# THE NUMBER OF GRAPHS WITH A GIVEN AUTOMORPHISM GROUP

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**1. Introduction.** In this paper, the graphs under consideration may have multiple edges but they do not have loops. We enumerate the number $N[H: n, p]$ of topologically distinct graphs with $n$ vertices and $p$ edges whose automorphism group is the permutation group $H$. As in (**5**), this enumeration is considered in the context of the theory of permutation representations of finite groups. We begin with some definitions and notation.

Let $\mathsf{N}$ denote the set of natural numbers $0, 1, 2, \ldots$, etc., and let $[\![x_1, x_2, \ldots, x_n]\!]$ denote an unordered $n$-tuple of elements from some set. Suppose $\mathfrak{G}$ is a permutation group of degree $q$ which permutes the elements of the set $X = \{a_1, a_2, \ldots, a_q\}$. The set of trivial orbits of $g \in \mathfrak{G}$ will be denoted by $T(g)$ and $|T(g)|$ will be denoted by $m(g)$. Let $K(\mathfrak{G})$ denote the *symmetric group of order* $q!$ which consists of all permutations of the elements of $X$. Usually, when the elements permuted by the symmetric group of degree $q$ and order $q!$ are not specified explicitly, it will be denoted by $\mathfrak{S}_q$. The permutation

$$\begin{pmatrix} a_1\, a_2 \ \ldots\ a_q \\ a_{i_1}\, a_{i_2} \ldots a_{i_q} \end{pmatrix}$$

belonging to $K(\mathfrak{G})$ will be denoted, when no ambiguity arises, simply by

$$\left(\begin{pmatrix} a_k \\ a_{i_k} \end{pmatrix}\right),$$

etc. Let $(\mathfrak{G})^p$, $p \in \mathsf{N}$, denote the group of permutations (see **1**, p. 300) of the homogeneous products of $p$ dimensions of the elements of $X$ induced by the permutations belonging to $\mathfrak{G}$ of elements of $X$. More precisely, suppose $a_s^k$, $a_s \in X$, denotes the unordered set $[\![a_s, a_s, \ldots, a_s]\!]$ of $k$ elements and $a_s^k g$, $g \in \mathfrak{G}$, denotes the unordered set $[\![a_s g, a_s g, \ldots, a_s g]\!]$ of $k$ elements. Then $(\mathfrak{G})^p$ is the permutation group, which permutes all elements of the form

$$[\![a_{11}^{p_1}, a_{22}^{p_2}, \ldots, a_{ss}^{p_s}]\!]$$

($p_i \in \mathsf{N}$, $p_1 + p_2 + \ldots + p_s = p$; $a_{ii} \in X$, $a_{ii} \neq a_{jj}$, $i \neq j$), consisting of permutations $\phi^p(g)$, $g \in \mathfrak{G}$, defined by

$$\phi^p(g) = \begin{pmatrix} [\![a_{11}^{p_1}, a_{22}^{p_2}, \ldots, a_{ss}^{p_s}]\!] \\ [\![a_{11}^{p_1}g, a_{22}^{p_2}g, \ldots, a_{ss}^{p_s}g]\!] \end{pmatrix}.$$

Received November 14, 1966.

*Example.* Suppose that $X = \{a, b, c\}$ and $\mathfrak{G} = \{g_1, g_2, g_3\}$, where $g_1 = (a)(b)(c)$, $g_2 = (a \quad b \quad c)$ and $g_3 = (a \quad c \quad b)$. Then

$$(\mathfrak{G})^2 = \{\phi^2(g_1), \phi^2(g_2), \phi^2(g_3)\},$$

where, in an obvious notation, $\phi^2(g_1) = (a^2)(b^2)(c^2)(ab)(bc)(ac)$, $\phi^2(g_2) = (a^2 \quad b^2 \quad c^2)(ab \quad bc \quad ac)$, and $\phi^2(g_3) = (a^2 \quad c^2 \quad b^2)(ab \quad ac \quad bc)$. A *permutation representation* $\mu$ of an abstract finite group $P$ of order $\pi$ is a homomorphism from $P$ into a permutation group $\mathfrak{G}$, e.g., $\phi^p\colon \mathfrak{G} \to (\mathfrak{G})^p$ is (see **1**, p. 300), a permutation representation of $\mathfrak{G}$ called, say, the *$(p)$-representation* of $\mathfrak{G}$. Now suppose $\mu_1\colon P \to \mathfrak{G}_1$ and $\mu_2\colon P \to \mathfrak{G}_2$ are permutation representations of $P$, and $\mathfrak{G}_1$ and $\mathfrak{G}_2$ permute the elements of the sets $A_1 = \{x_1, x_2, \ldots, x_\theta\}$, $A_2 = \{y_1, y_2, \ldots, y_\theta\}$, respectively. Then $\mu_1, \mu_2$ are *equivalent*, denoted by $\mu_1 \sim \mu_2$, if there exist mappings $\sigma\colon \mu_1(P) \to \mu_2(P)$, $\tau\colon A_1 \to A_2$ such that for every $x_i \in A_1$,

