A characterisation of regular *n*-gons via (in)commensurability

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1. *Introduction*

In Euclidean geometry, a regular polygon is equiangular (all angles are equal in size) and equilateral (all sides have the same length) polygon. So regular polygons should be thought of as special polygons.

Another relevant concept in geometry is incommensurability. Recall that two magnitudes are incommensurable if their ratio is not given by a pair of positive integers [1]. Incommensurability is a relevant topic not only in mathematics; its discovery represented an important moment in the development of human thought (see [2, p. 472]). It is no wonder that in Italian schools this concept is treated in both mathematics and philosophy.

In addition to the classic discovery of the incommensurability between side and diagonal of a square and side and diagonal of a regular pentagon, recent articles prove that commensurability between elements of a regular polygon does not occur frequently.

For those readers who are unfamiliar with these ideas, we remark that the only regular polygon which has a diagonal commensurable with its side is the regular hexagon. More generally, polygons admitting pairs of commensurable diagonals have been characterised, and in particular it has been proved that if the number of vertices of a regular polygon is not a multiple of 6, then its pairs of diagonals of different length are incommensurable (see [3] and [4]). Similarly, it has been shown recently that every pair of diagonals of different length of a Platonic solid are incommensurable (see [5]). Recently, see [6], it has been highlighted how the concept of incommensurability may help to understand not only specific irrational numbers such as ratios between diagonals of regular polygons, but probably also all real numbers. Referring to the study of diagonals of regular polygons we suggest the interesting article of [7]. We also highlight that these kinds of problems are connected to the study of methods to determine the trigonometric ratios (see [8]).

It should also be noted that commensurability can also be referred to by angular quantities. For example, it might be interesting for students to know that in every Pythagorean triangle the acute angles are not rational multiples of 360°, so that they are incommensurable.

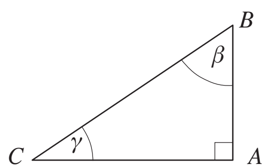

FIGURE 1: In a Pythagorean triangle, the acute angles are incommensurable.

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This easily follows from an Evans-Isaacs' result (see Theorem 3), and it has been object of other interesting investigations (see [9, Corollary 1], [10], [11] or [12, Section 2]).

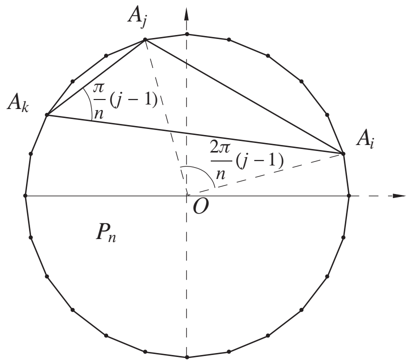

FIGURE 2: The relationship between the magnitude of the angles and the vertex indices of a regular n-gon.

We may think the vertices of a regular polygon as the nth roots of unity, so that the interior angles of any triangle $A_iA_jA_k$ whose vertices are among those of a regular polygon $P_n = (A_0, A_1, \dots, A_{n-1})$ are rational in degrees, that is rational multiples of π or (equivalently) of 360° (see Figure 3). In particular they are pairwise commensurable (see [3, Lemma 2.1], for example).

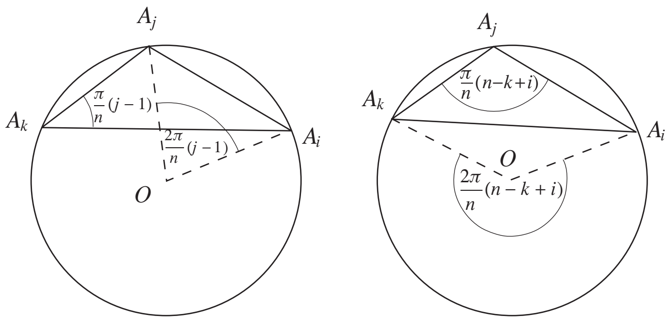

FIGURE 3: Left: $\angle A_iA_kA_j = \frac{1}{2}\angle A_iO_jA_j$. Right: $\angle A_kA_iA_i = \frac{1}{2}\angle A_kOA_i$.

