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LOCALLY TRIVIAL FIBRATIONS WITH SINGULAR 1-DIMENSIONAL STEIN FIBER OVER q-COMPLETE SPACES

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Abstract. In connection with Serre's problem, we consider a locally trivial analytic fibration $\pi : E \longrightarrow B$ of complex spaces with typical fiber X. We show that if X is a Stein curve and B is q-complete, then E is q-complete.

§1. Introduction

Let $\pi : E \rightarrow B$ be a locally trivial analytic fibration of complex spaces with Stein fiber X of dimension n.

The following question was raised by Serre [17]:

Under the above assumptions, does it follow that E is Stein if B is Stein?

The answer is 'Yes' for $n = 0$ (Stein [23] and Le Barz [12]) and $n = 1$ (Mok [13]. In fact, some partial results were previously proved by various authors, Siu [19], Sibony [18], Hirschowitz [9], etc).

However, for $n \geq 2$ there are counterexamples to Serre's question (see Skoda [21], Demailly [7], and Coeuré–Loeb [3]).

Related to this circle of ideas we study the case when the base B is q-complete. The normalization is chosen such that Stein spaces correspond to 1-complete spaces.

For $n = 0$, *i.e.*, E is a topological covering of B, Ballico [2] proved the q-completeness of E. This is a particular case of a result due to Vâjâitu [24] which gives that if $\pi : Y \rightarrow Z$ is a locally trivial analytic fibration with hyperconvex fibre and Z is q-complete, then Y is q-complete. (A complex space S is said to be *hyperconvex* if S is Stein and has a negative exhaustion function which is continuous and plurisubharmonic.)

For $n = 1$, in order to generalize Mok's result, Vâjâitu [26] showed if X is non-singular, E is q-complete if B is q-complete. It remained the open problem when X is a singular Stein curve.

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Our main result gives a complete answer to this question. It can be stated as follows.

THEOREM 1. *Let $\pi : E \to B$ be a locally trivial analytic fibration with typical fiber X. If X is a Stein curve and B is q-complete, then E is q-complete, too.*

We remark that for $q = 1$ the above theorem can be deduced from the case when the fiber is non-singular. This is due to the fact that the class of Stein spaces is invariant under finite holomorphic surjections (see Narasimhan [14]).

When $q > 1$ the situation is drastically different because it is not known (see Colţoiu [6]) if the following holds:

Let $p : Y \to Z$ be a finite surjective holomorphic map of complex spaces. Assume Y is q-complete. Does it follow that Z is q-complete?

(When Z is q-complete, then Y is q-complete. See Vâjâitu [25].)

To avoid this difficulty we use essentially an approximated extension of q-convex functions defined on complex subspaces with control of the directions of positivity of the extended function. Also the quasi-plurisubharmonic functions (Peternell [15]; see also Demailly [8]) will play an important rôle in the proof.

§2. Preliminaries

Throughout this paper all complex spaces are assumed to be reduced and with countable topology.

Let Y be a complex space and T_yY denotes the (Zariski) tangent space of Y at y. Set $TY = \cup_{y\in Y} T_yY$.

A subset $\mathcal{M} \subset TY$ is said to be *a linear set over* Y (*of codimension* $\leq q-1$) if for every point $y \in Y$, $\mathcal{M}_y := \mathcal{M} \cap T_yY \subset T_yY$ is a complex vector subspace (of codimension $\leq q-1$). If $W \subset Y$ is an open subset, we have an obvious definition of the restriction $\mathcal{M}_{|W}$.

Let $\pi : Z \to Y$ be an analytic morphism of complex spaces and $\mathcal{M}$ a linear set over Y. For every $z \in Z$ we have an induced $\mathbf{C}$-linear map of complex vector spaces $\pi_{*,z} : T_zZ \to T_yY$, where $y = \pi(z)$. We set

$$\pi^*\mathcal{M} := \bigcup_{z\in Z} (\pi_{*,z})^{-1}(\mathcal{M}_y).$$

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Clearly, $\pi^*\mathcal{M}$ is a linear set over Z. Moreover, if codim $(\mathcal{M}) \leq q-1$, then codim $(\pi^*\mathcal{M}) \leq q-1$.