$$(x_i\mu_1(r))\tau = (x_i\tau)(\mu_2(r)\sigma), \qquad r \in P.$$

Let $\mu\colon P \to \mathfrak{G}$ be a permutation representation of $P$. The *characteristic* $\chi(r)$, $r \in P$, of $r$ in $\mu$ is the number of trivial orbits of $\mu(r)$, i.e., $\chi(r) = m(\mu(r))$. The *character* $\chi$ of $P$ in $\mu$ is the set of characteristics $\chi(r)$, $r \in P$. If $\mu(P)$ consists of just the identity element of $\mathfrak{G}$, then $\mu$ is called the *unit representation* of $P$ and $\chi(r) = 1$, $r \in P$. In this case, $\chi$ is called the *unit character* of $P$ and is denoted by **1**. If $P_0$ is a subgroup of $P$, then the *mark* (see **1**, p. 236) $m(P_0; \mu)$ of $P_0$ in $\mu$ is defined by

$$m(P_0; \mu) = \Big|\bigcap_{r \in P_0} T(\mu(r))\Big|.$$

If $P_1, P_2, \ldots, P_\Omega$ are all the distinct, up to conjugacy, subgroups of $P$, then $m(\mu)$ is the set of marks $m(P_i; \mu)$, $i = 1, 2, \ldots, \Omega$. Suppose $\mu_1, \mu_2, \ldots, \mu_N$ are permutation representations of $P$ and $\chi_1, \chi_2, \ldots, \chi_N$ are the characters of $P$ in $\mu_1, \mu_2, \ldots, \mu_N$, respectively. The *scalar product* $(\chi_1, \chi_2, \ldots, \chi_N)$ of $\chi_1, \chi_2, \ldots, \chi_N$ is defined by

$$(\chi_1, \chi_2, \ldots, \chi_N) = \frac{1}{\pi} \sum_{r \in P} \chi_1(r)\chi_2(r) \ldots \chi_N(r).$$

We shall now restrict the discussion to the case when $P = \mathfrak{S}_n$. Suppose $H_1, H_2, \ldots, H_\omega$ are all the distinct, up to conjugacy, sugbroups of $\mathfrak{S}_n$. Let $S_i = \{H_i x_{i1}, H_i x_{i2}, \ldots, H_i x_{i\alpha_i}\}$ be the set of left cosets of $\mathfrak{S}_n$ with respect to $H_i$. Let $\mathfrak{S}_n{}^{H_i}$ denote the group of permutations

$$\mu_i(r) = \left(\begin{pmatrix} H_i x_{ik} \\ H_i x_{ik} r \end{pmatrix}\right), \qquad r \in \mathfrak{S}_n,\ 1 \leqq k \leqq \alpha_i,\ 1 \leqq i \leqq \omega.$$

$\mathfrak{S}_n{}^{H_i}$ is a transitive group and $\mu_i$ is called *the transitive permutation representation of $\mathfrak{S}_n$ induced by $H_i$*. Let

$$\phi_i^p\colon K(\mathfrak{S}_n^{H_i}) \to \{K(\mathfrak{S}_n^{H_i})\}^p$$

https://doi.org/10.4153/CJM-1968-103-x Published online by Cambridge University Press

denote the ($p$)-representation of $K(\mathfrak{S}_n^{H_i})$. Clearly, the composition $\phi_i^p \mu_i$ of $\phi_i^p$ and $\mu_i$ is again a permutation representation of $\mathfrak{S}_n$ called (see **4**) *the symmetrized Kronecker product representation of* $\mathfrak{S}_n$ *of dimension* $p$ *induced by* $H_i$. We write $\sigma_i^p = \phi_i^p \mu_i$, $1 \leqq i \leqq \omega$. Let $\chi_i$, $\chi_i^p$ denote the character of $\mathfrak{S}_n$ in $\mu_i$ and $\sigma_i^p$, respectively, and let $\bar{\chi}_i^p$ denote the character of $K(\mathfrak{S}_n^{H_i})$ in $\phi_i^p$. It is well known (see **2**, p. 273) that $\chi_i(r) = (n!h_\rho/|H_i|g_\rho)$, $r \in \mathfrak{S}_n$, where $C_\rho$ is the class of $\mathfrak{S}_n$, of order $g_\rho$, which contains $r$ and $h_\rho = |C_\rho \cap H_i|$.