This Article takes its motivation from the concept of commensurability as opposed to incommensurability. The basic idea is that commensurability is a phenomenon that in regular n-gons mostly appears just for pairs of congruent elements (sides, consecutive angles, diagonals), so that if we impose some 'conditions' of commensurability among elements of cyclic n-gons, we could expect some kind of 'regularity'. Which conditions should we consider?

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Looking at 'commensurability properties' of a regular polygon P_n, we can say that:

(i) P_n is cyclic.

(ii) Each pair of consecutive sides of P_n are commensurable.

(iii) Each pair of interior angles of triangles whose vertices are among the vertices of P_n are commensurable.

It is reasonable to ask whether the three above properties (i)–(iii), besides being necessary also express together a sufficient condition for a convex polygon to be regular. In this Article we will demonstrate that the answer is 'yes'.

Theorem 1: Let $P_n = (A_0, A_1, A_2, \ldots, A_{n-1})$ be a convex n-gon. Then P_n is regular if, and only if, the conditions (i)−(iii) hold.

Of course, if a polygon meets only one of the three conditions (i)−(iii) it is not necessarily regular; in fact, easy examples also show that even two of these conditions are not sufficient for a polygon to be regular (see Examples 1, 2 and 3).

A more in-depth analysis will be devoted to the study of regular polygons with an odd number of sides. In this case we have the following elegant characterisation:

Theorem 2: Let n be an odd positve integer. A cyclic n-gon is regular if, and only if, each pair of its consecutive sides and each pair of its consecutive angles are commensurable.

We remark that every triangle is a cyclic polygon, so that it trivially satisfies the condition (i). Therefore, the above result can be read as an extension of the following well-known characterisation of equilateral triangles in terms of commensurability.

Proposition 1.3: [see [13, p.228] or [14, Proposition 1]] Every triangle all of whose sides have rational length and all of whose angles are rational in degrees must be equilateral.

2. *The result*

First, we introduce a result due to Evans and Isaacs (see [14]), which will be is useful for our purpose: We can state their theorem as follows:

Theorem 3 (Evans-Isaacs): If Δ is a non-isosceles triangle with two or more rational sides and with all angles rational (measured in degrees), then Δ has angles 30°, 60° and 90°.

Remark: Note that if a triangle T has two commensurable sides and its interior angles are rational in degrees, then it is similar to a triangle Δ which satisfies the hypothesis of the Evans-Isaacs theorem.

In view of the Remark, the Evans-Isaacs result can also be stated as follows:

If T is a non-isosceles triangle with two or more commensurable sides and with all angles rational (measured in degrees), then T has angles 30°, 60° and 90°. In particular the hypothenuse is twice the length of a leg and it is incommensurable with the other one.

We are now in a position to prove our first characterisation (Theorem 1):

Proof: Let P_n be a regular polygon. Then the first two conditions (i) and (ii) are trivially satisfied. The third property derives from the fact that each angles of the triangle $A_kA_iA_j$ is of the type $k\frac{\pi}{n}$, where k is an integer (see Figure 3).

Conversely, let P_n be a convex polygon, and suppose that the three conditions (i)–(iii) hold. Then P_n is inscribed in a circle C, so that to prove that P_n is regular it will be sufficient to verify that the measure of each pair of consecutive sides of P_n coincide. In this order put $l_1 = A_0A_1$ and $l_2 = A_1A_2$, and show that $l_1 = l_2$ (see Figure 4).

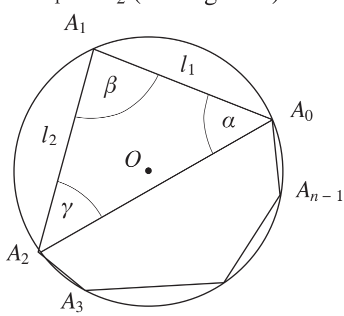

FIGURE 4: Triangles inscribed in cyclic polygons.

By hypothesis l_1 / l_2 is rational and the interior angles α, β and γ of the triangle T, $A_0A_1A_2$ are pairwise commensurable, so that $\alpha = r_1\beta$ and $\alpha = r_2\gamma$, for suitable rational numbers r_1 and r_2. It follows that

$$180° = \alpha + \beta + \gamma = \alpha\left(1 + r_1^{-1} + r_2^{-1}\right),$$

so that α and hence β and γ are rational in degrees.