A (local) chart of Y at a point $y \in Y$ is a holomorphic embedding $\iota : U \to \widehat{U}$, where $U \ni y$ is an open subset of Y and $\widehat{U}$ is an open subset of some euclidean space $\mathbf{C}^n$. Holomorphic embedding means that $\iota(U)$ is an analytic subset of $\widehat{U}$ and the induced map $\iota : U \to \iota(U)$ is biholomorphic.

Suppose $\iota : U \to \widehat{U}$ is a local chart at y; then the differential map $\iota_{*,y} : T_yY \to \mathbf{C}^n$ of ι at y is an injective homomorphism of complex vector spaces.

Let $D \subset \mathbf{C}^n$ be an open subset. A function $\varphi \in C^\infty(D, \mathbf{R})$ is said to be *q-convex* if the quadratic form

$$L(\varphi, z)(\xi) = \sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(z)\xi_i\bar{\xi}_j,\ \xi \in \mathbf{C}^n,$$

has at least $n-q+1$ positive eigenvalues for every $z \in D$, or equivalently, there exists a family $\{M_z\}_{z\in D}$ of $(n-q+1)$-dimensional complex vector subspaces of $\mathbf{C}^n$ such that $L(\varphi, z)_{|M_z}$ is a positive definite form for all $z \in D$.

Let Y be a complex space. A function $\varphi \in C^\infty(Y, \mathbf{R})$ is said to be *q-convex* if every point of Y admits a local chart $\iota : U \to \widehat{U} \subset \mathbf{C}^n$ such that there is an extension $\widehat{\varphi} \in C^\infty(\widehat{U}, \mathbf{R})$ of $\varphi_{|U}$ which is q-convex on $\widehat{U}$. (This definition does not depend on the local embeddings.)

We say that Y is *q-complete* if there exists a q-convex function $\varphi \in C^\infty(Y, \mathbf{R})$ which is exhaustive, *i.e.*, the sublevel sets $\{\varphi < c\}, c \in \mathbf{R}$, are relatively compact in Y.

The following is due to Peternell [15].

DEFINITION 1. Let Y be a complex space, $W \subset Y$ an open set, $\mathcal{M}$ a linear set over W, and $\varphi \in C^\infty(W, \mathbf{R})$.

(a) Let $y \in W$. Then we say that φ is *weakly 1-convex with respect to* $\mathcal{M}_y$ if there are: a local chart $\iota : U \to \widehat{U}$ of Y with $y \in U \subset W$, $\widehat{U} \subset \mathbf{C}^n$ open set, and an extension $\widehat{\varphi} \in C^\infty(\widehat{U}, \mathbf{R})$ of $\varphi_{|U}$ such that $L(\widehat{\varphi}, \iota(y))(\iota_{*,y}\xi) \geq 0$ for every $\xi \in \mathcal{M}_y$.

We say that φ is *weakly 1-convex with respect to* $\mathcal{M}$ if φ is weakly 1-convex with respect to $\mathcal{M}_y$ for every $y \in W$.

(b) The function φ is said to be *1-convex with respect to* $\mathcal{M}$ if every point of W admits an open neighborhood $U \subset W$ such that there exists a 1-convex function θ on U with $\varphi - \theta$ weakly 1-convex with respect to $\mathcal{M}_{|U}$.

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It is not difficult to see that the extension $\widehat{\varphi}$ of φ is irrelevant for the above definition. In particular, if the functions φ and ψ are (weakly) 1-convex with respect to $\mathcal{M}$, so is their sum $\varphi + \psi$.

Definition 2. Let Y be a complex space and $\mathcal{M}$ a linear set over Y. We denote by $\mathcal{B}(Y, \mathcal{M})$ the set of all $\varphi \in C^o(Y, \mathbf{R})$ such that every point of Y admits an open neighborhood D on which there are functions $f_1, \ldots, f_k \in C^\infty(D, \mathbf{R})$ which are 1-convex with respect to $\mathcal{M}_{|_D}$ and

$$\varphi_{|_D} = \max\{f_1, \ldots, f_k\}.$$

From [24] and [4] we quote

Proposition 1. *Let $\mathcal{M}$ be a linear set over a complex space Y and $\varphi \in \mathcal{B}(Y, \mathcal{M})$. Then for every $\eta \in C^o(Y, \mathbf{R})$, $\eta > 0$, there exists $\widetilde{\varphi} \in C^\infty(Y, \mathbf{R})$ which is 1-convex with respect to $\mathcal{M}$ and*

$$\varphi \leq \widetilde{\varphi} < \varphi + \eta.$$

In particular, if $\mathcal{M}$ has codimension $\leq q - 1$, then $\widetilde{\varphi}$ is q-convex.

