

Bessel-integral functions

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§1. *Summary.* In a very remarkable work on the operational Calculus, Dr Balth. van der Pol¹ has introduced a new function, playing with respect to Bessel function of order zero the same part as the cosine- or sine-integral with respect to the ordinary cosine or sine. He showed that this function—which he called *Bessel-integral function*—can be used to express the differential coefficient of any Bessel function with respect to its index. But he did not investigate the further properties of his new function. I propose to give here some of them, which appear to be interesting, and to introduce and study the functions connected, in the same way, with Bessel functions of any order.

§2. *Definition of the function of order zero.*

It is well known that the *cosine-integral*² is defined by the formula

$$\text{ci}(x) = - \int_x^\infty \frac{\cos u}{u} du,$$

and the *sine-integral* by

$$\text{si}(x) = - \int_x^\infty \frac{\sin u}{u} du = \int_0^x \frac{\sin u}{u} du - \frac{\pi}{2}.$$

Following Dr Van der Pol,³ we shall define the *Bessel-integral function of order zero* by

$$Ji_0(x) = - \int_x^\infty \frac{J_0(u)}{u} du,$$

¹ *Philosophical Magazine*, 8 (1929), 861-898, (887).

² For properties of cosine- and sine-integrals, the reader is referred to Niels Nielsen, *Theorie des Integrallogarithmus* (Leipzig, 1906). Nielsen's notation will be used through this paper. As regards Bessel functions, we shall follow Watson's notation.

³ Dr van der Pol uses the simple notation $Ji(x)$. As we shall deal with functions connected with the Bessel function of order n , we find it convenient to denote van der Pol's function by $Ji_0(x)$, thus introducing the *order* of Bessel-integral functions.

where J_0 is Bessel function of order zero. It is readily seen that this integral is convergent. We shall now investigate the properties of this function.

§ 3. *Integral representation and various expressions.*

Let us start from Parseval's integral for the Bessel function of order zero, namely

$$J_0(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin \theta) d\theta.$$

Writing u in place of x , dividing both members by u , and integrating with respect to u from x to ∞ , we obtain

$$\int_x^\infty \frac{J_0(u)}{u} du = \frac{1}{\pi} \int_x^\infty \frac{du}{u} \int_0^\pi \cos(u \sin \theta) d\theta$$

or

$$Ji_0(x) = -\frac{1}{\pi} \int_0^\pi d\theta \int_x^\infty \frac{\cos(u \sin \theta)}{u} du.$$

But we have

$$\begin{aligned} -\int_x^\infty \frac{\cos(u \sin \theta)}{u} du &= -\int_x^\infty \frac{\cos(u \sin \theta)}{u \sin \theta} d(u \sin \theta) \\ &= ci(x \sin \theta). \end{aligned}$$

Hence the interesting formula, a generalization of Parseval's integral,

$$Ji_0(x) = \frac{1}{\pi} \int_0^\pi ci(x \sin \theta) d\theta. \tag{1}$$

Various results can be deduced from it. It is known that

$$ci(x) = \gamma + \log x + \int_0^x \frac{\cos t - 1}{t} dt, \tag{2}$$

γ being Euler's constant, so that

$$\begin{aligned} ci(x \sin \theta) &= \gamma + \log(x \sin \theta) + \int_0^{x \sin \theta} \frac{\cos t - 1}{t} dt, \\ &= \gamma + \log(x \sin \theta) + \int_0^x \frac{\cos(u \sin \theta) - 1}{u} du. \end{aligned}$$

If we integrate from 0 to π with respect to θ , we obtain, after dividing by π ,

$$Ji_0(x) = \gamma + \frac{1}{\pi} \int_0^\pi \log(x \sin \theta) d\theta + \int_0^x \frac{J_0(u) - 1}{u} du.$$