By contradiction assume that T is not isosceles. Then T satisfies the condition of Evans-Isaacs' (Theorem 3), so that the angles of T $\left(A_0, A_1, A_2\right)$ are 30°, 60°, 90°, and hence it would appear that T is inscribed in a semicircle of C, and in particular the legs of l_1 and l_2 must be incommensurable. This contradicts the second commensurability property, (ii). Each pair of consecutive sides of P_n are commensurable, set out in the list above Theorem 1. Thus T is an isosceles triangle.

If $l_1 \neq l_2$, then one of the two cases occurs: $\left|A_0A_2\right| = l_1$ or $\left|A_0A_2\right| = l_2$. By hypothesis l_1 and l_2 are commensurable, so that in both the cases the third side A_0A_2 is commensurable with l_1 and l_2. It turns out that the triangle

T is similar to a triangle with rational sides and rational angles α, β (in degrees), so that T is equilateral (see for example [13, p. 228] or [14, Proposition 1]). This contradiction shows that $l_1 = l_2$.

Similarly, it can be proved that every two consecutive sides of P_n are congruent, and hence the cyclic polygon P_n is regular.

The following examples shows that the conditions (i), (ii), (iii) are independent.

Example 1: Any rhombus (different from a square) whose interior angles are rational in degrees is not cyclic (so that it does not satisfy (i)); on the other hand, it trivially satisfies conditions (ii) and (iii).

Example 2: By Proposition 1.3, every non-equilateral triangle all of whose angles are rational in degrees must have at least a non-commensurable pair of consecutive sides. So that, for example, the classical triangle 30°, 60°, 90°, verifies conditions (i) and (iii) but does not satisfy the condition (ii).

We note that it is not difficult to give examples of quadrilaterals of this type (see Example 4). Instead, it appears more difficult to find cyclic n-gons ($n > 4$) which satisfy the condition (iii), and which have non-commensurable pairs of consecutive sides.

Example 3: To determine cyclic polygons with commensurable consecutive sides, but which do not verify (iii), it is sufficient to consider any Pythagorean triangle (see [10, p. 191), for example the 3, 4, 5 triangle.

3. *Polygons with an odd number of vertices*

We will now see that a further characterisation can be obtained in the case of polygons with an odd number of vertices. Indeed, in this case (n odd), condition (iii) of main Theorem 1 admits interesting equivalent formulations.

Lemma: Let $P_n = (A_0, A_1, \dots, A_{n-1})$ be a cyclic polygons with an odd number of vertices. The following conditions are equivalent:

(a) The condition (iii) of Theorem 1: Each pair of interior angles of a whichever triangle whose vertices are among the vertices of P_n are commensurable.

(b) For each triplet of distinct vertices A_k, A_i, A_j of the polygon P_n, the interior angles of the triangle $T(A_k, A_i, A_j)$ are rational in degrees.

(c) Every pair of consecutive interior angles of P_n are commensurable.

(d) Every pair of consecutive central angles $\angle A_iOA_{i+1}$ and $\angle A_{i+1}OA_{i+2}$ ($i = 0, \dots, n-1$) of P_n are commensurable.

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Proof: We will prove a chain of implications.

(a ⇒ b) Let α, β and γ be the internal angles of the triangle $T\ (A_k, A_i, A_j)$, then by hypothesis $\alpha = r_1\beta$ and $\alpha = r_2\gamma$, for suitable rational numbers r_1 and r_2. Then $180° = \alpha + \beta + \gamma = \alpha\left(1 + r_1^{-1} + r_2^{-2}\right)$, so that α and hence β and γ are rational in degrees.

(b ⇒ c) Let A_iA_{i+1} and $A_{i+1}A_{i+2}$ be two consecutive sides of P_n. By hypothesis the interior angles of the triangle $A_iA_{i+1}A_{i+2}$ are rational in degrees. It follows that every pair of interior angles of P_n are commensurable. In particular the condition (c) is verified.

(c ⇒ d) Suppose now that (c) is satisfied and put $\alpha_i = \angle A_{i-1}A_iA_{i+1}$. It follows that, for every $i = 0, 1, \ldots, n - 1$, two consecutive angles α_i and α_{i+1} are commensurable, so that there exist suitable rational numbers k_i such that

$$\alpha_0 = k_0\alpha_1,\ \alpha_1 = k_1\alpha_2,\ \alpha_2 = k_2\alpha_3,\ \ldots,$$

$$\alpha_i = k_i\alpha_{i+1},\ \ldots,\ \alpha_{n-1} = k_{n-1}\alpha_0.$$

On the other hand the sum of the interior angles of P_n is $(n - 2)\,180°$, then

$$(n - 2)\,180° = \alpha_0 + \ \ldots + \alpha_{n+1}$$

$$= \alpha_0\left(1 + k_0^{-1} + (k_0k_1)^{-1} + \ \ldots\ + (k_0k_1\ldots k_{n-1})^{-1}\right),$$

so that α_0 and hence every α_i is rational in degrees.

