Canad. Math. Bull. Vol. 17 (5), 1975

ON SOME NUMBERS RELATED TO THE BELL NUMBERS

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ABSTRACT. The Bell numbers B_n can be defined by $B_n=\sum_{k=1}^{n} S(n,k)$ where $S(n,k)$ is the Stirling number of the second kind. In this note we employ a technique developed by Rota (which formalizes the umbral calculus) to derive a veriety of facts concerning the related numbers $F_n=\sum_{k=1}^{n} k!S(n,k)$ and polynomials $F_n(x)=\sum_{k=1}^{n} k!S(n,k)x^k$.

1. The Stirling numbers of the second kind, denoted here by $S(n,k)$, occur frequently in the literature of combinatorial theory and the calculus of finite differences. In the latter $S(n,k)$ is related to the difference operator Δ; in more precise terms, and using the terminology introduced by Mullin and Rota in [9], $S(n,k)$ are the connection constants relating the basic sequences x^n and $(x)_n$. That is to say, $x^n=\sum_{k=0}^{n} S(n,k)(x)_k$, where $(x)_k$ is the falling factorial given by the formula $(x)_k=x(x-1)\cdots(x-k+1)$. In the combinatorial literature $S(n,k)$ occurs in a variety of interesting contexts. In classical distribution and occupancy problems ("balls in boxes") $S(n,k)$ counts the number of ways of placing n distinct objects into k non-distinct boxes with no box left empty [5, 8, 10]. Obviously equivalent is the interpretation of $S(n,k)$ as the number of partitions of an n-set X into k non-empty (disjoint) subsets, or blocks, where a partition of X is a family of disjoint (nonempty) subsets of X whose union is X. From the latter interpretation of $S(n,k)$ it follows immediately that the numbers $B_n=\sum_{k=0}^{n} S(n,k)$ can be interpreted combinatorially as the number of distinct partitions of a set of n elements. The numbers B_n are alternately called the Bell numbers (after E. T. Bell) or the exponential numbers (Bell's terminology) and have been the object of much study for more than forty years. The sequence B_n can be characterized by its exponential generating function, well-known to be $\exp(e^x-1)$; alternatively B_n satisfy the recursion

$$B_{n+1}=\sum_{k=0}^{n}\binom{n}{k}B_k, \qquad B_0=1. \tag{1}$$

The exponential polynomials $B_n(x)$ are related to the Bell numbers, and are given by the formula $B_n(x)=\sum_{k=0}^{n} S(n,k)x^k$. These polynomials have been studied extensively by a great many authors (see the references cited in [13]).

Received by the editors April 9, 1973 and, in revised form, May 13, 1974.

Related to the Stirling numbers $S(n, k)$ are the numbers $k!S(n, k)$, that is, the Stirling numbers weighted by the appropriate factorial. These numbers too admit a variety of combinatorial interpretations, all of which are closely linked to the interpretations of $S(n, k)$. For example, in occupancy and distribution problems, $k!S(n, k)$ counts the number of ways of placing n distinct balls into k distinct boxes. In terms of partitions of an n-set, $k!S(n, k)$ is the number of distinct *ordered* partitions with k blocks. Other interpretations include the number of surjections from an n-set onto a k-set [1] and the number of ordered nontrivial factorizations into k factors of a square-free integer which is the product of n distinct primes [7, 8].

It is perhaps curious that there appear to be very much fewer references to the related numbers $F_n=\sum_{k=0}^n k!S(n, k)$ or the polynomials $F_n(x)=\sum_{k=0}^n k!S(n, k)x^k$. The numbers F_n can be given a combinatorial interpretation analogous to the one for the Bell numbers B_n: if X is an n-set then F_n is the number of distinct *ordered* partitions of X. These numbers are discussed in [6] in connection with preference arrangements; the recursion for F_n is derived, as well as the exponential generating function and an asymptotic estimate. The latter two are also derived in [7] using a different technique.

In the present paper we derive a variety of results, some apparently unnoticed, concerning F_n and $F_n(x)$. In doing so we make use of a particularly elegant technique developed by Rota in [13] and further extended in [9] to provide a rigorous formulation for the Blissard, or symbolic, calculus. By use of this approach we somewhat simplify (and unify) the derivation of these facts as well as provide some interesting and natural links to the Eulerian numbers.

