

Form its *characteristic equation*

$$f(k) = 0$$

by putting

$$x'_\beta = kx_\beta, (\beta = 1, 2, \dots n),$$

and eliminating the x_β . The roots of $f(k)$ are clearly the above k_α ; and on substituting $k_\alpha x'_\beta$ for x'_β in equations (1) we find that they are satisfied, provided

$$\frac{x_1}{\rho^{-\alpha}} = \frac{x_2}{\rho^{-2\alpha}} = \dots = \frac{x_{n-1}}{\rho^{-(n-1)\alpha}} = \frac{x_n}{1}.$$

These values of $x_1, \dots x_n$ define an invariant point* or *pole* P_α of the substitution. By varying α we obtain n poles $P_1, P_2, \dots P_n$ forming a complete set of poles, with their coordinates *entirely independent of the coefficients* a_r . With the one restriction that D must not vanish, the a_r may have any real or complex values, and every substitution of the form (1) possesses the common system of invariant points $P_1, \dots P_n$.

The corresponding linear invariants, or *invariant* $(n - 2)$ *planes*, also possessed in common, are easily seen to be

$$\xi_\alpha \equiv \sum_{\beta=1}^n \rho^{\alpha\beta} x_\beta = 0, (\alpha = 1, 2, \dots n) \dots \dots \dots (2)$$

§2. The result of successive applications of substitutions of the class (1), whether alike or different, is always another substitution of the class, the *multipliers* (or roots of the characteristic equation) for the resultant being the products of corresponding multipliers for the components. The substitution inverse to (1) is also of the same class.

There will be differences of *type* † within the class, according to the system of equalities among the multipliers. The minimum system of invariant points and $(n - 2)$ -planes, viz. the P_α and the ξ_α , belongs to all; but these will not exhaust the invariant points and $(n - 2)$ -planes of a substitution having one or more equalities among

* *Linear Substitutions and their Invariants*, §2. *Proc. Edin. Math. Soc.*, Vol. XXX.

† *Ibid.* §9.

The only difference between (4) and (3) is that on the left of (4) the order of the first $(n - 1)$ variables x_1, \dots, x_{n-1} is reversed.

Now let x''_γ ($\gamma = 1, 2, \dots, n$) be the result of two applications of $S(\rho)$. Then

$$S^2(\rho). \quad x''_\gamma = n^{-\frac{1}{2}} \sum_{\beta=1}^n \rho^{\gamma\beta} x'_\beta \text{ by (3),}$$

$$= x_{n-\gamma} \text{ by (4);}$$

the original variables being restored, but, with the exception of the last, which remains unchanged, *reversed in order*. It follows that $S^4(\rho)$ is equivalent to the identical substitution, and that $S(\rho)$ is of order four. Further, since the multiplier with which the variables are restored is unity, we have

$$\Delta_n^4(\rho) = 1,$$

$$\Delta_n(\rho) = \exp(\frac{1}{2}p\pi i), \dots\dots\dots(5),$$

p having one of the values 0, 1, 2, 3.

§ 4. $S(\rho)$ being of order four, the roots of its characteristic equation must be fourth roots of unity (*Ibid*, § 5), whence

$$\Delta_n(\rho, k) \equiv (-)^n (k - 1)^q (k + 1)^r (k - i)^s (k + i)^t,$$

where $q + r + s + t = n. \dots\dots\dots(6)$

In finding the indices, we shall confine ourselves to a specific value of ρ ,

$$\rho = \exp(2\pi i/n).$$

The more general case need not detain us, for other values of ρ lead to the same quantities x'_a in (3), only in a different order.

By multiplication of determinants we find for

$$\left. \begin{aligned} n = 2m + 1, \Delta_n(\rho, k)\Delta_n(\rho, -k) &= -(k^2 - 1)^{m+1}(k^2 + 1)^m \\ &= 2m, & & = (k^2 - 1)^{m+1}(k^2 + 1)^{m-1} \end{aligned} \right\} \dots(7)$$

By a theorem of Gauss (see *Mathews, Theory of Numbers*, §184), writing

$$S = \sum_{s=0}^{n-1} \exp(2s^2\pi i/n),$$

we have for

$$\begin{aligned} n \equiv 0 \pmod{4}, \quad S &= (1 + i) \sqrt{n} \\ &\equiv 1 & & = \sqrt{n} \\ &\equiv 2 & & = 0 \\ &\equiv 3 & & = i \sqrt{n}. \end{aligned}$$

