On the natural completion
functor for Cauchy spaces

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The natural completion functor on the category of $c^\wedge$-embedded
Cauchy spaces induces a completion functor on a category of
sequential Cauchy spaces which preserves sequential regularity.

1. Introduction

This work is a continuation of the Cauchy space completion theory
developed in [3], and the reader is asked to refer to this source for
terminology and notation used without being defined in this paper. *The
assumption is made throughout this paper that all Cauchy spaces are Cauchy-
separated unless otherwise indicated.*

In Section 2, we show that the natural completion developed in [3]
determines a completion functor, denoted by N , on the category of CCH
of $c^\wedge$-embedded Cauchy spaces which preserves such properties as total
boundedness and uniformizability. In Section 3, it is shown that the
category SCH of sequential Cauchy spaces is a subcategory of CCH , and
that N is a completion functor when restricted to SCH . In Section 4,
the functor N is composed with two modification functors to produce a
completion functor N_S on the category SRCH of sequentially regular
Cauchy spaces.

2. The completion functor N

The term "convergence space" will be identified with "complete Cauchy
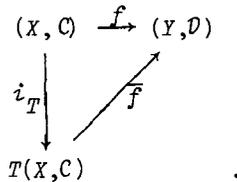space". Thus every convergence space (X, q) is ω-Hausdorff and

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satisfies Fischer's axiom

$$(F \to x \quad \text{and} \quad G \to x \quad \text{implies} \quad F \cap G \to x) \ .$$

Let **CHY** be the category with Cauchy spaces as objects and Cauchy-continuous maps as morphisms. Let **LCH** be any full subcategory of **CHY** , and let **LCH*** be the full subcategory of complete objects in **LCH** . The Extension of Identity Principle states that two continuous mappings of a convergence space into a Hausdorff convergence space which agree on a dense subset are identical; from this it follows that a morphism $f : (X, C) \to (Y, \mathcal{D})$ in **CHY** is an epimorphism if $f(X)$ is dense in $(Y, \mathcal{D})$.

DEFINITION 2.1. A *completion functor* T on **LCH** is a covariant functor $T : \text{LCH} \to \text{LCH*}$ with the following properties:

(1) for each $(X, C) \in \text{LCH}$, there is an embedding
$i_T : (X, C) \to T(X, C)$ such that $i_T(X)$ is dense in
$T(X, C)$;

(2) if f is a Cauchy-continuous mapping of (X, C) into
$(Y, \mathcal{D}) \to \text{LCH*}$, then there is a unique Cauchy-continuous
mapping $\overline{f}$ of $T(X, C)$ into $(Y, \mathcal{D})$ such that the
following diagram commutes;

$$\begin{array}{ccc} (X, C) & \xrightarrow{\ f\ } & (Y, \mathcal{D}) \\ {\scriptstyle i_T}\downarrow & \nearrow\!\!\overline{f} & \\ T(X, C) & & \end{array} \quad .$$

Then $\left(T(X, C), i_T\right)$ is called a T-completion of (X, C) .

In other words, i_T is an epi-reflection of **LCH** into **LCH*** . Note that $T(X, C)$ is (up to a commuting homeomorphism) uniquely determined. If $g : \left(X_1, C_1\right) \to \left(X_2, C_2\right)$ is a morphism in **LCH** , then $Tg = \overline{g}$, where $\overline{g} : T\left(X_1, C_1\right) \to T\left(X_2, C_2\right)$ is the unique Cauchy-continuous extension whose existence is guaranteed by (2). The same functor symbol is used to represent transformations of both objects and morphisms; we shall also use the same functor symbol when the domain of the functor is restricted to a

subcategory.

Let CCH be the full subcategory of CHY consisting of the $c^\wedge$-embedded Cauchy spaces. If (X, C) is an object in CCH , let $N(X, C) = \hom^\wedge C^\wedge(X, C)$ be the *natural completion* described in [3]. If $f : (X_1, C_1) \to (X_2, C_2)$ is a morphism in CCH , define $N(f) = \overline{f}$, where $\overline{f} : N(X_1, C_1) \to N(X_2, C_2)$ is the unique Cauchy-continuous extension of f whose existence is established by Theorem 2.4 of [3]. Using these and other results from [3], we immediately obtain the following result.

PROPOSITION 2.2. *N is a completion functor on* CCH .

The functor N will be called the *natural completion functor*. For any $(X, C) \in CCH$, $N(X, C)$ is known to be a c-embedded convergence space; in particular $N(X, C)$ is Hausdorff and ω-regular. The latter property is stronger than ordinary regularity for convergence spaces. In the remainder of this section, we examine some additional properties of the functor N .