Now

$$\frac{1}{\pi} \int_0^\pi \log(x \sin \theta) d\theta = \log x + \frac{1}{\pi} \int_0^\pi \log \sin \theta d\theta$$

and a classical result is

$$\frac{1}{\pi} \int_0^\pi \log \sin \theta \, d\theta = -\log 2.$$

So we have the formula

$$Ji_0(x) = \gamma + \log \frac{x}{2} + \int_0^x \frac{J_0(u) - 1}{u} \, du. \tag{3}$$

From (2) and (3) we have

$$Ji_0(x) - ci(x) = \int_0^x \frac{J_0(u) - \cos u}{u} \, du - \log 2, \tag{4}$$

so that

$$Ji_0(0) - ci(0) = -\log 2,$$

a result found by Dr van der Pol, and written by him in the following form

$$\int_0^\infty \frac{J_0(x) - \cos x}{x} \, dx = \log 2.$$

We can easily generalize it by writing

$$\int_0^\infty \frac{J_0(ax) - \cos bx}{x} \, dx = \log \frac{2b}{a}.$$

From formula (3), we can obtain an expansion for Ji_0 , by expanding the Bessel function J_0 in ascending powers of u , and integrating: we find

$$Ji_0(x) = \gamma + \log \frac{x}{2} + \sum_{s=1}^\infty \frac{(-1)^s (\frac{1}{2}x)^{2s}}{s! s! 2s}$$

or, introducing a hypergeometric function,

$$Ji_0(x) = \gamma + \log \frac{1}{2}x - \frac{1}{8}x^2 {}_2F_3(1, 1; 2, 2, 2; -x^2/4).$$

Using some results stated by Nielsen on the cosine-integral, we can also obtain, through an integrating process the following results

$$Ji_0(x) - \log x = \frac{2x \sin \pi x}{\pi} (\gamma - \log 2) + \frac{2x \sin \pi x}{\pi} \sum_{s=1}^\infty \frac{(-1)^{s-1}}{s^2 - x^2} [Ji_0(s) - \log s],$$

$$\lim_{n \rightarrow \infty} \sum_{s=n+1}^\infty \frac{J_0(sx/n)}{s} = -Ji_0(x).$$

§ 4. *Function of second kind.* If we apply the same integrating process to the formula

$$J_0(x) = \frac{2}{\pi} \int_0^\infty \sin(x \cosh t) dt,$$

we find readily

$$Ji_0(x) = \frac{2}{\pi} \int_0^\infty \text{si}(x \cosh t) dt.$$

Now, Y_0 being Bessel function of second kind, we have

$$Y_0(x) = -\frac{2}{\pi} \int_0^\infty \cos(x \cosh t) dt$$

so that if we introduce the new function

$$Yi_0(x) = -\int_x^\infty \frac{Y_0(u)}{u} du,$$

we shall be able to write

$$Yi_0(x) = -\frac{2}{\pi} \int_0^\infty \text{ci}(x \cosh t) dt.$$

This function $Yi_0(x)$ can be called *Bessel-integral function of second kind* (and order zero).

We have the two important formulae, the first of which can be immediately deduced from (3),

$$\frac{d Ji_0(x)}{dx} = \frac{J_0(x)}{x},$$

$$\frac{d Yi_0(x)}{dx} = \frac{Y_0(x)}{x}.$$

Using them, it is easy to write the differential equation of the third order satisfied by Ji_0 : we find

$$x^2 y''' + 3xy'' + (x^2 + 1)y' = 0,$$

and this equation has the three independent solutions $Ji_0(x)$, $Yi_0(x)$ and constant, just as the equation

$$xy''' + 2y'' + xy' = 0$$

is satisfied by $\text{ci}(x)$, $\text{si}(x)$ and constant.