Now the coefficient of k^{n-1} in $\Delta_n(\rho, k)$ is

$$(-)^{n-1}n^{-1}S;$$

and with $n = 2m + 1$, $\Delta_n(\rho, k)$ must by (7) have the form

$$\Delta_n(\rho, k) = -(k-1)^\alpha(k+1)^{m+1-\alpha}(k-i)^\beta(k+i)^{m-\beta}.$$

Picking out the coefficient of the second-highest power of k , we have for

$$\begin{aligned} n \equiv 1 \pmod{4}, & (2\alpha - m - 1) + i(2\beta - m) = 1, \alpha = \frac{1}{2}m + 1, \beta = \frac{1}{2}m; \\ n \equiv 3 & (2\alpha - m - 1) + i(2\beta - m) = i, \alpha = \frac{1}{2}(m + 1) = \beta. \end{aligned}$$

Again with $n = 2m$, $\Delta_n(\rho, k)$ is of form

$$\Delta_n(\rho, k) = (k-1)^\alpha(k+1)^{m+1-\alpha}(k-i)^\beta(k+i)^{m-1-\beta},$$

whence for

$$\begin{aligned} n \equiv 0 \pmod{4}, & \alpha = \frac{1}{2}m + 1, \beta = \frac{1}{2}m; \\ & \equiv 2 \quad \alpha = \frac{1}{2}(m + 1), \beta = \frac{1}{2}(m - 1). \end{aligned}$$

The following table summarises these results, and also gives the values of p in (5):

n	$\Delta_n(\rho, k)$	$p \equiv \pmod{4}$
$4s$	$(k-1)^{s+1}(k+1)^s(k-i)^s(k+i)^{s-1}$	$2s + 1$
$4s + 1$	$-(k-1)^{s+1}(k+1)^s(k-i)^s(k+i)^s$	$2s$
$4s + 2$	$(k-1)^{s+1}(k+1)^{s+1}(k-i)^s(k+i)^s$	$2s + 2$
$4s + 3$	$-(k-1)^{s+1}(k+1)^{s+1}(k-i)^{s+1}(k+i)^s$	$2s + 3$

§5. Taking q, r, s, t as in (6), the canonical form of $S(\rho)$ is clearly the following:

$$\begin{aligned} \xi'_\alpha &= \xi_\alpha, & (\alpha = 1, 2, \dots, q), \\ \xi'_\beta &= -\xi_\beta, & (\beta = q + 1, \dots, q + r), \\ \xi'_\gamma &= i\xi_\gamma, & (\gamma = q + r + 1, \dots, q + r + s), \\ \xi'_\delta &= -i\xi_\delta, & (\delta = q + r + s + 1, \dots, n), \end{aligned}$$

the ξ being a set, not in general unique, of linear invariants of $S(\rho)$. It remains to express the ξ in terms of the original coordinates x .

A separation can be made between the ξ with real and those with imaginary multipliers; the former appearing as the invariants of a new substitution in $(q + \tau)$ variables, and the latter in the same relation to another substitution in $(s + t)$ variables. The reduction shows that the ξ are in all cases *real* linear functions of the x ; thus the linear invariants, and therefore also the poles of $S(\rho)$ are all real.

An invariant of $S(\rho)$ with real multiplier (± 1) is an invariant of $S^2(\rho)$ with multiplier $+1$, and therefore by (4) is unaltered by interchange throughout of x_a, x_{n-a} , ($a = 1, 2, \dots, n - 1$). An invariant of $S(\rho)$ with imaginary multiplier $(\pm i)$ is an invariant of $S^2(\rho)$ with multiplier -1 , and therefore simply changes sign on interchange throughout of x_a, x_{n-a} .

Write
$$\begin{aligned} x_a + x_{n-a} &= y_a, \\ x_a - x_{n-a} &= z_a, \end{aligned}$$

where a takes all integral values from 1 to m or $m - 1$ according as $n = 2m + 1$ or $2m$. In the latter case write $x_m = y_m$, and in both cases $x_n = y_{m+1}$. Then the invariants with real multipliers are linear functions of the y , those with imaginary multipliers of the z .

