ON TAKING MIXING-LENGTH THEORY SERIOUSLY

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There has been progress in convection theory in the past decade, mainly in the problem of mild convection. Yet, we are still not able to cope with vigorous convection such as we face in the envelopes of late-type stars. Most astrophysicists therefore use mixing-length theory and get on with calculating their models. As this situation may continue for a while, it may be a good thing to consider what mixing-length theory really is and to see whether it can be taken seriously as a physical model for stellar convection.

Different authors mean different things when they speak of mixing-length theory. Here, we interpret the theory in terms of the specific model in which a star is composed of a background fluid through which discrete, well-defined parcels of fluid move. These parcels may be thought of as quasiparticles whose number density is sufficiently high that they constitute a second fluid permeating the background fluid. The convective model is therefore a two-fluid model loosely resembling the composite of radiation and matter familiar in astrophysics, except that the quasiparticle fluid is more complicated than the photon gas.

In applying this model we must write down equations of motion for the quasiparticles. We have to specify the nature of the quasiparticles, and most people, with varying degrees of explicitness, treat them as idealizations of the buoyant thermals described by meteorologists. Fortunately, there is by now some guidance provided by laboratory data on the motion of isolated thermals in both laminar and turbulent fluids. Turner (1963, 1973) has described these experiments and has outlined the simple theory which has been evolved to describe them. In particular, he assumes that the thermals are small compared with any scale heights so that gradients across them, both inside and just outside, may be neglected. Only in their vertical motion do they sense the presence of the ambient temperature gradient.

Turner's description allows for turbulent exchange of heat, momentum, and mass between a quasiparticle and the ambient medium. With some slight modifications of his discussions we may derive the following set of equations governing the motion of quasiparticles. We display these just to give some idea of their form:

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$$\frac{dm}{dt} = \tilde{\rho}\left[\Sigma_1 |\underset{\sim}{u}-\underset{\sim}{U}| - \Sigma|\underset{\sim}{v}|\right] \quad , \tag{1}$$

$$m \frac{d\underset{\sim}{u}}{dt} = \tilde{g}(m-\tilde{m}) + \frac{d}{dt}(\tilde{m}\,\underset{\sim}{U})$$
$$+ \rho_0\, \Sigma_1 |\underset{\sim}{u}-\underset{\sim}{U}|(\underset{\sim}{u}-\underset{\sim}{U}) - \tilde{m}\mu(\underset{\sim}{u}-\underset{\sim}{U}) \quad , \tag{2}$$

$$\frac{dh}{dt} = \frac{w}{\rho}\frac{d\tilde{p}}{dz} - \left[\tilde{\rho}\, \Sigma_1 |\underset{\sim}{u}-\underset{\sim}{U}| + q\right](h-\tilde{h}) \quad , \tag{3}$$

$$\frac{d\underset{\sim}{x}}{dt} = \underset{\sim}{u} \quad . \tag{4}$$

Here m, $\underset{\sim}{x} = (x,y,z)$, $\underset{\sim}{u} = (u,v,w)$ and h are the mass, position, velocity and specific enthalpy of a thermal, $\tilde{\rho}$, $\tilde{p}$, $\tilde{h}$ and $\underset{\sim}{U}$ are the local means of density, pressure, enthalpy and velocity of the ambient medium at x. Σ_1 and Σ are cross sections for entrainment and erosion (both of the order of the geometrical cross section of the thermal), $\underset{\sim}{v}$ is the ambient turbulent velocity at $\underset{\sim}{x}$, $\tilde{m}$ is ambient mass displaced by thermal, $\underset{\sim}{\tilde{g}}$ is the acceleration of gravity corrected for the hydrodynamic mass of the thermal, q^{-1} is a thermal decay time allowing for radiative and turbulent exchanges, and μ^{-1} is a similar viscous decay time. Evidently these formulae must contain some fudge factors to be obtained by comparison with measurements, be they experimental, meteorological, or astrophysical.