From [15] we have:

Proposition 2. *Let Y be a complex space and $\varphi \in C^\infty(Y, \mathbf{R})$ a q-convex function. Then there is a linear set $\mathcal{M}$ over Y of codimension $\leq q-1$ such that φ is 1-convex with respect to $\mathcal{M}$.*

Motivated by Propositions 1 and 2, we say that a complex space Y is 1-*complete with respect to a linear set $\mathcal{M}$ over Y* if there exists an exhaustion function $\varphi \in \mathcal{B}(Y, \mathcal{M})$. Consequently a complex space Y is q-complete if, and only if, Y is 1-complete with respect to some linear set $\mathcal{M}$ of codimension $\leq q - 1$.

Let us recall that a Stein space S is said to be *hyperconvex* if there is a smooth plurisubharmonic exhaustion function $\varphi : S \to (-\infty, 0)$.

In [24] the following result has been proved:

Proposition 3. *Let $\pi : E \to B$ be a locally trivial analytic fibration with hyperconvex fibre. If B is 1-complete with respect to a linear set $\mathcal{M}$ over B, then E is 1-complete with respect to $\pi^*\mathcal{M}$. In particular if B is q-complete, then E is q-complete.*

From this it follows

Corollary 1. *Let $\pi : E \to B$ be a covering space with q-complete base B. Let $\mathcal{M}$ be a linear set over B of codimension $\leq q-1$ such that B is 1-complete with respect to $\mathcal{M}$. Then there exists $\mu : E \to \mathbf{R}$ a smooth exhaustion function which is 1-convex with respect to $\pi^*\mathcal{M}$. In particular E is q-complete.*

We shall also need the following result of M. Peternell ([15], Satz 3.1).

Proposition 4. *Let Y be a complex space and $A \subset Y$ a closed analytic subset. Then there is a function $h \in C^\infty(Y, \mathbf{R})$ such that:*

a) $h \geq 0$, $\{h = 0\} = A$.

b) *For every $y \in Y$ there exists an open neighborhood U of y and $\theta \in C^\infty(U, \mathbf{R})$ such that*

$$\log(h_{|_{U \setminus A}}) + \theta_{|_{U \setminus A}}$$

is 1-convex.

Remark 1. The function $\log h$ is locally equal to the sum of a plurisubharmonic function and a smooth function. Such a function is called in Demailly [8] a *quasi-plurisubharmonic function*.

§3. Construction of an auxiliary fibration

We recall that a complex space X is called *hyperbolic* (in the sense of Kobayashi) if the Kobayashi semidistance d_X is a distance. See the book of S. Lang [11].

Examples and properties.

1) $\mathbf{C} \setminus \{p, q\}$ with $p, q \in \mathbf{C}$, $p \neq q$, is hyperbolic.

2) Any open subset of a hyperbolic space is hyperbolic.

3) Let $\pi : X' \to X$ be a covering of complex spaces. Then X' is hyperbolic if and only if X is hyperbolic.

4) Any relatively compact open subset of $\mathbf{C}^n$ is hyperbolic.

A proof of these facts may be found in Kobayashi [10] and Lang [11].

Let us recall also the following result. (See Siu [20], p. 176 and Royden [16], p. 311.)

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Lemma 1. *Let F be a hyperbolic manifold and W a connected complex space. Let $f : W \times F \to F$ be a holomorphic map such that for some $w_o \in W$ the restriction of f to $\{w_o\} \times F$ is biholomorphic onto F. Then f is independent of the variable in W, i.e., $f(w, x) = f(w_o, x)$ for all $w \in W$ and $x \in F$.*

Let now X be a Stein space of pure dimension 1 (Stein curve) and W a connected complex space.