**2. The main theorem.** Let $\mu$ be a permutation representation of $\mathfrak{S}_n$ and let $\chi$ be the character of $\mathfrak{S}_n$ in $\mu$. If $T$ is a transitive constituent of $\mu(\mathfrak{S}_n)$, let $\bar{\mu}_{(T)}(r)$ denote the *restriction of* $\mu(r)$ *to a permutation on the elements of* $T$, e.g., if $\mu(\mathfrak{S}_n)$ permutes the elements: 1, 2, 3, 4, 5, 6, 7; $T = \{4, 5, 6, 7\}$ and $\mu(r) = (12)(3)(456)(7)$, then $\bar{\mu}_{(T)}(r) = (456)(7)$. Therefore, by definition, $\bar{\mu}_{(T)}$ is a transitive permutation representation of $\mathfrak{S}_n$.

LEMMA 1. (See **3**, p. 57.) $\bar{\mu}_{(T)} \sim \mu_a$, $1 \leqq a \leqq \omega$, *where* $\mu(H_a)$ *is the stabilizer of some element of* $T$.

*Remark* 1. $\mu_a$ is called *a transitive constituent of* $\mu$. Suppose $\mu(\mathfrak{S}_n)$ has transitive constituents $T_1, T_2, \ldots, T_\theta$ and $\bar{\mu}_{(T_i)} \sim \mu_{\beta_i}$, $\beta_i \in \mathsf{N}$, $1 \leqq \beta_i \leqq \omega$, $1 \leqq i \leqq \theta$. $\{\mu_{\beta_i}: i = 1, 2, \ldots, \theta\}$ is called *the decomposition of* $\mu$ *into its transitive constituents* and we write $\mu = \sum_{i=1}^{\theta} \mu_{\beta_i}$. The decomposition of $\mu$ is unique up to equivalence.

The following lemma is well known and follows immediately from the definitions.

LEMMA 2.

$$\chi(r) = \sum_{i=1}^{\theta} \chi_{\beta_i}(r), \qquad r \in \mathfrak{S}_n. \tag{1}$$

*Remark* 2. (1) is usually written as $\chi = \sum_{i=1}^{\theta} \chi_{\beta_i}$. $\chi_{\beta_i}$ is called *a transitive constituent of* $\chi$ *and* $\{\chi_{\beta_i}: i = 1, 2, \ldots, \theta\}$ is called the *decomposition of* $\chi$ *into its transitive constituents*. Equivalent representations of $\mathfrak{S}_n$ have the same character, therefore, the decomposition of $\chi$ is unique. The following lemma is well known.

LEMMA 3. (See **3**, p. 280.) $(\chi, \mathbf{1}) = \theta$, *where* $\theta$ *is the number of transitive constituents of* $\mu$ (*and* $\chi$).

Let $N[n, p]$ denote the number of topologically distinct graphs with $n$ vertices and $p$ edges and $N[H: n, p]$ the number of such graphs with automorphism group $H$. Suppose $H_L$, $1 \leqq L \leqq \omega$ is the subgroup of $\mathfrak{S}_n$ permutationally isomorphic to the direct product $\mathfrak{S}_{n-2}\mathfrak{S}_2$ of $\mathfrak{S}_{n-2}$ and $\mathfrak{S}_2$, then we have the following lemma.

LEMMA 4. $N[n, p] = (\chi_L^p, \mathbf{1})$.

*Proof.* $H_L$ is permutationally isomorphic to the automorphism group of a graph with $n$ vertices consisting of one edge and $n - 2$ isolated vertices. Now, applying Theorem 3 of (**5**), the lemma follows immediately.

THEOREM 1. $\chi_L^p = \sum_{i=1}^{\omega} a_i \chi_i$ *if and only if* $a_i = N[H_i\colon n, p]$.

