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ARTICLE

The integral Chow ring of $\mathcal{M}_{1,n}$ for $n = 3, \ldots, 10$

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Abstract
We compute the integral Chow ring of the moduli stack of smooth elliptic curves with n marked points for $3 \le n \le 10$.

Contents

1. Introduction

1.1. Contents

The main result of this paper is the following:

Theorem 1.1. *Let λ_1 be the first Chern class of the Hodge bundle. Then over a field of characteristic not equal to 2 or 3,*

(a) $\mathrm{CH}(\mathcal{M}_{1,3}) = \mathbb{Z}[\lambda_1]/(12\lambda_1, 6\lambda_1^2)$
(b) $\mathrm{CH}(\mathcal{M}_{1,4}) = \mathbb{Z}[\lambda_1]/(12\lambda_1, 2\lambda_1^2)$
(c) $\mathrm{CH}(\mathcal{M}_{1,n}) = \mathbb{Z}[\lambda_1]/(12\lambda_1, \lambda_1^2)$*, for* $n = 5, \ldots, 10$.

We open by reviewing some essential background: the Weierstrass form for elliptic curves, the Chow rings of $\mathcal{M}_{1,1}$ and $\mathcal{M}_{1,2}$, and higher Chow groups with ℓ-adic coefficients. We then compute the integral Chow rings of $\mathcal{M}_{1,n}$ for $3 \leq n \leq 10$ over a (not-necessarily algebraically closed) field k with char $k \neq 2, 3$ by using higher Chow groups with ℓ-adic coefficients in the base case $n = 3$, and then leveraging this information for larger n. This extends Belorousski's computation of the *rational* Chow ring of these stacks [2]. Along the way, we also prove the rationality of $\mathcal{M}_{1,n}$ for $3 \leq n \leq 10$, which was previously only known in the case $\Bbbk = \bar{\Bbbk}$, char $\Bbbk = 0$, and analyze the notion of the *integral tautological ring* of $\mathcal{M}_{1,n}$.

1.2. History

In [16], Mumford introduced the study of the intersection theory of the coarse moduli space of genus g curves, $\overline{M}_g$. This space is singular, and its Chow ring cannot be defined with integer coefficients, but the singularities are mild enough that it can be defined with rational coefficients (the *rational Chow ring*). Extending this notion, the rational Chow rings of the moduli stacks of genus g stable (resp. smooth) n-pointed curves, denoted $\overline{\mathcal{M}}_{g,n}$ (resp. $\mathcal{M}_{g,n}$), have been computed for many (g, n) [2, 5, 9, 10, 13, 16, 17].

However, using rational coefficients eliminates all torsion, and so ignores a rich part of the structure of the space. Enabled by the extension of the definition of integral Chow rings to quotient stacks by Totaro [20] and Edidin-Graham [8], Vistoli and Edidin-Graham computed the integral Chow rings of $\mathcal{M}_2$ [21], $\mathcal{M}_{1,1}$ and $\overline{\mathcal{M}}_{1,1}$ [8]. Then progress froze until the recent development of new techniques for computing with integral coefficients, such as the patching lemma of [7] and the *higher Chow groups with ℓ-adic coefficients* of [15]. See the below table for a list of currently known values.

Table 1. *All currently known integral Chow rings of $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$*

genus	moduli	reference
$g = 0$	$\mathcal{M}_{0,n}$, $n \geq 3$	classical
	$\overline{\mathcal{M}}_{0,n}$, $n \geq 3$	[14]
$g = 1$	$\mathcal{M}_{1,1}$	[8]
	$\overline{\mathcal{M}}_{1,1}$	[8]
	$\mathcal{M}_{1,2}$	[12]
	$\overline{\mathcal{M}}_{1,2}$	[6, 12]
	$\overline{\mathcal{M}}_{1,3}$	[1, 3]
	$\overline{\mathcal{M}}_{1,4}$	[1]
	$\mathcal{M}_{1,n}$, $3 \leq n \leq 10$	[–]
$g = 2$	$\mathcal{M}_2$	[21]
	$\overline{\mathcal{M}}_2$	[7, 15]
	$\mathcal{M}_{2,1}$	[18]
	$\overline{\mathcal{M}}_{2,1}$	[6]

1.3. The patching problem

One powerful tool for computing Chow rings is the *excision exact sequence*. Given a closed substack $p : Z \to X$ with complement U, there is an exact sequence

$$\mathrm{CH}(Z) \xrightarrow{p_*} \mathrm{CH}(X) \to \mathrm{CH}(U) \to 0.$$

This sequence allows one to compute the Chow ring of an open locus when the Chow ring of its complement and of the whole space are known. However, we frequently find ourselves in the opposite situation: when dealing with complicated objects stratified by simpler ones, we may be able to compute the Chow rings of Z and its complement U, and need to patch these together to get the Chow ring of X.

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This may be referred to as the *patching problem*, and solving it is the crux of many Chow computations. The above-mentioned new techniques, the patching lemma and higher Chow groups with ℓ-adic coefficients, give methods for solving the patching problem and have fueled the recent explosion in progress in computing integral Chow rings.

1.4. Conventions

For the remainder of this paper, all schemes and stacks are over a fixed field $\Bbbk$ of characteristic not equal to 2 or 3.