A Cauchy space is said to be *totally bounded* if each ultra-filter is a Cauchy filter.

LEMMA 2.3. *A regular completion of a totally bounded Cauchy space is totally bounded.*

Proof. Let $\big((X^*, C^*), j\big)$ be a regular completion of (X, C) . If G is an ultrafilter on X^* , then $\operatorname{cl} j(X) = X^*$ implies that there is an ultrafilter F on X^* containing $j(X)$ such that $G \geq \operatorname{cl} F$. If (X, C) is totally bounded, then F' , the restriction of F to X , is in C , and so $j(F') = F \in C^*$. Since (X^*, C^*) is regular, $\operatorname{cl} F \in C^*$, and therefore $G \in C^*$.

A Cauchy space (X, C) is said to be *uniformizable* if C is the set of all Cauchy filters for some uniformity on X , and *completely regular* if (X, C) is a Cauchy subspace of a completely regular topological space (considered as a complete Cauchy space). If UCH and QCH are the full subcategories of CHY consisting of the uniformizable and completely regular Cauchy spaces, respectively, then it is clear that $UCH \subset QCH \subset CCH$. Since there are completely regular topological spaces which have no compatible complete uniformity, a completely regular Cauchy space need not be uniformizable.

An object in either **UCH** or **QCH** has a strict regular completion within its own category. If $(X, C) \in$ **QCH** then such a completion is obtained by taking the closure of (X, C) in the completely regular topological space in which (X, C) is, by definition, embedded. For $(X, C) \in$ **UCH**, a strict regular completion in **UCH** can be similarly obtained by first completing the associated uniformity.

THEOREM 2.4. *The completion functor N preserves total boundedness, complete regularity, and uniformizability.*

Proof. For total boundedness, the assertion follows from Lemma 2.3. For $(X, C) \in$ **QCH**, let (X^*, C^*) denote the strict completion of (X, C) described in the preceding paragraph. It follows from Proposition 1.5, [3], that $N(X, C)$ and $N(X^*, C^*)$ are homeomorphic spaces. Also, since (X^*, C^*) is complete and $c\hat{}$-embedded, $N(X^*, C^*) = (X^*, C^*)$. Therefore, $N(X, C)$ is completely regular. The same reasoning is applicable if $(X, C) \in$ **UCH**.

From Theorem 2.4, it follows that every Hausdorff compactification of a completely regular topological space can be obtained by choosing an appropriate Cauchy structure and applying the natural completion functor.

3.   Sequential Cauchy spaces

One branch of convergence space theory is concerned with sequences rather than filters. The Cauchy space axioms can be readily adapted to the sequential setting, and indeed this has been done in [1]. Since we are interested in using the natural completion functor N as a tool for studying completions of sequential Cauchy spaces, it is more convenient in this case to axiomatize sequential Cauchy spaces by means of filters generated by sequences. We shall refer to any filter generated by a sequence as a *sequential filter*.

DEFINITION 3.1. A Cauchy (X, C) is *sequential* if, for each $F \in C$, there is a sequential filter $G \in C$ such that $G \leq F$.

A complete sequential Cauchy space can be taken as the definition of a *sequential convergence space*. Let **SCH** denote the full subcategory of **CHY** consisting of sequential Cauchy spaces.

PROPOSITION 3.2. *Each sequential Cauchy space has a strict*

sequential completion.

Proof. Let X^* be the set of all Cauchy equivalence classes of members of C, and define $j : X \to X^*$ by: $j(x) = [\dot{x}]$, the equivalence class containing the fixed ultrafilter generated by $\{x\}$. Define a convergence structure q on X^* as follows:

  (1) if $[\dot{x}] \in j(X)$, then a filter H q-converges to $[\dot{x}]$ if and only if there is a sequential filter $F \to x$ in (X, C) such that $H \geq j(F)$;

  (2) if $\alpha \in X^* - j(X)$, then H q-converges to α if and only if there is a sequential filter $F \to \alpha$ such that $H \geq j(F) \cap \dot{\alpha}$.

One can easily verify that (X^*, q) is a sequential convergence space in which (X, C) is densely embedded by the mapping j. If C^* is the complete Cauchy structure on X^* consisting of the q-convergent filters, then $\big((X^*, C^*), j\big)$ is thus a sequential completion of (X, C). The completion is trivially strict, since the only members of C^* containing $X^* - j(X)$ are fixed ultrafilters.

COROLLARY 3.3. *The sequential Cauchy spaces are precisely the Cauchy subspaces of the sequential convergence spaces.*

THEOREM 3.4. SCH $\subset$ CCH .