§ 5. *Properties of definite integrals.* We can use the Ji_0 function to express some integrals connected with Bessel functions. For instance, let us consider the integral

$$\int_0^x J_1(t) \log t dt,$$

where J_1 is Bessel function of order unity. As we have

$$J_1(t) = -J'_0(t),$$

we can integrate by parts, and use formula (3) to obtain

$$\int_0^x J_1(t) \log t \, dt = Ji_0(x) - J_0(x) \log x - \gamma + \log 2.$$

The same process of integrating by parts enables us to find the value of the integral

$$\int_0^x Ji_0(t) \, dt$$

which can be written

$$x Ji_0(x) - \int_0^x J_0(t) \, dt$$

so that we have

$$\int_0^x Ji_0(t) \, dt = x Ji_0(x) - 2[J_1(x) + J_3(x) + J_5(x) + \dots].$$

§ 6. *Functions of order n.* We shall now introduce *Bessel-integral functions of order n*, by the formula

$$Ji_n(x) = - \int_x^\infty \frac{J_n(u)}{u} \, du. \tag{5}$$

As Weber showed, we have

$$\int_0^\infty \frac{J_n(u)}{u} \, du = \frac{1}{n},$$

so that we can write

$$Ji_n(x) = \int_0^x \frac{J_n(u)}{u} \, du - \frac{1}{n}. \tag{6}$$

This integral is finite, $J_n(u)/u$ containing only integer powers of u : and we can infer from it that the integral in (5) is convergent.

These two definitions lead us to a number of formulæ introducing the Ji functions: we shall write some of them without giving the demonstrations, which are easy.

For instance, using one of the recurrence-formulæ of Bessel functions, we obtain

$$(n - 1) Ji_{n-1}(x) - (n + 1) Ji_{n+1}(x) = \frac{2n}{x} J_n(x);$$

and, as

$$\frac{d Ji_n(x)}{dx} = \frac{J_n(x)}{x},$$

we can write

$$(n - 1) Ji_{n-1}(x) - (n + 1) Ji_{n+1}(x) = 2n Ji'_n(x),$$

which is a recurrence-formula between Bessel-integral functions and their derivates.

In the particular case $n=1$, it gives

$$Ji_2(x) = -\frac{J_1(x)}{x}. \tag{7}$$

From the recurrence-formula and the definition, we can obtain the following expansions

$$\begin{aligned} nJi_n(x) &= -1 + J_n(x) + 2[J_{n+2}(x) + J_{n+4}(x) + \dots], \\ mJi_{2m} &= -\frac{1}{x} [J_1 + 3J_3 + \dots + (2m - 1) J_{2m-1}], \\ mJi_{2m}^i &= -\frac{1}{2}J_0 - J_2 - J_4 - \dots - J_{2m-2} - \frac{1}{2}J_{2m}, \\ (2m + 1) Ji_{2m+1} &= -Ji_1 + \frac{4}{x} (J_2 + 2J_4 + \dots + mJ_{2m}), \end{aligned}$$

and the two results

$$\begin{aligned} Ji_{2p}(x) &= Ji_{2p}(-x), \\ Ji_{2p+1}(x) + Ji_{2p+1}(-x) &= -\frac{2}{2p + 1}. \end{aligned}$$

Formula (5) can be extended to Bessel functions whose order is not an integer. Specially interesting is the case $n = \frac{1}{2}$. We have then

$$Ji_{\frac{1}{2}}(x) = -\int_x^\infty \frac{J_{\frac{1}{2}}(u)}{u} du = -\sqrt{\frac{2}{\pi}} \int_x^\infty \frac{\sin u}{u \sqrt{u}} du.$$

If we integrate by parts, we obtain

$$Ji_{\frac{1}{2}}(x) = -2\sqrt{\frac{2}{\pi}} \frac{\sin x}{\sqrt{x}} - 2\sqrt{\frac{2}{\pi}} \int_x^\infty \frac{\cos u}{\sqrt{u}} du.$$