But putting

$$\left. \begin{aligned} \rho^a + \rho^{-a} &\equiv 2\cos(2\pi a/n) = \sigma_a \\ -i(\rho^a - \rho^{-a}) &\equiv 2\sin(2\pi a/n) = \tau_a \end{aligned} \right\}, \dots\dots\dots(8)$$

and taking

(i) $n = 2m + 1$, we have from (3),

$$\left. \begin{aligned} \sqrt{n} \cdot y'_a &= \sum_{\beta=1}^m \sigma_{a\beta} y_\beta + 2y_{m+1}, \quad (a = 1, 2 \dots m) \\ \sqrt{n} \cdot y'_{m+1} &= \sum_{\beta=1}^m y_\beta + y_{m+1} \end{aligned} \right\} \dots\dots\dots(9)$$

and

$$\sqrt{n} \cdot z'_a = \sum_{\beta=1}^m i\tau_{a\beta} z_\beta, \quad (a = 1, 2, \dots m) \dots\dots\dots(10)$$

The former is a substitution of order 2 in the $(m + 1)$ variables $y_1 \dots y_{m+1}$, of which the invariants, when expressed in terms of the x , are those of (3) with real multipliers, and have real coefficients throughout. The invariants of (10) are likewise identical with

multiplier, those given are not unique, and any linear functions of them would serve equally well. The form

$$\sum_{a=1}^n x_a + \sqrt{n} \cdot x_n$$

which appears first in each set, is at once seen to be invariant from (3). It is to be remembered that the σ, τ depend on the value of n .

Since the determinant of (3) is symmetrical, the pole corresponding to any invariant

$$\sum_{a=1}^n p_a x_a$$

can be immediately deduced: it is in fact the point with co-ordinates $(p_1, p_2 \dots p_n)$. The extension to the case of equal multipliers is easily made.

Relations come to light in the process of finding the forms, which are exemplified in the case of $n=7$. For this case the substitution (9) will have as an invariant with multiplier +1 the function

$$\sum_{a=1}^4 p_a y_a$$

provided

$$\begin{array}{llll} -(\sqrt{7} - \sigma_1)p_1 + & \sigma_2 p_2 + & \sigma_3 p_3 + & p_4 = 0 \\ \sigma_2 p_1 - (\sqrt{7} - \sigma_3)p_2 + & \sigma_1 p_3 + & & p_4 = 0 \\ \sigma_3 p_1 + & \sigma_1 p_2 - (\sqrt{7} - \sigma_2)p_3 + & & p_4 = 0 \\ 2p_1 + & 2p_2 + & 2p_3 - (\sqrt{7} - 1)p_4 = 0. & \end{array}$$

Since there are two invariants of this kind, only two of these four equations can be independent; in other words, each first minor of the determinant of the coefficients is zero. This can be verified in virtue of the relations

$$\begin{aligned} \sigma_1 + \sigma_2 + \sigma_3 &= -1, \\ \sigma_2 \sigma_3 + \sigma_3 \sigma_1 + \sigma_1 \sigma_2 &= -2; \end{aligned}$$

and the statement remains true when the sign of the radical is changed.

Similarly the conditions that

$$\sum_{a=1}^3 p_a z_a$$

be an invariant of (10) for this case, with multiplier +i, are

$$\begin{aligned}
 -(\sqrt{7} - \tau_1)p_1 + \tau_2 p_2 + \tau_3 p_3 &= 0 \\
 \tau_2 p_1 - (\sqrt{7} + \tau_3)p_2 - \tau_1 p_3 &= 0 \\
 \tau_3 p_1 - \tau_1 p_2 - (\sqrt{7} - \tau_2)p_3 &= 0,
 \end{aligned}$$

only one of which can be independent. This is evident on remembering that

$$\begin{aligned}
 \tau_1 + \tau_2 - \tau_3 &= \sqrt{7}, \\
 \tau_2 \tau_3 + \tau_3 \tau_1 - \tau_1 \tau_2 &= 0,
 \end{aligned}$$

but is no longer true when the sign of the radical in the three conditions is changed.

