finally, the results in [4] could be used to show that, in Theorems 3 and 4, "connected set" can be replaced by "unbounded arc" if f_1 and f_2 are real analytic (where "unbounded arc" is defined in [4]).

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Department of Mathematics,
University of New England,
Armidale,
New South Wales.

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