In this last integral, let us make $u = t^2$: it becomes

$$2 \int_{\sqrt{x}}^\infty \cos t^2 dt.$$

But this is one of Fresnel integrals,

$$C(x) = \int_x^\infty \cos t^2 dt$$

so that we have

$$Ji_{\frac{1}{2}}(x) = -2\sqrt{\frac{2}{\pi}} \frac{\sin x}{\sqrt{x}} - 4\sqrt{\frac{2}{\pi}} C(\sqrt{x})$$

a formula which can also be written

$$Ji_{\frac{1}{2}}(x) = -2J_{\frac{1}{2}}(x) - 4\sqrt{\frac{2}{\pi}} C(\sqrt{x}).$$

§7. *Integral representations.* If we start from the well known formulae for Bessel functions

$$J_{2p}(x) = \frac{1}{\pi} \int_0^\pi \cos 2p\theta \cos(x \sin \theta) d\theta,$$

$$J_{2p+1}(x) = \frac{1}{\pi} \int_0^\pi \sin(2p+1)\theta \sin(x \sin \theta) d\theta,$$

we obtain, following the integrating process already used, the two expressions

$$Ji_{2p}(x) = \frac{1}{\pi} \int_0^\pi \cos 2p\theta \operatorname{ci}(x \sin \theta) d\theta,$$

$$Ji_{2p+1}(x) = \frac{1}{\pi} \int_0^\pi \sin(2p+1)\theta \operatorname{si}(x \sin \theta) d\theta.$$

If, in the first of these formulae, we take $p=1$, we obtain an expression for Ji_2 , i.e., through (7), for J_1 , namely

$$J_1(x) = -\frac{x}{\pi} \int_0^\pi \cos 2\theta \operatorname{ci}(x \sin \theta) d\theta.$$

This formula, expressing Bessel function of order unity by use of the cosine-integral, seems to be new.

§8. *Expansions in series of Bessel-integral functions.* A number of interesting expansions containing Bessel-integral functions can be found by transforming some known properties of Bessel functions.

Let us start from the Schlömilch formula

$$e^{tu(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(u).$$

Dividing by u and integrating with respect to u , we have

$$\begin{aligned} \int_x^\infty \frac{e^{tu(t-1/t)}}{u} du &= \sum t^n \int_x^\infty \frac{J_n(u)}{u} du \\ &= -\sum t^n Ji_n(x). \end{aligned}$$

But we know that the *logarithmic-integral function* is defined by

$$\text{li}(e^{ax}) = - \int_x^\infty \frac{e^{au}}{u} du,$$

and we can write the expansion

$$\text{li}\{e^{\frac{1}{2}x(t-1/\theta)}\} = \sum_{n=-\infty}^{+\infty} t^n J_i_n(x),$$

a curious generalization of the Schlömilch formula.

Considering now Neumann's expansions

$$\begin{aligned} \cos x &= J_0(x) - 2J_2(x) + 2J_4(x) - \dots, \\ \sin x &= 2J_1(x) - 2J_3(x) + 2J_5(x) - \dots, \end{aligned}$$

we obtain, using again the integrating process, the two following expansions for the cosine- and sine-integrals

$$\begin{aligned} \text{ci}(x) &= J_{i_0}(x) - 2J_{i_2}(x) + 2J_{i_4}(x) - \dots, \\ \text{si}(x) &= 2J_{i_1}(x) - 2J_{i_3}(x) + 2J_{i_5}(x) - \dots \end{aligned}$$

Similarly, using another Neumann's expansion for the function $\cos(x \sin \theta)$, we are able to write

$$\text{ci}(x \sin \theta) = J_{i_0}(x) + 2J_{i_2}(x) \cos 2\theta + 2J_{i_4}(x) \cos 4\theta + \dots,$$

or

$$\text{ci}(x \cos \theta) = J_{i_0}(x) - 2J_{i_2}(x) \cos 2\theta + 2J_{i_4}(x) \cos 4\theta - \dots,$$

and analogous expansions for the sine-integral.