2. Recall from [13] that for any indeterminate u we can write

$$\sum_{\pi} (u)_{N(\pi)} = u^n, \tag{2}$$

where $(u)_{N(\pi)}=u(u-1)\cdots(u-N(\pi)+1)$ and where π is any partition of an n-set and $N(\pi)$ is the number of blocks of π. Notice that $1\leq N(\pi)\leq n$ and further that the number of partitions with exactly $k=N(\pi)$ blocks is $S(n, k)$.

Let V be the real vector space of polynomials in the variable u and define the linear functional L on the basis $\{(u)_k : k\geq 0\}$ of V by

$$L(1) = 1, \qquad L((u)_k) = x^k k!, \qquad k = 1, 2, 3, \ldots$$

where x is any (fixed) real number. Then L can be extended uniquely to all of V. Applying L to (2) we obtain

$$\begin{aligned} L(u^n) &= \sum_{\pi} L((u)_{N(\pi)}) \\ &= \sum_{\pi} x^{N(\pi)}(N(\pi))! \\ &= \sum_{k=1}^{n} k!\, S(n, k)x^k; \end{aligned}$$

https://doi.org/10.4153/CMB-1974-132-8 Published online by Cambridge University Press

that is, for x a (fixed) arbitrary real number,

$$F_n(x) = L(u^n). \tag{3}$$

From the definition of L we have

$$\begin{aligned} L((u)_{n+1}) &= x^{n+1}(n+1)! \\ &= x(n+1)L((u)_n) \\ &= xL((n+1)(u)_n) \\ &= xL(\Delta(u)_{n+1}) \end{aligned}$$

where Δ is the difference operator defined by $\Delta f(u)=f(u+1)-f(u)$. It follows that if $p(u)$ is any polynomial,

$$L(p(u)) = x[p(0)+L(\Delta p(u))]. \tag{4}$$

Set $p(u)=u^n$; then $\Delta p(u)=\sum_{j=0}^{n-1} \binom{n}{j}u^j$, and (4) becomes

$$\begin{aligned} L(u^n) &= xL\left(\sum_{j=0}^{n-1}\binom{n}{j}u^j\right) \\ &= x\sum_{j=0}^{n-1}\binom{n}{j}L(u^j). \end{aligned} \tag{5}$$

Thus we have the recurrence

$$F_n(x) = x\sum_{j=0}^{n-1}\binom{n}{j}F_j(x). \tag{6}$$

Notice that since (6) holds for an arbitrary but fixed real number x it follows that the *polynomials* $F_n(x)$ defined by $F_n(x)=\sum_{k=1}^{n} k!S(n, k)x^k$ satisfy (6). Specializing (6) to the case $x=1$ yields (upon adding F_n to both sides)

$$2F_n = \sum_{j=0}^{n}\binom{n}{j}F_j; \tag{7}$$

using the usual notation of the "umbral" calculus, we have

$$2F_n = (1+F)^n, \qquad F^j \equiv F_j. \tag{8}$$

The relation (7) appears in [6].

3. Using (3) we can follow a procedure similar to that given in [13] to obtain the exponential generating function of $F_n(x)$ (and thence F_n) in a formalistic way:

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{F_n(x)}{n!}t^n &= \sum_{n=0}^{\infty}\frac{L(u^n)}{n!}t^n \\ &= L(e^{ut}). \end{aligned}$$

Put $e^t=1+v$ and expand $(1+v)^u$ using the binomial theorem:

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{F_n(x)}{n!}t^n &= L((1+v)^u)\\ &= L\left(\sum_{n=0}^{\infty}\frac{(u)_n}{n!}v^n\right)\\ &= \sum_{n=0}^{\infty}\frac{L((u)_n)}{n!}v^n\\ &= \sum_{n=0}^{\infty}x^n v^n\\ &= \frac{1}{1-xv}.\end{aligned}$$

Hence,

$$\sum_{n=0}^{\infty}\frac{F_n(x)}{n!}t^n = \frac{1}{1-x(e^t-1)}. \tag{9}$$

Specializing to the case $x=1$ yields

$$\sum_{n=0}^{\infty}\frac{F_n}{n!}t^n = \frac{1}{2-e^t}, \tag{10}$$

which appears in [6, 7]. Alternatively, setting $x=-1$ the right-hand side becomes $e^{-t}=\sum_{n=0}^{\infty}((-1)^n/n!)t^n$. Noting that $F_n(-1)=\sum_{k=0}^{n}(-1)^k k!S(n,k)$ we have proved

$$\sum_{k=0}^{n}(-1)^k k!\,S(n,k) = (-1)^n. \tag{11}$$

This formula is not new; see for example, [8, p. 170].