*Proof.* We recall that $\sigma_L^p\colon \mathfrak{S}_n \to \{K(\mathfrak{S}_n^{H_L})\}^p$. Suppose $\chi_L^p = \sum_{i=1}^{\omega} a_i\chi_i$. Let $T_{11}, T_{12}, \ldots, T_{1a_1}, T_{21}, T_{22}, \ldots, T_{2a_2}, \ldots, T_{\omega 1}, T_{\omega 2}, \ldots, T_{\omega a_\omega}$ be the transitive constituents of $\sigma_L^p(\mathfrak{S}_n)$ and suppose, from Lemma 1, that $\bar{\sigma}^p_{L(T_{ij})} \sim \mu_i$, $j = 1, 2, \ldots, a_i$, $1 \leqq i \leqq \omega$, when $\bar{\sigma}_L^p(H_i)$ is the stabilizer of some element of $T_{ij}$. Let this element be

$$U \equiv [\![H_L^{p_1}x_{Lc_1}, H_L^{p_2}x_{Lc_2}, \ldots, H_L^{p_t}x_{Lc_t}]\!], \qquad H_Lx_{Lc_k} \in S_L,$$

$p_i \in \mathsf{N}$, $p_1 + p_2 + \ldots + p_t = p$. Thus, for every $h \in H_i$, $U\sigma_L^p(h) = U$, i.e.,

$$(2) \qquad [\![H_L^{p_1}x_{Lc_1}h, H_L^{p_2}x_{Lc_2}h, \ldots, H_L^{p_t}x_{Lc_t}h]\!] = U.$$

Let $N$ be the set $\{u_1, u_2, \ldots, u_n\}$, and suppose $N \,\&\, N$ denotes the set consisting of $[\![u_i, u_j]\!]$, $i, j = 1, 2, \ldots, n$. Suppose $\mathfrak{S}_n$ permutes the elements of $N$ and, in particular, suppose $\mathfrak{S}_2$ permutes the elements $u_1, u_2$ and $\mathfrak{S}_{n-2}$ the elements $u_3, u_4, \ldots, u_n$. A mapping $\psi\colon S_L \to N \,\&\, N$ from $S_L$ into $N \,\&\, N$ is defined as follows:

$$H_Lx_{Lb}\psi = [\![u_i, u_j]\!], \qquad 1 \leqq b \leqq \alpha_L,$$

if $[\![u_1hx_{Lb}, u_2hx_{Lb}]\!] = [\![u_i, u_j]\!]$, $h \in H_L$ $(= \mathfrak{S}_{n-2}\mathfrak{S}_2)$. This mapping is well-defined, one-to-one and onto, and induces, in a natural way, a one-to-one mapping $\psi^p$ (say) of the set of elements permuted by $\sigma_L^p(\mathfrak{S}_n)$ onto the set of sets each consisting of $p$ elements of $N \,\&\, N$. If $G$ is the graph with vertex set $N(G) \equiv N$ and edge set $E(G) \equiv \psi^p(u)$, we write $G = G(u)$. Thus $G(u)$ is a graph with $n$ vertices and $p$ edges. If $u$ and $u'$ are elements permuted by $\sigma_L^p(\mathfrak{S}_n)$, then, by definition, $G(u)$ and $G(u')$ are topologically similar if and only if $u$ and $u'$ belong to the same transitive constituent. Furthermore, from equation (2), $H_i$ is the automorphism group of $G(u)$. Therefore, there exists at least $a_i$ topologically distinct graphs with $n$ vertices, $p$ edges and automorphism group equal to $H_i$. Since, from Lemmas 3 and 4, $N[n, p] = \sum_{i=1}^{\omega} a_i$, it follows that $N[H_i\colon n, p] = a_i$, $1 \leqq i \leqq \omega$. Since $\psi^p$ is one-to-one and onto, the converse of the theorem is also true. This completes the proof of the theorem.

## 3. The marks of a subgroup in $\sigma_L^p$.

In Theorem 1 it was proved that if $\chi_L^p = \sum_{i=1}^{\omega} a_i\chi_i$, then $a_i = N[H_i\colon n, p]$. To obtain the value of $a_i$, $1 \leqq i \leqq \omega$, a knowledge of $\chi_L^p$ is insufficient. However, we note the following theorem.

THEOREM 2. (See **1**, p. 238.) $\chi_L^p = \sum_{i=1}^{\omega} a_i\chi_i$ *if and only if*

$$m(\sigma_L^p) = \sum_{i=1}^{\omega} a_i\, m(\mu_i)$$

(*i.e., if and only if* $m(H_j; \sigma_L^p) = \sum_{i=1}^{\omega} a_i m(H_j; \mu_i)$, $1 \leqq j \leqq \omega$).

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*Remark* 3. In (**1**, p. 238) and (**2**) it was shown that if the sets of marks $m(\sigma_L^p)$, $m(\mu_i)$, $i = 1, 2, \ldots, \omega$, are known, then they determine the $a_i$'s uniquely. This is essentially due to the "triangular nature" of *the table of marks* $m(H_i; \mu_j)$, $i, j = 1, 2, \ldots, \omega$, illustrated in the example below. We will assume the marks $m(H_i; \mu_j)$, $i, j = 1, 2, \ldots, \omega$, are known. We now show how to calculate the marks $m(H_i; \sigma_L^p)$, $i = 1, 2, \ldots, \omega$. Once these are known, it is a simple matter (see the example below) to determine the $a_i$'s and hence, by Theorem 1, to determine $N[H_i; n, p]$, $i = 1, 2, \ldots, \omega$. Some further definitions and notation are now required.