In astrophysical treatments of convection many of the effects modelled in these equations have been included. Turbulent exchange of momentum between fluid elements and the ambient medium was included in the early theories (e.g. Prandtl 1932, Biermann 1932, Siedentopf 1933) and Öpik (1950) allowed for turbulent exchange of heat. Ulrich (1970a,b) has adopted the formulation of Morton, Taylor, and Turner (1956) in his studies. However, when astrophysicists use these equations of motion they generally replace them by algebraic equations; that is they essentially replace d/dt by w/ℓ where ℓ is a length to be specified. This gives rise to the usual local mixing-length treatment. Sometimes, some or all of these algebraic equations are averaged over height with some arbitrary weight function to produce a nonlocal extension of the theory (e.g. Ulrich 1976).

Such reductions of the dynamical equations for thermals have not been favoured in the meteorological literature. Certainly, they are not suitable for use by anyone interested in studying the interaction of stellar pulsations with convection. An alternative procedure, first attempted by Priestley (1953, 1954, 1959) for hydrostatic convective layers, is to solve the differential equations and use them together with some hypotheses about the distribution of initial conditions of quasiparticles to compute the heat flux. This has also been attempted for linear pulsation theory (Gough 1977). But in both instances one has to build in some information about the number density of each kind of quasiparticle at each height, generally by specifying creation rates. This becomes quite an undertaking for the nonlinear pulsation problem and even the formulation of the calculation has not been agreed upon. The manner of incorporating the dynamical equations into the convection theory thus poses a major difficulty in applying this kind of model. As we have hinted, it requires a prescription of the number of quasiparticles for each value of the parameters, and this distribution must be specified in a way that is compatible with the dynamics.

Formulated in this way, the model resembles kinetic theory and, in an attempt to capitalize on this, a transport equation was written down as if the quasiparticles satisfied Hamiltonian dynamics (Spiegel 1963). Deviations from this ideal behaviour were compensated for by introducing a source term in the transport equation. A modification was suggested by Castor (unpublished manuscript) who renounced the simple form provided by Hamiltonian dynamics and wrote a continuity equation for the one-particle distribution in the phase space of the quasiparticles. The phase space was enlarged over the usual six dimensional μ-space of position and velocity to include the temperature of a single quasiparticle as a phase parameter. In doing this one loses the volume-preserving feature of the phase fluid, which raises questions about the meaning of the approach, especially when one attempts coordinate transformations. Yet it seems to us a useful thing to write a continuity equation for the phase space density of quasiparticles and, for the present, ignore some of the niceties. We modify Castor's choice and use specific enthalpy (rather than temperature) of the quasiparticle as a variable and add an additional phase parameter, the quasiparticle mass. We have then an eight-dimensional phase space in which the density of representative points is $f(\underset{\sim}{x},\underset{\sim}{u},h,m;t)$. The continuity equation satisfied by f is:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x_i}(\dot{x}_i f) + \frac{\partial}{\partial u_i}(\dot{u}_i f) + \frac{\partial}{\partial h}(\dot{h} f) + \frac{\partial}{\partial m}(\dot{m} f) = \left(\frac{\partial f}{\partial t}\right)_{coll} , \qquad (5)$$

where dots denote differentiation with respect to time and a collision term has been introduced. The collision term is supposed to express the turbulent destruction and creation of quasiparticles; through this term we may represent our crude understanding of turbulence. It seems inadvisable to use a form like the Boltzmann collision integral since the interactions are probably not dominated by two-body collisions. Instead, it is perhaps best to include a loss term like $-f/\tau$ to represent the destruction of quasiparticles, where τ is a time required for the quasiparticle to travel its own diameter. This term then embodies a basic idea of mixing-length theory. But what about the creation term?

The generation of new quasiparticles is not really understood, and to quantify it, a specific model is needed. Often one imagines that quasiparticles grow from small fluctuations because of the instability mechanism. However, in a turbulent medium the fluctuations are not small. In the quasiparticle picture we think of the new quasiparticles as decay products of the old ones to represent the turbulent cascade process. Their development through instability is already included in the dynamical equations.