From (9) we derive a remarkable representation of $F_n(x)$ for $x\neq -1$ as an infinite series:

$$\begin{aligned}\frac{1}{1-x(e^t-1)} &= \frac{1}{(1+x)\left(1-\dfrac{x}{1+x}e^t\right)}\\ &= \frac{1}{1+x}\sum_{k=0}^{\infty}\left(\frac{x}{1+x}e^t\right)^k\\ &= \frac{1}{1+x}\sum_{k=0}^{\infty}\left(\frac{x}{1+x}\right)^k\sum_{n=0}^{\infty}\frac{(kt)^n}{n!}\\ &= \frac{1}{1+x}\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{k=0}^{\infty}\left(\frac{x}{1+x}\right)^k k^n\end{aligned}$$

and hence we conclude that

$$F_n(x) = \frac{1}{1+x}\sum_{k=0}^{\infty}\left(\frac{x}{1+x}\right)^k k^n. \tag{12}$$

The case $x=1$ is known ([6, 7]):

$$F_n = \frac{1}{2}\sum_{k=0}^{\infty}\frac{k^n}{2^k}. \tag{13}$$

Notice that (12) is not meaningful for $x \leq -\frac{1}{2}$ since the right-hand side does not converge. Relation (13) is the formal analogue for F_n of the Dobinski formula [4] for the Bell numbers B_n while (12) is the analogue of a formula for the exponential polynomials appearing in [9, p. 205].

4. The linear functional L can be used to relate the polynomial $F_n(x)$ to the Eulerian numbers [2], denoted here by $a_{n,k}$. Recall the famous formula of Worpitzky [15]:

$$u^n = \sum_{k=1}^{n} a_{n,k} \binom{u+k-1}{n}. \tag{14}$$

Applying L to both sides of (14) gives

$$L(u^n) = \sum_{k=1}^{n} \frac{a_{n,k}}{n!} L((u+k-1)_n) \tag{15}$$

But $(u+k-1)_n = \sum_{r=0}^{n} \binom{n}{r}(u)_r(k-1)_{n-r}$ and so simplifying (15) we obtain

$$\begin{aligned} L(u^n) &= \sum_{k=1}^{n} a_{n,k} \sum_{r=0}^{n} \frac{(k-1)_{n-r}}{(n-r)!} x^r \\ &= \sum_{k=1}^{n} a_{n,k} \sum_{r=0}^{n} \binom{k-1}{n-r} x^r, \end{aligned}$$

hence we conclude that

$$F_n(x) = \sum_{k=1}^{n} a_{n,k} x^{n-k+1}(1+x)^{k-1}. \tag{16}$$

I have been unable to find (16) in the literature, although the special case with $x=1$ is known; see, for example, [11, p. 89] where the formula

$$F_n = \sum_{k=1}^{n} a_{n,k} 2^{k-1} \tag{17}$$

is derived by means of generating functions.

Using a probabilistic interpretation for the Eulerian numbers given in [14] we can rewrite (17) as

$$2\frac{F_n}{n!} = \sum_{k=1}^{n} p_k 2^k \tag{18}$$

where p_k is the probability that the sum S_n of n independent uniform random variables on [0, 1) lies between $k-1$ and k. It follows that

$$2\frac{F_n}{n!} = \sum_{k=1}^{n} 2^k [G_n(k) - G_n(k-1)] \geq \int_0^n 2^t \, dG_n(t) \tag{19}$$

where G_n is the distribution function of S_n. Evaluating the integral in (19) recursively using integration by parts we can show that

$$\int_0^n 2^t \, dG_n(t) = \left(\frac{1}{\ln 2}\right)^n, \tag{20}$$

https://doi.org/10.4153/CMB-1974-132-8 Published online by Cambridge University Press

hence we obtain a lower bound for F_n. In fact it has been shown elsewhere [6, 7] that

$$2\frac{F_n}{n!} \sim \left(\frac{1}{\ln 2}\right)^{n+1} \tag{21}$$

so that the inequality (19) is not really very good There does not appear to be any way to obtain (21) directly from (19).

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