Clearly $\angle A_{i-1}OA_{i+1} = 360° - 2\alpha_i$ (see Figure 5) so that $\angle A_{i-1}OA_{i+1}$ is rational in degrees for every i.

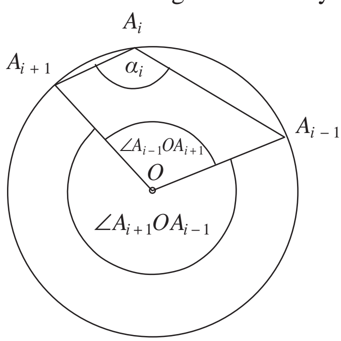

FIGURE 5: The angle α_i is rational in degrees, and hence the angle $\angle A_{i-1}OA_{i+1}$ is likewise rational in degrees.

By hypothesis $n = 2m + 1$, so that

$$\angle A_0OA_{2m} = \angle A_0OA_2 + \angle A_2OA_4 + \ \ldots + \angle A_{2i}OA_{2(i+1)} + \ldots + \angle A_{2(m-1)}OA_{2m}$$

is rational in degrees. It follows that $\angle A_{n-1}OA_0 = 360° - \angle A_0OA_{2m}$ is rational in degrees (see Figure 6).

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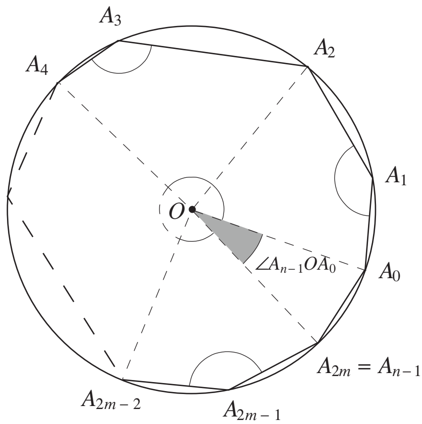

FIGURE 6: Angular relationships in an n-gon P_n with an odd number of vertices: the central angle determined by the consecutive vertices A_{n-1} and A_0, can be expressed in terms of $\angle A_1$, $\angle A_3$, … , $\angle A_{2n-1}$.

Similarly, we can show that $\angle A_0OA_1$, $\angle A_1OA_2$, … are rational in degrees. In particular every pair of consecutive central angles $\angle A_iOA_{i+1}$ and $\angle A_{i+1}OA_{i+2}$ $(i = 0, \ldots, n-1)$ of P_n are commensurable. Thus (c) $\Rightarrow$ (d).

(d $\Rightarrow$ a) Now, suppose that (d) is satisfied. For every $i = 0, \ldots, n-1$, put $\beta_i = \angle A_iOA_{i+1}$. By hypothesis there exist rational numbers $h_0, \ldots, h_{n-1}$ such that

$$\beta_0 = h_0\beta_1, \beta_1 = h_1\beta_2, \beta_2 = h_2\beta_3, \ldots \beta_i = h_i\beta_{i+1}, \ldots \beta_{n-1} = h_{n-1}\beta_0,$$

On the other hand $\sum \beta_i = 360°$, then

$$360° = \beta_0 + \ldots + \beta_{n-1}$$
$$= \beta_0\left(1 + h_0^{-1} + (h_0h_1)^{-1} + \ldots + (h_0h_1\ldots h_{n-1})^{-1}\right),$$

so that β_0 and hence every β_i is rational in degrees.

Consider now a triangle T (A_i, A_j, A_k) whose vertices are among those of the vertices of the polygon P_n. Let u and v be positive integers such that $A_j = A_{i+u}$ and $A_j = A_{i+v}$. Clearly $\angle A_iOA_{i+u} = \beta_i + \ldots + \beta_{i+u-1}$ and $\angle A_iOA_{i+v} = \beta_i + \ldots + \beta_{i+v-1}$, so that $\angle A_iOA_j$ and $\angle A_iOA_k$ are both rational in degrees. It follows that $\angle A_iA_kA_j$ and $\angle A_kA_jA_i$ are rational in degrees (see Figure 3), i.e. they are commensurable.