§6. The application of the substitution (3) to the transformation of algebraic equations may be indicated. The equations

$$\begin{aligned}
 \phi_n(x) &\equiv \prod_{a=1}^n (x - x_a) = 0, \\
 \psi_n(x) &\equiv \prod_{a=1}^n (x - x'_a) = 0
 \end{aligned}$$

are reciprocally related, the roots of each being derived from those of the other by the same linear substitution.

Consider the effect of increasing the roots of $\phi_n(x)$ by the constant quantity λ . It is clear that all the roots of $\psi_n(x)$ will remain unchanged, except x'_n , which will be increased by $\sqrt{n} \cdot \lambda$.

If therefore we choose $\lambda = -n^{-1} \sum_{a=1}^n x_a$, two things will happen :

- (i) the term in x^{n-1} will drop out from $\phi_n(x)$;
- (ii) the new x'_n will vanish, and $\psi_n(x)$ be replaced by the function of degree one lower

$$\psi_{n-1}(x) \equiv \prod_{a=1}^{n-1} (x - x'_a).$$

Were it possible to obtain the x'_a in terms of the coefficients of $\phi_n(x)$, $\phi_n(x)$ could be solved. In the case of the cubic

$$\phi_3(x) \equiv x^3 + 3px + q = 0,$$

(3) reduces, after the change indicated, to the equations

$$\begin{aligned}
 x'_1 &= 3^{-1}(\omega x_1 + \omega^2 x_2 + x_3), \\
 x'_2 &= 3^{-1}(\omega^2 x_1 + \omega x_2 + x_3), \\
 0 = x'_3 &= 3^{-1}(x_1 + x_2 + x_3) ;
 \end{aligned}$$

TABLE OF LINEAR INVARIANTS OF $S(\rho)$.

n	mult + l	mult + i
2	$\Sigma x + \sqrt{2} \cdot x_2$	none
3	$\Sigma x + \sqrt{3} \cdot x_3$	$x_1 - x_2$
4	$\Sigma x + \sqrt{4} \cdot x_4$ $x_2 + x_4$	$x_1 - x_3$
5	$\Sigma x + \sqrt{5} \cdot x_5$ $(x_1 + x_4) - (x_2 + x_3)$	$(\tau_1 + \sqrt{5})(x_1 - x_4) + \tau_2(x_2 - x_3)$
6	$\Sigma x + \sqrt{6} \cdot x_6$ $(2 + \sqrt{6})(x_1 + x_3) - \sqrt{6}(x_2 + x_4) - 4x_3$	$(\tau_1 + \sqrt{6})(x_1 - x_3) + \tau_2(x_2 - x_4)$ (here $\tau_1 = \tau_2 = +\sqrt{3}$)
7	$\Sigma x + \sqrt{7} \cdot x_7$ $(\sigma_1 - \sigma_2 + \sqrt{7})(x_1 + x_6) + (\sigma_2 - \sigma_3 - \sqrt{7})(x_2 + x_5) + (\sigma_3 - \sigma_1)(x_3 + x_4)$	$\tau_2(x_1 - x_6) - (\tau_1 - \sqrt{7})(x_2 - x_5)$ $(x_1 - x_6) + (x_2 - x_5) - (x_3 - x_4)$
8	$\Sigma x + \sqrt{8} \cdot x_8$ $4(x_1 + x_7) - \sqrt{8} \cdot x_4 + \sqrt{8} \cdot x_5$ $(x_1 + x_7) - (x_3 + x_5)$	$4(x_1 - x_7) + \sqrt{8}(x_2 - x_6)$ $(x_1 - x_7) + (x_3 - x_5)$

whence

$$x_1' x_2' = -\sum x_2 x_3 = -3p,$$

$$x_1'^3 + x_2'^3 = 3 \sqrt{3} \cdot x_1 x_2 x_3 = -3 \sqrt{3} \cdot q,$$

leading to the usual solution. In the case of the biquadratic also the method leads to Euler's reducing cubic.

Section III.—Triangles in Multiple Perspective. Geometrical Interpretation of Substitution (1) for n = 3.