Suppose $P_1, P_2, \ldots, P_q$ are the transitive constituents of $\mu_L(H_a)$, $1 \leqq a \leqq \omega$, where $P_i = \{d_{i1}, d_{i2}, \ldots, d_{i\gamma_i}\}$, $d_{ij} \in S_L$, $j = 1, 2, \ldots, \gamma_i$, $\sum_{i=1}^q \gamma_i = \alpha_L$. Let $\xi(a)$ be the permutation defined by

$$\xi(a) \equiv (d_{11}, d_{12}, \ldots, d_{1\gamma_1})(d_{21}, d_{22}, \ldots, d_{2\gamma_2}) \ldots (d_{q1}, d_{q2}, \ldots, d_{q\gamma_q}).$$

Thus $\xi(a) \in K(\mathfrak{S}_n^{H_L})$ but is not necessarily in $\mathfrak{S}_n^{H_L}$. $\xi(a)$ will be called *the $H_a$-induced permutation of* $K(\mathfrak{S}_n^{H_L})$. Note that if $\xi(a) \in \mathfrak{S}_n^{H_L}$, then there exists $r \in \mathfrak{S}_n$ such that $\mu_L(r) = \xi(a)$ and in this case $m(\xi(a)) = \chi_L(r)$. Finally, suppose *the cycle-index* $Z[\mathfrak{S}_p]$ *of* $\mathfrak{S}_p$ is defined by:

$$Z[\mathfrak{S}_p] = \frac{1}{p!} \sum_{[j]} A_{j_1 j_2 \ldots j_p} f_1^{j_1} f_2^{j_2} \ldots f_p^{j_p}, \tag{3}$$

where the summation is over all partitions $[j]$ of $p$ and

$$A_{j_1 j_2 \ldots j_p} = \frac{p!}{1^{j_1} 2^{j_2} \ldots p^{j_p} j_1! j_2! \ldots j_p!},$$

then $Z[\mathfrak{S}_p; m(\xi(a))]$ denotes the natural number obtained by writing $f^t = m(\xi^t(a))$, $t = 1, 2, \ldots, p$, in (3).

LEMMA 5. (See **4**, p. 90.)

$$\bar{\chi}_L^p(\xi(a)) = Z[\mathfrak{S}_p; m(\xi(a))], \qquad 1 \leqq a \leqq \omega.$$

THEOREM 3.

$$m(H_a; \sigma_L^p) = Z[\mathfrak{S}_p; m(\xi(a))], \qquad 1 \leqq a \leqq \omega.$$

*Proof.* Let $u$ be an element permuted by $(\mathfrak{S}_n^{H_L})^p$. For example, suppose

$$u \equiv [\![ d_{11}^{p_{11}}, d_{12}^{p_{12}}, \ldots, d_{1\gamma_1}^{p_{1\gamma_1}}, d_{21}^{p_{21}}, d_{22}^{p_{22}}, \ldots, d_{2\gamma_2}^{p_{2\gamma_2}}, \ldots, d_{q1}^{p_{q1}}, d_{q2}^{p_{q2}}, \ldots, d_{q\gamma_q}^{p_{q\gamma_q}} ]\!],$$

$$p_{ij} \in \mathbf{N}, \quad \sum_{i=1}^{q} \sum_{j=1}^{\gamma_i} p_{ij} = p.$$

Assume, furthermore, that

$$u\sigma_L^p(h) = [\![ d_{11}^{p_{11}}h, d_{12}^{p_{12}}h, \ldots, d_{1\gamma_1}^{p_{1\gamma_1}}h, d_{21}^{p_{21}}h, d_{22}^{p_{22}}h, \ldots, d_{2\gamma_2}^{p_{2\gamma_2}}h, \ldots, d_{q1}^{p_{q1}}h, d_{q2}^{p_{q2}}h, \ldots, d_{q\gamma_q}^{p_{q\gamma_q}}h ]\!] = u \quad \text{for every } h \in H_a. \tag{4}$$