The problem, of course, is that we do not know much about the decay products following the destruction of the quasiparticles, and this is the first clear difficulty that must be faced in completing the theory. It is becoming increasingly clear in turbulence theory that the turbulent spectrum is strongly influenced by the number of decay products in the breakup of a quasiparticle, and possible models have been discussed which may provide guidance (cf. Frisch 1977). We shall not offer any preferences in the present discussion. Our aim instead is to bring out the points at which physical assumptions are needed to make the mixing-length model cogent.

Once a form for $(\partial f/\partial t)_{coll}$ is decided, the remaining difficulties are computational. This is not to belittle them; they are fierce and a moderately reasonable approximation scheme is not immediately apparent. The computational methods depend on the way one uses equation (5), and that has to be discussed next.

We believe that it would be sensible to try to construct moment equations from equation (5). For example, multiplication of equation (5) by m followed by integration over $d_5\Omega = d_3\underset{\sim}{u}\, dm\, dh$ gives

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot \underset{\sim}{F}_m = \int \left(\frac{\partial f}{\partial t}\right)_{coll} m d_5\Omega - \int \frac{\dot{m}}{m} f d_5\Omega \quad , \tag{6}$$

where

$$\rho_m = \int m f d_5\Omega \tag{7}$$

is the mass density of the gas of quasiparticles and

$$\underset{\sim}{F}_m = \int m \underset{\sim}{u} f d_5\Omega \tag{8}$$

is the mass flux of the gas. The last term on the right of equation (6) represents the mass exchanged with the background fluid by entrainment and erosion.

One may compute other moment equations, but we shall not do that here. We should however mention that the number of moments goes up faster than in ordinary kinetic theory or transfer theory. Quantities like fluxes of enthalpy and mechanical energy arise and there is the all-important turbulent stress tensor:

$$T_{ij} = \int m u_i u_j f d_5\Omega \quad . \tag{9}$$

Once a hierarchy of moment equations has been written down [a skeleton version has recently been studied by Stellingwerf (private communication)] the problem of closing it off must be faced. A possible approach, resembling the moment method, is to decide on an approximation for f and use that guess, for that is all it is at present, to get approximate expressions for the higher moments in terms of the lower moments. Once this is done, a last problem of principle remains. One must still decide how to describe that part of the fluid that does not move in quasiparticles. Should this be thought of as a zero fluctuation condensate of the quasiparticle gas? Or should one describe the background as an ordinary laminar fluid acted on by the stresses and so forth generated by the quasiparticles? The latter course seems decidedly preferable to us, especially for treating penetrative convection, where most of the matter may be in the background fluid. If that is accepted, the next course of action is to write the dynamical equations for the background fluid including the mass, momentum, and energy sources indicated by the moment equations of the quasiparticle gas. Then, in principle, one has a complete set of equations for the dynamics of a star with turbulent convection, but for the present without rotation or magnetic field.

Now we have to come to the key question: is this what has to be done or are we to be saved from it by a 'real theory' starting from the full fluid equations? We think that the immediate prospects for a sound fluid dynamical approach are not bright. And even the approximations to such an approach as are on the horizon promise to be far more demanding computationally than the scheme summarized here.

At present, untold computing hours are being lavished on stellar models using a mixing-length theory whose reliability is untested off the main sequence. It seems to us that if this situation is to continue it would be well to take the mixing-length theory seriously. In particular, one should be clear on the turbulence model one is using and not simply alter the standard formulae according to whim, as is often done in the literature. We are not saying that alternative general structures to that given here may not be preferable. Nor are the procedures we outline meant to be rigid. The present version of a mixing-length procedure is a synthesis of ingredients existing in the literature and we have done no more than put it together to show that a cogent discussion of mixing-length theory is possible. We have especially tried to show where the physics is missing and to indicate a framework for including it. The resulting equations are in principle capable of dealing with many of the problems of current interest, such as the nonlinear interaction of pulsation and convection. Those coping with such questions are all too familiar with many of the problems we have raised. But they, as we ourselves, have sometimes dealt with these problems piecemeal and have not tried to put them into context by working with a concrete general model. We are claiming here that the specification of such a model is possible and desirable and that if one can be constructed, stellar convection theory may begin to seem more rational.

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