We assume that a biholomorphic map $\Phi : W \times X \to W \times X$ is given such that the diagram

$$\begin{array}{ccc} W \times X & \xrightarrow{\ \Phi\ } & W \times X \\ & {\scriptstyle pr}\searrow \quad \swarrow{\scriptstyle pr} & \\ & W & \end{array}$$

is commutative. So, for every $w \in W$ we have an automorphism of X, $\Phi_w : X \to X$, given by $\Phi_w(x) = \Phi(w, x)$.

Let $\nu : \widetilde{X} \to X$ be the normalization map. Every $\Phi_w, w \in W$, lifts to a unique automorphism $\widetilde{\Phi}_w$ of $\widetilde{X}$ such that the diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\ \widetilde{\Phi}_w\ } & \widetilde{X} \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \nu} \\ X & \xrightarrow[\ \Phi_w\ ]{} & X \end{array}$$

is commutative.

The maps $\{\widetilde{\Phi}_w\}_{w \in W}$ define a unique map $\widetilde{\Phi} : W \times \widetilde{X} \to W \times \widetilde{X}$ and we have a commutative diagram

$$\begin{array}{ccc} W \times \widetilde{X} & \xrightarrow{\ \widetilde{\Phi}\ } & W \times \widetilde{X} \\ {\scriptstyle \mathrm{id} \times \nu}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \times \nu} \\ W \times X & \xrightarrow[\ \Phi\ ]{} & W \times X. \end{array} \tag{1}$$

We show

Lemma 2. *$\widetilde{\Phi}$ is biholomorphic.*

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This will be proved in two steps.

Step 1. $\widetilde{\Phi}$ *is biholomorphic if* X *is irreducible.*

We may assume that $S := \mathrm{Sing}(X) \neq \emptyset$. Put $\widetilde{S} = \nu^{-1}(S)$. Consider the commutative diagram of isomorphisms

$$\begin{array}{ccc} W \times (\widetilde{X} \setminus \widetilde{S}) & \xrightarrow{\widetilde{\Phi}_1} & W \times (\widetilde{X} \setminus \widetilde{S}) \\ \downarrow{\scriptstyle \mathrm{id} \times \nu} & & \downarrow{\scriptstyle \mathrm{id} \times \nu} \\ W \times (X \setminus S) & \xrightarrow{\Phi_1} & W \times (X \setminus S) \end{array} \tag{2}$$

where $\Phi_1 := \Phi_{|_{W \times (X \setminus S)}}$ and $\widetilde{\Phi}_1 := \widetilde{\Phi}_{|_{W \times (\widetilde{X} \setminus \widetilde{S})}}$.

If $\mathrm{card}(\widetilde{S}) \geq 2$, then $\widetilde{X} \setminus \widetilde{S}$ is hyperbolic (by Examples 1), 2), and 3) in the beginning of this section). It follows from Lemma 1 that the maps Φ_1 and $\widetilde{\Phi}_1$ in diagram (2) are independent of $w \in W$, therefore also the maps Φ and $\widetilde{\Phi}$ in diagram (1) are independent of w. In particular $\widetilde{\Phi}$ is biholomorphic.

Similarly, if $\mathrm{card}(\widetilde{S}) = 1$ and $\widetilde{X} \neq \mathbf{C}$, it follows that $\widetilde{X} \setminus \widetilde{S}$ is hyperbolic and $\widetilde{\Phi}$ is biholomorphic (being independent of $w \in W$).

It remains to study the case when X has only one singular point, say $S = \{x_o\}$, at which X is locally irreducible (therefore $\mathrm{card}(\widetilde{S}) = 1$) and $\widetilde{X} = \mathbf{C}$ is its normalization. We may assume that $\nu^{-1}(x_o) = 0 \in \mathbf{C}$. It follows then easily that $\widetilde{\Phi}(w, \widetilde{x}) = f(w) \cdot \widetilde{x}$ with $f \in \mathcal{O}^*(W)$, so obviously $\widetilde{\Phi}$ is biholomorphic.

Step 2. $\widetilde{\Phi}$ *is biholomorphic for arbitrary* 1*-dimensional Stein space* X.