Proof. If $(X, C) \in$ SCH , then it is shown in the preceding proof that (X, C) is a Cauchy subspace of a sequential convergence space. What we are calling a "sequential convergence space" is described in [2] as an "ω-Hausdorff S-space satisfying Fischer's axiom", and it is shown in [2] that such spaces are c-embedded convergence spaces. From [3], it follows that a c-embedded convergence space is a complete $c^\wedge$-embedded Cauchy space, and hence a member of CCH . Since the $c^\wedge$-embedded property is hereditary, $(X, C) \in$ CCH .

Next, we consider the behavior of the completion functor N on the category SCH . It is not known whether the completion $N(X, C)$ is strict for an arbitrary object $(X, C) \in$ CCH , but it turns out that this completion is strict if $(X, C) \in$ SCH . This fact is established with the help of the following lemma, which shows that even in the general case the functor N exhibits a weak form of "strictness".

LEMMA 3.5. *Let* $(X, C) \in$ CCH , *let* $N(X, C) = \left(X_N, C_N\right)$, *and let* $i_N : (X, C) \to \left(X_N, C_N\right)$ *be the natural embedding. Also, let* q *be the convergence structure on* X_N *determined by* C_N . *Then, if* $F \to y$ *in* $\left(X_N, C_N\right)$, *there is a filter* $G \to y$ *in* $\left(X_N, C_N\right)$ *such that* $i_N(X) \in G$, *and* $F \geq \mathrm{cl}_{\omega q} G$.

Proof. Define a convergence structure p on X_N as follows: $F \to y$ in $\left(X_N, p\right)$ if there is a filter $G \to y$ in $\left(X_N, C_N\right)$ such that $i_N(X) \in G$, and $F \geq \mathrm{cl}_{\omega q} G$. Clearly $q \leq p$, and it is easy to verify that $\omega p = \omega q$. Thus, if $\mathcal{D}$ is the complete Cauchy structure on X_N consisting of all p-convergent filters, it follows $(X, \mathcal{D})$ is a $c^{\wedge}$- embedded completion of (X, C) . Thus $i_N : (X, C) \to \left(X_N, \mathcal{D}\right)$ has a Cauchy-continuous extension $\overline{i}_N : \left(X_N, C_N\right) \to \left(X_N, \mathcal{D}\right)$; this implies $C_N \subset \mathcal{D}$, which is equivalent to $p \leq q$. Therefore $\mathcal{D} = C_N$, and it follows that $N(X, C)$ satisfies the assertion of the lemma.

THEOREM 3.6. *If* $(X, C) \in$ SCH , *then* $N(X, C) \in$ SCH . *Indeed,* $N(X, C)$ *is a strict completion which is homeomorphic to the completion* (X^*, C^*) *described in Proposition* 3.2.

Proof. The first assertion follows from the second. To prove the second, it is sufficient to show that the completion $N(X, C)$ is strict, since the completion of Proposition 3.2 is also strict and regular, and it follows from Corollary 1.6 of [4] that two strict regular completions of the same Cauchy space are homeomorphic.

To verify that $N(X, C)$ is strict, we refer to the notation of Lemma 3.5. Let $F \to y$ in $\left(X_N, C_N\right)$. If $(X, C) \in$ SCH , then we may assume that the filter G described in the lemma is sequential. Then it is easy to see that $\mathrm{cl}_{\omega q} G = \mathrm{cl}_q G = G \cap \dot{y}$. Since $G \to y$ in $\left(X_N, C_N\right)$ and $F \geq \mathrm{cl}_q G$, it follows that $N(X, C)$ is a strict completion of (X, C) .

As a supplement to Theorem 2.4, we note that the completion functor N also preserves the sequential property.

4. Sequentially regular Cauchy spaces

A sequential convergence space (X, q) is said to be *sequentially regular* if (X, q) has the same convergence relative to sequential filters as does $(X, \omega q)$, where ωq is the completely regular modification of q. Sequential regularity is important in the construction of Novák's *sequential envelope* (see [5] and [6]). We extend this concept to sequential Cauchy spaces, and use a "translation" of the functor N to obtain a completion functor N_S on the category of sequentially regular Cauchy spaces which, under certain conditions, yields Novák's sequential envelope as its completion (see Example 4.6).

In defining sequential regularity for Cauchy spaces, we make use of the two modification functors defined below.