If, in the last formula, we take $\theta = \frac{\pi}{3}$, we have

$$\text{ci} \frac{x}{2} = J_{i_0}(x) + J_{i_2}(x) - J_{i_4}(x) + J_{i_6}(x) - J_{i_8}(x) + \dots,$$

and if we take $\theta = \frac{\pi}{4}$,

$$\text{ci} \frac{x}{\sqrt{2}} = J_{i_0}(x) - 2J_{i_4}(x) + 2J_{i_8}(x) - 2J_{i_{12}}(x) + \dots$$

Now, considering the expansion for $\text{ci}(x \sin \theta)$, and differentiating both members with respect to θ , we obtain an expansion for

$$\cos(x \sin \theta) \cot \theta,$$

which, making $\theta = \frac{\pi}{4}$, gives

$$\cos \frac{x}{\sqrt{2}} = -4 [J_{i_2}(x) - 3J_{i_6}(x) + 5J_{i_{10}}(x) - \dots].$$

If in the expansion obtained above for $\text{ci}(x \cos \theta)$ we write $\cos \theta = a$, and if we introduce Gegenbauer's polynomial of order zero, by

$$\cos n\theta = nC_n^0(a),$$

we have

$$\text{ci}(ax) = Ji_0(x) - 4C_2^0(a) Ji_2(x) + 8C_4^0(a) Ji_4(x) - \dots$$

§ 9. *Integral properties.* Nielsen has established the following integral formulae¹

$$\int_0^\infty \text{si}(t) J_0(2\sqrt{tx}) dt = -\frac{\sin x}{x}$$

$$\int_0^\infty \text{ci}(t) J_0(2\sqrt{tx}) dt = \frac{\cos x - 1}{x}$$

$$\int_0^\infty \text{si}(t) J_1(2\sqrt{tx}) \frac{dt}{\sqrt{t}} = -\frac{\frac{\pi}{2} + \text{si}(x)}{\sqrt{x}}$$

$$\int_0^\infty \text{ci}(t) J_1(2\sqrt{tx}) \frac{dt}{\sqrt{t}} = \frac{\text{ci}(x) - \gamma - \log x}{\sqrt{x}}$$

Dividing by x and integrating, we obtain

$$\int_0^\infty \text{si}(t) Ji_0(2\sqrt{tx}) dt = \frac{\sin x}{2x} - \frac{\text{ci}(x)}{2}$$

$$\int_0^\infty \text{ci}(t) Ji_0(2\sqrt{tx}) dt = \frac{1 - \cos x}{2x} - \frac{\text{si}(x)}{2}$$

$$\int_0^\infty \text{si}(t) Ji_1(2\sqrt{tx}) \frac{dt}{\sqrt{t}} = \frac{\frac{\pi}{2} + \text{si}(x) + 2 \sin x}{\sqrt{x}} + 4C(\sqrt{x})$$

$$\int_0^\infty \text{ci}(t) Ji_1(2\sqrt{tx}) \frac{dt}{\sqrt{t}} = \frac{\gamma + 2 + \log x - \text{ci}(x) - 2 \cos x}{\sqrt{x}} + 4S(\sqrt{x})$$

C and S being Fresnel integrals,

$$C(x) = \int_x^\infty \cos t^2 dt, \quad S(x) = \int_x^\infty \sin t^2 dt.$$

¹ The third of these formulae is given in Nielsen's book with a slight mistake: he wrote $\frac{\pi}{2} - \text{si}(x)$ instead of $\frac{\pi}{2} + \text{si}(x)$, which is the real value.

Let us also write the following integral, which can be derived from a similar one containing J_0 ,

$$\int_0^\infty e^{-at} J_{i_0}(bt) dt = \frac{1}{2a} \log \frac{\sqrt{a^2 + b^2} - a}{\sqrt{a^2 + b^2} + a}.$$

And, as a conclusion, we add the result obtained by Dr van der Pol

$$\frac{\partial J_n(x)}{\partial n} = \int_0^x J_{i_0}(x-t) J'_n(t) dt$$

which holds for $n \geq 1$.