Clearly we may assume that X has no isolated points. Let $X = \cup X_i$ be the decomposition of X into irreducible components. We claim first that for every index i there is a unique index j so that $\Phi(W \times X_i) = W \times X_j$. To show this, we let $\mathrm{Reg}(X) = \cup D_i$ be the decomposition into connected components with $X_i = \overline{D_i}$. Obviously, for each i there is a (unique) j such that $\Phi(W \times D_i) = W \times D_j$. Using the continuity of Φ the claim follows.

Now let $\widetilde{X} = \cup \widetilde{X_i}$ be the decomposition of $\widetilde{X}$ into connected components. Therefore $\nu(\widetilde{X_i}) = X_i$ and $\nu_{|_{\widetilde{X_i}}} : \widetilde{X_i} \to X_i$ is the normalization of X_i.

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From the above claim each connected component $W \times \widetilde{X}_i$ of $W \times \widetilde{X}$ corresponds by $\widetilde{\Phi}$ to a (unique) connected component $W \times \widetilde{X}_j$. Thus we have a commutative diagram

$$(3)\qquad \begin{array}{ccc} W \times \widetilde{X}_i & \xrightarrow{\widetilde{\Phi}_2} & W \times \widetilde{X}_j \\ {\scriptstyle \mathrm{id} \times \nu}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \times \nu} \\ W \times X_i & \xrightarrow[\Phi_2]{} & W \times X_j \end{array}$$

where $\Phi_2 := \Phi_{|_{W \times X_i}}$ and $\widetilde{\Phi}_2 := \widetilde{\Phi}_{|_{W \times \widetilde{X}_i}}$.

We fix some biholomorphic map $h : X_j \to X_i$ (*e.g.*, $h = \Phi_{w_o}^{-1}$ for some $w_o \in W$) and consider the commutative diagram

$$(4)\qquad \begin{array}{ccccc} W \times \widetilde{X}_i & \xrightarrow{\widetilde{\Phi}_2} & W \times \widetilde{X}_j & \xrightarrow{\mathrm{id} \times \widetilde{h}} & W \times \widetilde{X}_i \\ {\scriptstyle \mathrm{id} \times \nu}\downarrow & & {\scriptstyle \mathrm{id} \times \nu}\downarrow & & {\scriptstyle \mathrm{id} \times \nu}\downarrow \\ W \times X_i & \xrightarrow[\Phi_2]{} & W \times X_j & \xrightarrow[\mathrm{id} \times h]{} & W \times X_i. \end{array}$$

By step 1), $(\mathrm{id} \times \widetilde{h}) \circ \widetilde{\Phi}_2$ is biholomorphic, so $\widetilde{\Phi}_2$ is biholomorphic. It follows that $\widetilde{\Phi}$ is biholomorphic and the Lemma 2 is completely proved.

LEMMA 3. *Let $\pi : E \to B$ be a locally trivial holomorphic fibration with fibre X a pure 1-dimensional Stein space.*

Then there exist $\pi' : E' \to B$ a locally trivial holomorphic fibration with fibre $\widetilde{X}$ = the normalization of X and a holomorphic map $\tau : E' \to E$ with the following properties:

1) *The diagram*

$$\begin{array}{ccc} E' & \xrightarrow{\tau} & E \\ & {\scriptstyle \pi'}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & B & \end{array}$$

is commutative.

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2) *For every $b \in B$ the induced map*

$$\tau_b : E'_b \longrightarrow E_b$$

is the normalization map $\nu : \widetilde{X} \to X$.

3) *Let*

$$A := \bigcup_{b \in B} \operatorname{Sing}(E_b)$$

and $A' := \tau^{-1}(A)$. Then A and A' are closed analytic subsets of E and E' respectively and $\tau_{|_{E' \setminus A'}} : E' \setminus A' \to E \setminus A$ is biholomorphic.

Proof. Let $(W_i)_{i \in I}$ be a locally finite open covering of B such that $E_{|W_i}$ is trivial and $W_i \cap W_j$ is connected for every $i, j \in I$. We have the transition functions $\Phi_{ij} : (W_i \cap W_j) \times X \to (W_i \cap W_j) \times X$ which are biholomorphic and such that the diagram

$$\begin{array}{ccc} (W_i \cap W_j) \times X & \xrightarrow{\Phi_{ij}} & (W_i \cap W_j) \times X \\ & {\scriptstyle pr} \searrow \quad \swarrow {\scriptstyle pr} & \\ & W_i \cap W_j & \end{array}$$

is commutative.