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Then, by definition, $m(H_a; \sigma_L^p)$ is the number of elements $u$ which satisfy (4). Since $P_i$ is a transitive constituent of $\mu_L(H_a)$, for any two elements $d_{i\alpha}$ and $d_{i\beta}$ of $P_i$ there exists $h' \in H_a$ such that

$$(5) \qquad d_{i\alpha}\mu_L(h') = d_{i\alpha}h' = d_{i\beta}.$$

Therefore, from (4) and (5),

$$(6) \qquad p_{ij} = p_{ik}, \qquad 1 \leqq j, k \leqq \gamma_i,\ 1 \leqq i \leqq q.$$

On the other hand, if $u$ is an element for which (6) is satisfied, then $u\sigma_L^p(h) = u$ for every $h \in H_a$. Let $p_{ij} = p_{ik} = p_i$, $1 \leqq i \leqq q$, then $u$ may be denoted by

$$u \equiv [\![d_{11}^{p_1}, d_{12}^{p_1}, \ldots, d_{1\gamma_1}^{p_1}, d_{21}^{p_2}, d_{22}^{p_2}, \ldots, d_{2\gamma_2}^{p_2}, \ldots, d_{q1}^{p_q}, d_{q2}^{p_q}, \ldots, d_{q\gamma_q}^{p_q}]\!],$$

$$(7) \qquad \gamma_1 p_1 + \gamma_2 p_2 + \ldots + \gamma_q p_q = p.$$

However, if $u'$ is an element permuted by $(\mathfrak{S}_n^{H_L})^p$ then, from the definition of $\xi(a)$,

$$(8) \qquad u'\phi_L^p(\xi(a)) = u'$$

if and only if $u'$ is of the same form as $u$ (as denoted by equation (7)), i.e., $u'\phi_L^p(\xi(a)) = u'$ if and only if $u'\sigma_L^p(h) = u'$ for every $h \in H_a$. Therefore

$$m(H_a; \sigma_L^p) = \bar{\chi}_L^p(\xi(a)) = Z[\mathfrak{S}_p; m(\xi(a))]$$

(by Lemma 5). This completes the proof of the theorem.

**4. Example** ($n = 5$). Let $\mathfrak{S}_5$ permute the symbols $a, b, c, d,$ and $e$, and $\boldsymbol{1}$ denote the identity permutation. We denote: (i) the cyclic subgroup of $\mathfrak{S}_k$ generated by an element of cyclic decomposition $(j_1, j_2, \ldots, j_k)$ by $C(j_1, j_2, \ldots, j_k)$, e.g., $C(2^2)$ denotes the subgroup of $\mathfrak{S}_4$ generated by $(ab)(cd)$; (ii) the dihedral subgroups of orders 8 and 10 of $\mathfrak{S}_4$ and $\mathfrak{S}_5$ by $D_8$ and $D_{10}$, respectively; (iii) the alternating subgroups of $\mathfrak{S}_4$ and $\mathfrak{S}_5$ by $A_4$ and $A_5$, respectively; (iv) the direct product of groups $P_1$ and $P_2$ by $P_1P_2$. Then the distinct, up to conjugacy, subgroups of $\mathfrak{S}_5$ are: $H_1 = \{\boldsymbol{1}\}$; $H_2 = \mathfrak{S}_1^3\mathfrak{S}_2$; $H_3 = C(12^2)$; $H_4 = C(1^23)$; $H_5 = C(14)$; $H_6 = \{\boldsymbol{1}, (ab)(cd)(e), (ac)(bd)(e), (ad)(bc)(e)\}$; $H_7 = \mathfrak{S}_1\mathfrak{S}_2^2$; $H_8 = C(5)$; $H_9 = \mathfrak{S}_1^2\mathfrak{S}_3$; $H_{10} = C(3)\mathfrak{S}_2$; $H_{11} = \{\boldsymbol{1}, (abc)(d)(e), (acb)(d)(e), (ab)(c)(de), (ac)(b)(de), (a)(bc)(de)\}$; $H_{12} = \mathfrak{S}_1D_8$; $H_{13} = D_{10}$; $H_{14} = \mathfrak{S}_1A_4$; $H_{15} = \mathfrak{S}_2\mathfrak{S}_3$; $H_{16}$ is a metacyclic group of order 20 generated by $(abcd)(e)$ and $(aedcb)$; $H_{17} = \mathfrak{S}_1\mathfrak{S}_4$; $H_{18} = A_5$; $H_{19} = \mathfrak{S}_5$.