§ 7. The triangle $X_1X_2X_3$ is said to be in perspective with $Y_\alpha Y_\beta Y_\gamma$, the order of the vertices being significant, when $X_1Y_\alpha, X_2Y_\beta, X_3Y_\gamma$ meet in a point. Thus $X_1X_2X_3$ may be in perspective with another triangle in either of six different ways, according to the order of the vertices. Let the vertices, referred to $X_1X_2X_3$ as triangle of reference in any system of homogeneous point-coordinates, be

$$Y_r (x_{r1}, x_{r2}, x_{r3}), (r = 1, 2, 3);$$

then the six cases, with their conditions, fall into two sets of three as follows: $X_1X_2X_3$ is in perspective with

- | | | | |
|-------------------|----------|---|-------------|
| (i) $Y_1Y_2Y_3$ | provided | $x_{12}x_{23}x_{31} = x_{13}x_{21}x_{32}$ | } (A) |
| (ii) $Y_2Y_3Y_1$ | ,, | $x_{11}x_{22}x_{33} = x_{12}x_{23}x_{31}$ | |
| (iii) $Y_3Y_1Y_2$ | ,, | $x_{13}x_{21}x_{32} = x_{11}x_{22}x_{33}$ | |
| (iv) $Y_1Y_3Y_2$ | ,, | $x_{22}x_{31}x_{13} = x_{33}x_{12}x_{21}$ | |
| (v) $Y_3Y_2Y_1$ | ,, | $x_{33}x_{12}x_{21} = x_{11}x_{23}x_{32}$ | |
| (vi) $Y_2Y_1Y_3$ | ,, | $x_{11}x_{23}x_{32} = x_{22}x_{31}x_{13}$ | |

The poles of perspective will be called $Z_r (r = 1, 2, \dots 6)$. Sets (A), (B) are distinguished by the cyclic order of the vertices $Y_1Y_2Y_3$. In each set any two of the conditions involve the third. It is thus clear that

- (a) *triple perspective* will occur when two conditions of one set are fulfilled, and therefore two of the vertices (say Y_2Y_3) may be taken at random. The three poles of perspective are clearly the vertices of a third triangle in triple perspective with each of the other two, the poles for each pair being the vertices of the remaining one;
- (β) *quadruple perspective* occurs when three conditions, not all of one set, are fulfilled. The construction of such a case

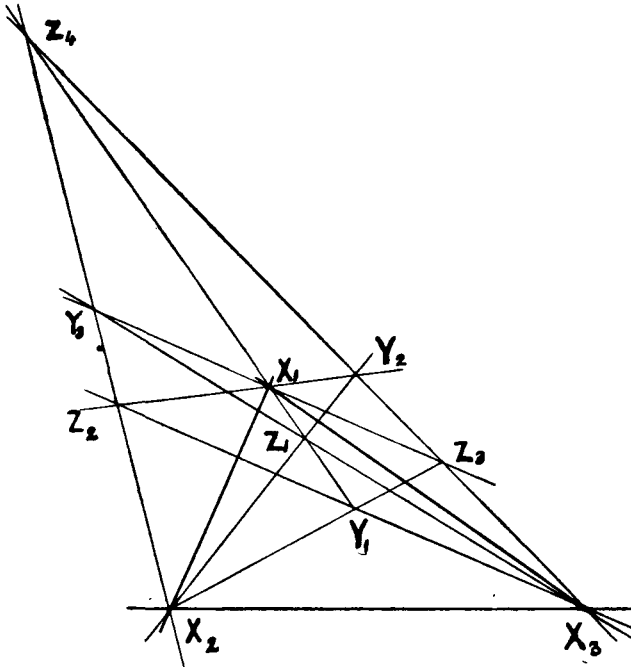
is a simple and instructive exercise in homogeneous coordinates.

One vertex (say Y_3) may be taken at random, and the system of coordinates so chosen that it becomes the point $(1, 1, 1)$. Thus if cases (i)–(iv) hold, we have for the coordinates of the other vertices

$$\begin{aligned} x_{12}x_{23} &= x_{13}x_{21} = x_{11}x_{22}, \\ x_{13}x_{22} &= x_{12}x_{21}, \end{aligned}$$

whence $\frac{x_{12}}{x_{13}} = \frac{x_{21}}{x_{23}} = \frac{x_{22}}{x_{21}} = \frac{x_{13}}{x_{11}} = \alpha$, say,

and Y_1, Y_2 are given by $Y_1(1, \alpha^2, \alpha)$, $Y_2(\alpha, \alpha^2, 1)$, α being arbitrary.