The proof is complete.

Now we can prove Theorem 2 introduced in the last part of Section 1.

Proof: The conditions are trivially necessary. Conversely, assume that P_n is cyclic (n-odd number), every pair of consecutive sides of P_n are commensurable and every pair of consecutive interior angles of P_n are commensurable. Then, by the Lemma, the polygon P_n also satisfies all the conditions (i)–(iii). Therefore it is regular by Theorem 1.

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We note that Theorem 2 fails to be true for polygons with an even number of sides. Indeed there exist easy examples of non-regular cyclic quadrilaterals whose sides are rational and whose interior angles are rational in degrees (see Figure 7).

Finally, we remark that the above example also shows that there exist cyclic quadrilaterals with commensurable consecutive sides which does not satisfies the condition (iii) of Theorem 1 (see also Example 3 in Section 2).

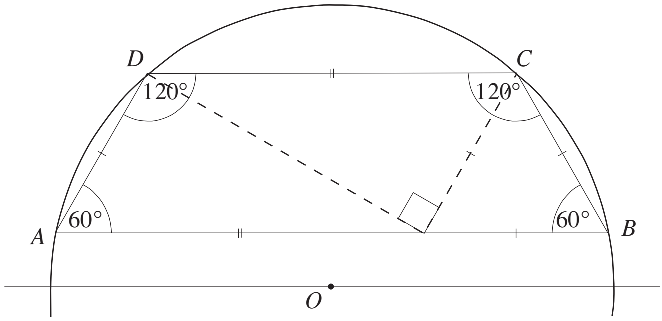

FIGURE 7: Here is an isosceles trapezium whose sides are pairwise commensurable and whose interior angles are pairwise commensurable.

4. *Comments and conclusions*

To have a more complete theoretical framework on this topic, it is worth remarking that cyclic quadrilaterals in which every pair of consecutive sides of P_n are commensurable and every pair of consecutive interior angles of P_n are commensurable were also studied by Parnami-Agrawal-Rajwade (see [15, Proposition 2]).

We conclude with two examples that may help the reader to appreciate the ideas of this Article.

Example 4 There exist non-regular pentagons in which every pair of consecutive sides of P_n are commensurable and every pair of consecutive interior angles of P_n are commensurable.

Hint: to find a non-regular pentagon having all consecutive sides pairwise commensurable and all interior angles rational in degrees, start from Figure 8, and work on one side.

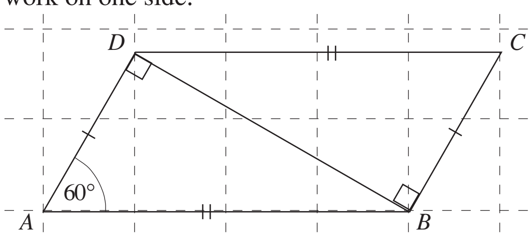

FIGURE 8: A parallelogram whose sides have rational length and whose interior angles are rational in degrees.

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Example 5: There exist cyclic quadrilaterals which verify the condition (iii) of Theorem 1 having non-commensurable pairs of consecutive sides.

Hint: consider two copies of a right triangle of the type 30-60-90 and glue them along their hypotenuse (see Figure 9).

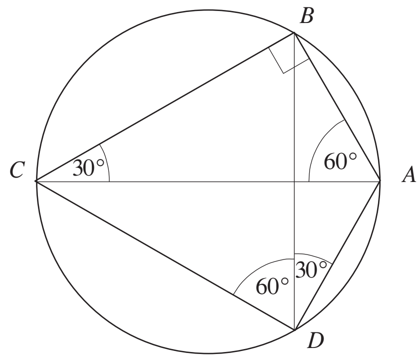

FIGURE 9: The kite shown in the figure satisfies the conditions (i) and (iii). Note that the consecutive sides *AB* and *BC* are incommensurable.

Finally an open question which is connected with Example 2.

Question: Are there cyclic n-gons ($n > 4$) which verify condition (iii) of Theorem 1 and which have non-commensurable pairs of consecutive sides?

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