Thus $H_L = \mathfrak{S}_{n-2}\mathfrak{S}_2$ is, when $n = 5$, the subgroup $H_{15}$. By inspection, the table of marks (see **1**, p. 241) for $\mathfrak{S}_5$ is as follows.

| | $H_1$ | $H_2$ | $H_3$ | $H_4$ | $H_5$ | $H_6$ | $H_7$ | $H_8$ | $H_9$ | $H_{10}$ | $H_{11}$ | $H_{12}$ | $H_{13}$ | $H_{14}$ | $H_{15}$ | $H_{16}$ | $H_{17}$ | $H_{18}$ | $H_{19}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mu_1$ | 120 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mu_2$ | 60 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mu_3$ | 60 | 0 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mu_4$ | 40 | 0 | 0 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mu_5$ | 30 | 0 | 2 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mu_6$ | 30 | 0 | 6 | 0 | 0 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mu_7$ | 30 | 6 | 2 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mu_8$ | 24 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mu_9$ | 20 | 6 | 0 | 2 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mu_{10}$ | 20 | 2 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mu_{11}$ | 20 | 0 | 4 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mu_{12}$ | 15 | 3 | 3 | 0 | 1 | 3 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mu_{13}$ | 12 | 0 | 4 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mu_{14}$ | 10 | 0 | 2 | 4 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 |
| $\mu_{15}$ | 10 | 4 | 2 | 1 | 0 | 0 | 2 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| $\mu_{16}$ | 6 | 0 | 2 | 0 | 2 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| $\mu_{17}$ | 5 | 3 | 1 | 2 | 1 | 1 | 1 | 0 | 2 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 |
| $\mu_{18}$ | 2 | 0 | 2 | 2 | 0 | 2 | 0 | 2 | 0 | 0 | 2 | 0 | 2 | 2 | 0 | 0 | 0 | 2 | 0 |
| $\mu_{19}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

(the entry in the $i$th row and $j$th column is $m(H_j; \mu_i)$).

Since $m(H_j; \mu_i) = 0, j > i$, the rows of the table are independent. The cyclic decomposition of the $H_a$-induced permutation $\xi(a)$ of $K(\mathfrak{S}_5^{H_{15}})$ will be denoted by $p[\xi(a)]$, $1 \leqq a \leqq 19$. The following results have been obtained:

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $1^{10}$ | $1^4 2^3$ | $1^2 2^4$ | $13^3$ | $24^2$ | $2^3 4$ | $1^2 2^2 4$ | $5^2$ | $13^3$ | 136 | 136 | $24^2$ | $5^2$ | 46 | 136 | 10 | 46 | 10 | 10 |
| 715 | 71 | 27 | 4 | 3 | 7 | 15 | 0 | 4 | 2 | 2 | 3 | 0 | 1 | 2 | 0 | 1 | 0 | 0 |
| 2002 | 140 | 42 | 4 | 0 | 0 | 22 | 2 | 4 | 2 | 2 | 0 | 2 | 0 | 2 | 0 | 0 | 0 | 0 |
| 5005 | 259 | 77 | 10 | 3 | 13 | 35 | 0 | 10 | 4 | 4 | 3 | 0 | 1 | 4 | 0 | 1 | 0 | 0 |
| 11440 | 448 | 112 | 10 | 0 | 0 | 48 | 0 | 10 | 4 | 4 | 0 | 0 | 0 | 4 | 0 | 0 | 0 | 0 |

(where the entry in the $i$th row and $j$th column is $m(H_j; \sigma_{15}^i)$, $i \neq 1$, and $p[\xi(j)]$ when $i = 1$).

As an example, in order to calculate $m(H_9; \sigma_{15}^5)$ we note, from Theorem 3, that

$$m(H_9; \sigma_{15}^5) = Z[\mathfrak{S}_5; m(\xi(9))].$$

Now

$$Z[\mathfrak{S}_5] = (5!)^{-1}\{f_1^5 + 10f_1^3 f_2 + 15 f_1 f_2^2 + 20 f_1^2 f_3 + 30 f_1 f_4 + 24 f_5 + 20 f_2 f_3\}$$

and, by inspection, $p[\xi(9)] = (13^3)$. Therefore, writing $f_1 = m(\xi(9)) = 1$; $f_2 = m(\xi^2(9)) = 1$; $f_3 = m(\xi^3(9)) = 10$; $f_4 = m(\xi^4(9)) = 1$; $f_5 = m(\xi^5(9)) = 1$, we obtain

$$m(H_9; \sigma_{15}^5) = Z[\mathfrak{S}_5; m(\xi(9))] = (5!)^{-1}\{1^5 + 10.1^3.1 + 15.1.1^2 + 20.1^2.10 + 30.1.1 + 24.1 + 20.1.10\} = 480/5! = 4.$$

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Furthermore, it is very easily verified from the table of marks that

$$m(\sigma_{15}^{5}) = 6m(\mu_1) + 12m(\mu_2) + 4m(\mu_3) + 9m(\mu_7) + m(\mu_9) + m(\mu_{13}) + 2m(\mu_{15}).$$