Triangles $X_1X_2X_3, Y_1Y_2Y_3$ in quadruple perspective.

[The figure is given for an arbitrary position of Y_3 , and $\alpha = -2$. The collinearity of $Z_1Z_2Z_3$ is a coincidence, due to the value of α .

chosen. For it is easily found that the coordinates of the poles are

$$Z_1(\alpha, \alpha, 1), Z_2(1, \alpha^2, 1), Z_3(1, \alpha, \alpha), Z_4(1, \alpha, 1);$$

and the determinant of the coordinates of the first three = $\alpha(\alpha - 1)^2(\alpha + 2)$, vanishing for $\alpha = -2$.]

(γ) *Sextuple* perspective occurs on the fulfilment of all six conditions, but, there being only four independent, one vertex Y_3 can still be taken at random. The conditions yield

$$\frac{x_{11}}{x_{12}} = \frac{x_{12}}{x_{13}} = \frac{x_{13}}{x_{11}} = \frac{x_{22}}{x_{21}} = \frac{x_{23}}{x_{22}} = \frac{x_{21}}{x_{23}} \\ = \text{(clearly) } 1, \omega, \text{ or } \omega^2.$$

Unity is rejected as reducing $Y_1Y_2Y_3$ to a point. We take the value ω , and obtain the triangle $Y_1(\omega^2, \omega, 1), Y_2(\omega, \omega^2, 1), Y_3(1, 1, 1)$; the value ω^2 would simply interchange Y_1, Y_2 . Thus *there is only one triangle in sextuple perspective with the triangle of reference, and with a vertex at an arbitrary point. If that point is the real point (1, 1, 1), the other two vertices are imaginary points on the real line*

$$x_1 + x_2 + x_3 = 0.$$

§8. Consider the substitution (1) for $n = 3$, applied to the system of coordinates here in use. It is evident that $Y_1Y_2Y_3$ is coincident with the invariant triangle of the substitution. We shall therefore revert to our previous notation and call it the triangle $P_1P_2P_3$. The six poles, in the order of §7, are

- (i) $Q_3(1, 1, \omega^2)$, (iv) $R_3(1, 1, \omega)$,
- (ii) $Q_2(1, \omega^2, 1)$, (v) $R_2(1, \omega, 1)$,
- (iii) $Q_1(\omega^2, 1, 1)$, (vi) $R_1(\omega, 1, 1)$.

To pass from one system of homogeneous coordinates to another, with the same triangle of reference, we simply alter the coordinates of each point in definite ratios, so that, e.g. the old coordinates of a point having been x_1, x_2, x_3 , its new ones are $(\alpha x_1, \beta x_2, \gamma x_3)$, α, β, γ being constant for all points. Thus the triangle $T_1(\omega^2\alpha, \omega\beta, \gamma) T_2(\omega\alpha, \omega^2\beta, \gamma) T_3(\alpha, \beta, \gamma)$, whatever the system of coordinates, is in sextuple perspective with the triangle of reference, and can be made to coincide with any triangle in this relation, by giving suitable values to $\alpha:\beta:\gamma$. Now the two sets

of multipliers $\alpha:\beta:\gamma = \omega^2:1:1$ and $\omega:1:1$ respectively change the points

$$P_1P_2P_3 \quad Q_1Q_2Q_3 \quad R_1R_2R_3$$

into

$$Q_3Q_2Q_1 \quad R_1R_3R_2 \quad P_3P_1P_2,$$

and

$$R_2R_3R_1 \quad P_3P_2P_1 \quad Q_1Q_3Q_2.$$

Hence each of the triangles $P_1P_2P_3$, $Q_1Q_2Q_3$, $R_1R_2R_3$ is in sextuple perspective with the triangle of reference, and the poles of perspective in each case are the vertices of the remaining two.