DEFINITION 4.1. The sequential modification functor S is a covariant functor $S : \mathbf{CCH} \to \mathbf{SCH}$ defined as follows: for an object $(X, C) \in \mathbf{CCH}$, $S(X, C) = (X, C_S)$, where $G \in C_S$ means that $G \in C$ and there is a sequential filter $F \in C$ such that $F \leq G$; for a morphism $f \in \mathbf{CCH}$, $Sf = f$.

In other words, $S(X, C)$ has the coarsest sequential Cauchy structure on the same underlying set which is finer than C.

DEFINITION 4.2. The completely regular modification functor Q is a covariant functor $Q : \mathbf{CCH} \to \mathbf{QCH}$ defined as follows: for an object $(X, C) \in \mathbf{CCH}$, $Q(X, C) = (X, C_Q)$, where C_Q is the finest completely regular Cauchy structure on X coarser than C; for a morphism $f \in \mathbf{CCH}$, $Qf = f$.

The space $Q(X, C)$ can be described as follows: let $N(X, C) = (X_N, C_N)$, and let q be the convergence structure corresponding to the complete Cauchy structure C_N; if we assume that (X, C) is a subspace of (X_N, q), then $Q(X, C)$ is the restriction of $(X_N, \omega q)$ to the set X. It follows that $Q(X, C)$ is a dense Cauchy subspace of the completely regular topological space $(X_N, \omega q)$, which can also be described as $QN(X, C)$, and consequently $NQ(X, C) = QN(X, C)$. Thus we have established the following result.

PROPOSITION 4.3. *For any object* $(X, C) \in$ CCH ,
$QN(X, C) = NQ(X, C)$.

DEFINITION 4.4. A sequential Cauchy space (X, C) is *sequentially
regular* if $SQ(X, C) = (X, C)$.

Let SRCH denote the full subcategory of SCH whose objects are the
sequentially regular Cauchy spaces. Note that $SQ(X, C) \in$ SRCH for any
object $(X, C) \in$ SCH ; thus the composite functor SQ may be regarded as
the "sequentially regular modification functor" which maps SCH into
SRCH .

Next we define the functor N_S : SRCH $\to$ SRCH as follows: N_S is the
restriction of the composite functor SNQ to the category SRCH . Note
that $N_S(X, C) = SNQ(X, C) = SQN(X, C)$ by Proposition 4.3; therefore
$N_S(X, C) \in$ SRCH , since $N(X, C) \in$ SCH by Theorem 3.5.

THEOREM 4.5. N_S *is a completion functor on* SRCH .

Proof. Let $(X, C) \in$ SRCH . As noted in the discussion following
Definition 4.2, $Q(X, C)$ is a dense subspace of $QN(X, C)$, and one can
routinely verify that $(X, C) = SQ(X, C)$ is a dense subspace of
$SQN(X, C) = N_S(X, C)$. Both of the functors S and Q preserve
completeness, and so $N_S(X, C) \in$ SRCH* and $\left(N_S(X, C), i_N\right)$ is a
completion of (X, C) . To verify the universal property, let
$f : (X, C) \to \left(X_1, C_1\right)$ be Cauchy-continuous, with $(X, C) \in$ SRCH and
$\left(X_1, C_1\right) \in$ SRCH* . In the following commutative diagram, $\overline{f} = Nf$, all
maps are morphisms, and each unlabelled map is the identity function:

$$
\begin{array}{ccccc}
(X,C) & \xrightarrow{\ i_N\ } & N(X,C) & \to & SQN(X,C) & = N_S(X,C) \\
\downarrow f & & \downarrow \overline{f} & & \downarrow \overline{f} \\
\left(X_1,C_1\right) & \longrightarrow & \left(X_1,C_1\right) & \to & SQ\left(X_1,C_1\right) & = \left(X_1,C_1\right) \ .
\end{array}
$$

This completes the proof.

EXAMPLE 4.6. Let (X, q) be a sequentially regular convergence
space. A filter F on X is said to be *fundamental* if $f(F)$ is

convergent for each real-valued continuous function f on (X, q) . Let C be the Cauchy structure on X consisting of all filters G on X which are finer than some fundamental sequential filter F ; it follows that $(X, C) \in$ SRCH . In this case, $Q(X, C) = \left(X, C_Q\right)$ has as its Cauchy filters the set of all fundamental filters on X . It can be shown that $N_S(X, C)$ is a subspace of Novák's sequential envelope $\sigma(X, C)$, and coincides with $\sigma(X, C)$ whenever (X, C) is dense (in our sense) in $\sigma(X, C)$.

It should be noted that the completion functor N_S on SRCH is not strict. Indeed, no strict completion functor exists on SRCH . The natural completion is, up to homeomorphism, the only strict completion of a sequential Cauchy space, and it does not preserve sequential regularity.

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