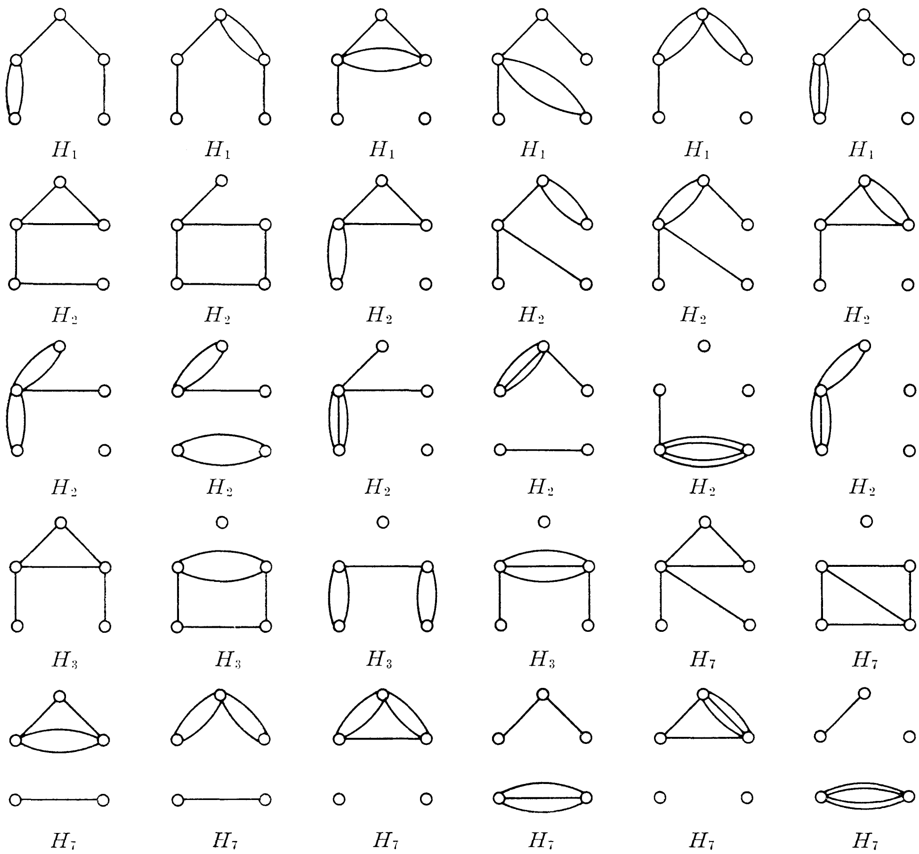
Then from Theorems 1 and 2, $N[H_1\colon 5, 5] = 6$, $N[H_2\colon 5, 5] = 12$, $N[H_3\colon 5, 5] = 4$, $N[H_7\colon 5, 5] = 9$, $N[H_9\colon 5, 5] = 1$, $N[H_{13}\colon 5, 5] = 1$, $N[H_{15}\colon 5, 5] = 2$, and, finally, $N[H_i\colon 5, 5] = 0$, $i = 4, 5, 6, 8, 10, 11, 12, 14, 16, 17, 18, 19$. These graphs are sketched below.

The following results have been obtained:

| $a$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $N[H_a\colon 5, 4]$ | 1 | 5 | 2 | 0 | 0 | 0 | 4 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 2 | 0 | 1 | 0 | 0 |
| $N[H_a\colon 5, 5]$ | 6 | 12 | 4 | 0 | 0 | 0 | 9 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 2 | 0 | 0 | 0 | 0 |
| $N[H_a\colon 5, 6]$ | 21 | 25 | 8 | 0 | 0 | 1 | 12 | 0 | 2 | 0 | 0 | 2 | 0 | 0 | 4 | 0 | 1 | 0 | 0 |
| $N[H_a\colon 5, 7]$ | 57 | 49 | 16 | 0 | 0 | 0 | 20 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 4 | 0 | 0 | 0 | 0 |

*Graphs with 5 vertices and 5 edges*

The automorphism group of the graph is written below each graph.

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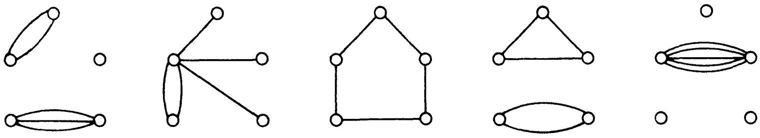

$H_7$ $H_9$ $H_{13}$ $H_{15}$ $H_{15}$

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