§9. The relation of $P_1P_2P_3$ to $X_1X_2X_3$ may be expressed by another geometrical analogy. A point and a line are called *harmonic pole and polar* with respect to a triangle in their plane when the line, and the joins of the point to the vertices, divide the three sides harmonically. If each vertex and opposite side of one triangle are harmonic pole and polar with respect to another, the former may be called *self-polar* with respect to the latter; and, as it easily follows from the conditions obtained below that the relation is reciprocal, the two triangles may be called *conjugate*.

To find the conditions that the three points

$$(x_{\alpha 1}, x_{\alpha 2}, x_{\alpha 3}), \quad (\alpha = 1, 2, 3),$$

form a triangle self-polar with respect to the triangle of reference.

Denoting by $X_{\alpha\beta}$ the cofactor of $x_{\alpha\beta}$ in the determinant Δ of the coefficients, and expressing that each point is the harmonic pole of the line joining the other two, we obtain the nine relations

$$x_{\alpha\beta} X_{\alpha\beta} = \text{const.} = \frac{1}{3}\Delta. \quad (\alpha, \beta = 1, 2, 3).$$

The two relations

$$x_{11}X_{11} = x_{12}X_{12}, \quad x_{33}X_{33} = x_{23}X_{23}$$

may be written respectively

$$x_{33}(x_{11}x_{22} + x_{12}x_{21}) = x_{23}(x_{12}x_{31} + x_{11}x_{32}),$$

$$x_{33}(x_{11}x_{22} - x_{12}x_{21}) = x_{23}(x_{12}x_{31} - x_{11}x_{32}),$$

whence, and by symmetry,

$$x_{11}x_{22}x_{33} = x_{23}x_{31}x_{12} = x_{32}x_{13}x_{21},$$

$$x_{11}x_{23}x_{32} = x_{22}x_{31}x_{13} = x_{33}x_{12}x_{21}.$$

These conditions, which satisfy all the nine relations, are exactly those for sextuple perspective (§ 7). Thus *triangles in sextuple perspective are conjugate, and vice versa.*

The triangle $P_1P_2P_3$, being uniquely determined as soon as the system of coordinates is specified, might be fitly called *the principal related triangle* for the system.

§ 10. The substitution (1) becomes for $n = 3$,

$$\left. \begin{aligned} x_1' &= a_1x_1 + a_2x_2 + a_3x_3 \\ x_2' &= a_3x_1 + a_1x_2 + a_2x_3 \\ x_3' &= a_2x_1 + a_3x_2 + a_1x_3 \end{aligned} \right\} \dots \dots \dots (13)$$

Taking x_1, x_2, x_3 as homogeneous point-coordinates, we must distinguish two possible interpretations. First, it might mean that the *point* whose coordinates are x_1, x_2, x_3 has to be shifted to (x_1', x_2', x_3') ; thus $(1, 0, 0)$ would be shifted to (a_1, a_3, a_2) , etc. Secondly, it might mean that the *line* whose equation is

$$l_1x_1 + l_2x_2 + l_3x_3 = 0$$

is to be shifted into the position of the line

$$l_1x_1' + l_2x_2' + l_3x_3'$$

$$\equiv (a_1l_1 + a_3l_2 + a_2l_3)x_1 + (a_2l_1 + a_1l_2 + a_3l_3)x_2 + (a_3l_1 + a_2l_2 + a_1l_3)x_3 = 0 ;$$

so that $x_1 = 0$ would become

$$a_1x_1 + a_2x_2 + a_3x_3 = 0,$$

and so on.

If in (13) we replace a_1, a_2, a_3 respectively by A_1, A_3, A_2 the cofactors of a_1, a_3, a_2 in the determinant of the coefficients, the *first* interpretation of the new substitution is geometrically equivalent to the *second* interpretation of (13). We can therefore afford to ignore the second interpretation as virtually involved in the first. It is also easy to see that the geometrical effects of (13) in its two interpretations are mutually inverse.

§ 11. With the first interpretation, then, (13) shifts $X_1X_2X_3$ into $Y_1(a_1, a_3, a_2), Y_3(a_2, a_1, a_3), Y_2(a_3, a_2, a_1)$. These coordinates satisfy equations (A) of § 7; hence *every triangle into which $X_1X_2X_3$ is changed by (13) is in triple perspective with $X_1X_2X_3$ itself.*

The poles are found to be

$$Z_1(a_1^{-1}, a_2^{-1}, a_3^{-1}), Z_2(a_3^{-1}, a_1^{-1}, a_2^{-1}), Z_3(a_2^{-1}, a_3^{-1}, a_1^{-1}),$$

satisfying equations (B) of § 7, and therefore forming a triangle in triple perspective with $X_1X_2X_3$, but in reversed cyclic order.