Therefore we have induced maps $\widetilde{\Phi}_{ij} : (W_i \cap W_j) \times \widetilde{X} \to (W_i \cap W_j) \times \widetilde{X}$ which are biholomorphisms by the previous lemma.

Then clearly $\{\widetilde{\Phi}_{ij}\}$ define the required holomorphic fibration $\pi' : E' \to B$. All other required properties in the lemma are easily verified.

Remark 2. The above two lemmas are trivial if we assume B to be normal. In this case (B normal) it is clear that $\widetilde{\Phi}_{ij}$ are biholomorphic and it is not necessary to make the assumption that the fiber X is a 1-dimensional Stein space (X may be an arbitrary complex space).

§4. The proof of the main result

In this section we shall prove the subsequent Theorem 1′ which clearly implies Theorem 1 already mentioned in the introduction.

THEOREM 1′. *Let $\pi : E \to B$ be a locally trivial analytic fibration with Stein fibre X of dimension 1 and assume that B is 1-complete with respect to a linear set $\mathcal{M}$ (over B). Then E is 1-complete with respect to $\pi^*\mathcal{M}$.*

In particular, E is q-complete if B is q-complete.

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Proof. When the fiber is non-singular Theorem $1'$ is proved in [26].

Subsequently we deal with the singular fibre X.

Let $\pi' : E' \to B$ be a fibration with the properties stated in Lemma 3.

Denote $p := \pi_{|_A} : A \to B$ which is a covering map. In fact A can be described locally over B as follows: Let $E_{|_U} \simeq U \times X$ be a local trivialization. Then $A \cap \pi^{-1}(U) \simeq U \times \mathrm{Sing}(X)$ and $\mathrm{Sing}(X)$ is a discrete subset of X, say $\mathrm{Sing}(X) = \{a_j\}_{j\in J}$, since X is one dimensional. By Corollary 1 there exists a smooth exhaustion function $\mu : A \to \mathbf{R}$ such that μ is 1-convex with respect to $p^*\mathcal{M}$.

We shall prove the following statement:

(♣) Let $\eta : A \to (0, \infty)$ be any continuous function. Then there exists an open neighborhood V of A in E and a smooth function $\widetilde{\mu} : V \to \mathbf{R}$ which is 1-convex with respect to $(\pi^*\mathcal{M})_{|_V}$ and

$$\mu \leq \widetilde{\mu} < \mu + \eta$$

on A.

To prove (♣) we follow an idea from (Colţoiu [5], Lemma 3); but we refine it in order to get extensions with controlled positivity directions which are necessary for our patching process.

We fix a non-negative smooth strictly subharmonic function $f : X \to \mathbf{R}$ such that $\mathrm{Sing}(X) = \{f = 0\}$.

Let also $\{U_i\}_{i\in I}$ and $\{W_i\}_{i\in I}$ be locally finite open coverings of B such that $U_i \subset\subset W_i \subset\subset B$ and E is trivial near $\overline{W}_i$. Now select $\theta_i \in C^\infty(B, \mathbf{R})$ with $\theta_i > 0$ on $\overline{U}_i$ and $\theta_i < 0$ on ∂W_i.

We have $E_{|_{W_i}} \simeq W_i \times X$ and $E_{|_{W_i}}$ contains the sequence of mutually disjoint closed analytic subsets $W_i \times \{a_j\}$, $j \in J$. On $W_i \times \{a_j\}$ we consider the restriction of μ and perturb it with $\epsilon_{ij}\theta_i \circ \pi$. More precisely, we define near $\overline{W}_i \times \{a_j\}$

$$\mu_{ij} = \mu + \epsilon_{ij}\theta_i \circ \pi$$

where $\epsilon_{ij} > 0$ are small enough constants to be chosen later. For every $x \in A$ we set $\mu_1(x) = \max\{\mu_{ij}(x)\,;\,(i,j) \in H(x)\}$ where $H(x) = \{(i,j) \in I \times J\,;\, x \in W_i \times \{a_j\}\}$. If ϵ_{ij} are small enough, then μ_1 is continuous on A, $\mu_1 \in \mathcal{B}(A, p^*\mathcal{M})$, and $\mu \leq \mu_1 < \mu + \eta$ on A.