The condition that Y_1, Y_2, Y_3 should satisfy (iv) in addition to (i)–(iii) is

$$a_2^3 = a_3^3;$$

thus the equality of two coefficients in the substitution involves quadruple perspective between $X_1X_2X_3$ and every triangle obtained from it by the substitution; and the same holds if two coefficients have the ratio ω or ω^2 , though the triangle is then imaginary. Whatever the system of point-coordinates and the value of α , the triangle $(\alpha, 1, 1), (1, 1, \alpha), (1, \alpha, 1)$ is in quadruple perspective in ways (i) to (iv) with $X_1X_2X_3$.

§ 12. The invariant points and lines of (13), forming the *invariant triangle*, are

$$\left. \begin{aligned} P_1(\omega^2, \omega, 1), & \quad \sqrt{3} \cdot \xi_1 = \omega x_1 + \omega^2 x_2 + x_3 \\ P_2(\omega, \omega^2, 1), & \quad \sqrt{3} \cdot \xi_2 = \omega^2 x_1 + \omega x_2 + x_3 \\ P_3(1, 1, 1), & \quad \sqrt{3} \cdot \xi_3 = x_1 + x_2 + x_3 \end{aligned} \right\} \dots\dots\dots(14)$$

This triangle is what we have called (§ 9) *the principal related triangle* for the particular system of point-coordinates in use. It is in sextuple perspective with, and conjugate to, the triangle of reference (§§ 8, 9); and not only so, but, as perspective and harmonic properties are unaltered by a linear transformation, it *stands in the same relations to every triangle obtainable by (13) from the triangle of reference*. When we say “by (13),” we mean by every substitution of the *form* (13), the result being independent of the coefficients of (13).

§ 13. The geometrical interpretation of $S(\rho)$ for $n=3$, or of $S(\omega)$, is plain: it changes the triangle of reference $X_1X_2X_3$ into the *principal related triangle* $P_1P_2P_3$ for the coordinates in use, the order of the vertices being significant. $S(\omega^2)$ will change $X_1X_2X_3$ into $P_2P_1P_3$, and substitutions of like form will change $X_1X_2X_3$ into each of the other four ways in which $P_1P_2P_3$ may be named. Each pair of substitutions, such as $S(\omega)$ and $S(\omega^2)$, gives *one* invariant triangle, and the nine vertices are as follows, the three

last deduced from the table of §5, the others by cyclic interchange :—

$$\begin{aligned} S_{11}(0, 1, -1), & \quad S_{12}(\sqrt{3}+1, 1, 1), & \quad S_{13}(-\sqrt{3}+1, 1, 1); \\ S_{21}(1, -\sqrt{3}+1, 1), & \quad S_{22}(-1, 0, 1), & \quad S_{23}(1, \sqrt{3}+1, 1); \\ S_{31}(1, 1, \sqrt{3}+1), & \quad S_{32}(1, 1, -\sqrt{3}+1), & \quad S_{33}(1, -1, 0). \end{aligned}$$

They are all real, the three diagonal ones lying in a straight line, and the others, in two sets of three, forming triangles in triple perspective with $X_1X_2X_3$.

Under repetitions of $S(\omega)$, $X_1X_2X_3$ passes through the successive stages $P_1P_2P_3$, $X_1X_2X_3$, $P_2P_1P_3$, and so back to $X_1X_2X_3$, the points S_{31} , S_{32} , S_{33} remaining fixed; and similarly in the other cases.*

* Since writing the above, I have found the following references to triangles in multiple perspective :—

Homologous Triangles, by Thomas Muir, M.A. [*Mess. Math.* 2 (1873)]. “Triply homologous” triangles are discussed, the vertices being taken in the same *cyclic* order throughout.

Perspective Dreiecke und Tetraeder, by Edmund Hess [*Math. Annalen* 28 (1887)]. Here sextuply perspective triangles are introduced, the standpoint being geometrical, and the application different from that of the present paper.