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Moreover, on $\partial(W_i \times \{a_j\})$ one has

$$(*) \qquad \mu_1 > \mu_{ij}$$

for every indices $(i,j) \in I \times J$.

We shall prove that μ_1 has an extension $\widetilde{\mu_1}$ to a neighborhood V of A such that $\widetilde{\mu_1} \in \mathcal{B}(V, \pi^*\mathcal{M})$ and this clearly will conclude the proof of (♣) in view of the approximation Proposition 1.

For this we choose open neighborhoods $D_j \subset\subset X$ of the points a_j such that $\overline{D_j} \cap \overline{D_{j'}} = \emptyset$ if $j \neq j'$. The functions μ_{ij} defined on $W_i \times \{a_j\}$ can be extended to smooth functions $\widetilde{\mu_{ij}}$ on $W_i \times D_j$ which are 1-convex with respect to $\pi^*\mathcal{M}$. Indeed, if p'_{ij} and p''_{ij} denote the projections of $W_i \times D_j$ on $W_i \times \{a_j\}$ and on D_j respectively, then one may set

$$\widetilde{\mu_{ij}} := \mu_{ij} \circ p'_{ij} + f_{|D_j} \circ p''_{ij}.$$

Put

$$\Omega := \bigcup_{(i,j) \in I \times J} W_i \times D_j$$

and for $x \in \Omega$, $\widetilde{\mu_1}(x) = \sup\{\widetilde{\mu_{ij}}(x)\,;\,(i,j) \in \Gamma(x)\}$ where $\Gamma(x) = \{(i,j) \in I \times J\,;\, x \in W_i \times D_j\}$.

If $V \subset \Omega$ is a small enough open neighborhood of A, it follows then from $(*)$ that $\widetilde{\mu_1}$ is continuous on V and in fact $\widetilde{\mu_1} \in \mathcal{B}(V, \pi^*\mathcal{M})$, whence the proof of statement (♣).

We now go back to the proof of Theorem 1′. Since Theorem 1′ holds for E', there exists a smooth exhaustion function $\psi' : E' \to \mathbf{R}$ which is 1-convex with respect to $(\pi')^*\mathcal{M}$.

We fix some smooth function $\widetilde{\mu} > 0$ defined near $\overline{V}$, where V is a sufficiently small open neighborhood of A such that $\widetilde{\mu}_{|\overline{V}}$ is proper and $\widetilde{\mu}$ is 1-convex with respect to $\pi^*\mathcal{M}$ near $\overline{V}$.

By Proposition 4 there is a quasi-plurisubharmonic function $\beta : E \to [-\infty, \infty)$ with $A = \{\beta = -\infty\}$. Also we may assume $\beta = 0$ on $E \setminus V$. Then $\beta' = \beta \circ \tau$ is quasi-plurisubharmonic on E' and $A' := \tau^{-1}(A) = \{\beta' = -\infty\}$. Since ψ' is a smooth exhaustion function on E' which is 1-convex with respect to $(\pi')^*\mathcal{M}$, there is a strictly increasing smooth convex function $\delta : (0,\infty) \to (0,\infty)$ such that $\delta \circ \psi' + \beta'$ is 1-convex with respect to $(\pi')^*\mathcal{M}$ on $E' \setminus A'$, and

$$\delta \circ \psi' + \beta' > \widetilde{\mu} \circ \tau$$

on $\tau^{-1}(\partial V)$.

Now E is covered by the open subsets $V_1 := V$ and $V_2 := E \setminus A$. On V_1 we consider the function $\varphi_1 = \widetilde{\mu}$ and on V_2 the function $\varphi_2 = \delta \circ \psi' \circ \tau^{-1} + \beta$ and we define the function $\psi_1 : E \to \mathbf{R}$ given by $\psi_1(x) := \max\{\varphi_k(x)\,;\, k \in K(x)\}$, where $K = \{1, 2\}$ and $K(x) = \{k \in K\,;\, x \in V_k\}$.

Then ψ_1 is a continuous exhaustion function on E and $\psi_1 \in \mathcal{B}(E, \pi^*\mathcal{M})$. Thus the proof of Theorem $1'$ is complete.

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