2. The $\mathcal{M}_{1,1}$ and $\mathcal{M}_{1,2}$ cases

Our analysis of $\mathcal{M}_{1,n}$ for higher n depends in multiple places on $\mathcal{M}_{1,1}$ and $\mathcal{M}_{1,2}$, so we first review their structure, which is essentially a corollary of the Weierstrass form for elliptic curves. The Chow ring of $\mathcal{M}_{1,1}$ was first computed in [8] and $\mathcal{M}_{1,2}$ in [12].

2.1. The Weierstrass form

We open with the classically known Weierstrass form for elliptic curves.

Theorem 2.1 (Weierstrass). *Any one-pointed smooth elliptic curve over a field $\Bbbk$ of characteristic not equal to 2 or 3 can be written in the form $y^2z = x^3 + axz^2 + bz^3$, where the marked point is the point at infinity* $[0 : 1 : 0]$. *Moreover, if we denote such a curve by $C_{(a,b)}$, then*

$$C_{(a,b)} \cong C_{(a',b')} \quad \text{if and only if} \quad (a', b') = (t^{-4}a, t^{-6}b).$$

The isomorphism between these curves is given by

$$[x : y : z] \mapsto [t^{-2}x : t^{-3}y : z].$$

An elliptic curve is smooth if and only if $D = 4a^3 + 27b^2 \neq 0$, nodal if and only if $D = 0$ and $(a, b) \neq (0, 0)$, and cuspidal if and only if $(a, b) = (0, 0)$. Lastly, we have

$$H^0(\omega_C) = \left\langle \frac{dx}{y} \right\rangle.$$

Rephrasing this gives the following corollaries:

Corollary 2.2. *The Weierstrass form gives an isomorphism*

$$\mathcal{M}_{1,1} \cong \left[\frac{\mathbb{A}^2 \setminus V(D)}{\mathbb{G}_m}\right],$$

where the $\mathbb{G}_m$ action has weight $(-4, -6)$ and $D = 4a^3 + 27b^2$.

Corollary 2.3. *We have that $\mathcal{M}_{1,2}$ is isomorphic to an open substack of a vector bundle over $B\mathbb{G}_m$.*

Proof. From the Weierstrass form, a two-pointed smooth elliptic curve is determined, up to scaling, by a choice of (a, b) and (x, y) such that

$$y^2 = x^3 + ax + b \quad \text{and} \quad D \neq 0.$$

We can solve for b to see that a, x, y vary freely, provided that $D \neq 0$, where

$$D = 4a^3 + 27b^2 = 4a^3 + 27(y^2 - (x^3 + ax))^2.$$

Since $\mathbb{G}_m$ acts with weights $-4, -2, -3$ on a, x, y, we conclude that $\mathcal{M}_{1,2}$ is open in $\left[\frac{\mathbb{A}^3_{a,x,y}}{\mathbb{G}_m}\right]$, where $\mathbb{G}_m$ acts with the above weights. ◻

Corollary 2.4. *The rings* $\mathrm{CH}(\mathcal{M}_{1,1})$ *and* $\mathrm{CH}(\mathcal{M}_{1,2})$ *are both quotients of* $\mathbb{Z}[x]/(12x)$.

Proof. This follows from Corollaries 2.2 and 2.3, along with the fact that D has weight 12 under the $\mathbb{G}_m$ action. ◻

Corollary 2.5. *The generator of the Chow ring of* $\mathcal{M}_{1,1}$ *and* $\mathcal{M}_{1,2}$ *is* λ_1*, the first Chern class of the Hodge bundle.*

Proof. Since $\mathbb{G}_m$ acts with weights -2 and -3 on x and y, respectively, we see that $\frac{dx}{y}$ has weight 1 under the $\mathbb{G}_m$ action. Hence, under the pullback map $\mathrm{CH}(B\mathbb{G}_m) \to \mathrm{CH}(\widetilde{\mathcal{M}}_{1,1})$, we have $x \mapsto \lambda_1$. Since by the previous Corollary $\mathrm{CH}(\mathcal{M}_{1,1})$ and $\mathrm{CH}(\mathcal{M}_{1,2})$ are generated by the pullback of x, we see that they are generated by λ_1. ◻

Corollary 2.6. *The pullback of* $x \in \mathrm{CH}(B\mathbb{G}_m)$ *to the moduli stacks of pointed elliptic curves used in this paper (*$\widetilde{\mathcal{M}}^r_{1,n}$*,* $\mathcal{X}$*,* $\mathcal{U}_3$*,* $\mathcal{U}'_2$*,* U_n*,* U'_n*,* V_n *and* V'_n*, most of which are defined later) is* λ_1.

Proof. Let $\mathcal{Z}$ be any of the above stacks. Then $\mathcal{Z}$ admits a morphism to $\widetilde{\mathcal{M}}^2_{1,1}$ given by forgetting all but the first marked point, which yields the following diagram:

$$\mathcal{Z} \longrightarrow \widetilde{\mathcal{M}}^2_{1,1} \longrightarrow B\mathbb{G}_m.$$

As noted in the proof of Corollary 2.5, $x \in \mathrm{CH}(B\mathbb{G}_m)$ pulls back to $\lambda_1 \in \mathrm{CH}(\widetilde{\mathcal{M}}^2_{1,1})$. Since the Hodge bundle pulls back to the Hodge bundle, we see that the pullback of x to $\mathcal{Z}$ is λ_1. ◻

Theorem 2.7. *Let* λ_1 *be the first Chern class of the Hodge bundle. Over a field of characteristic not equal to 2 or 3,*

(a) $\mathrm{CH}(\mathcal{M}_{1,1}) \cong \mathbb{Z}[\lambda_1]/(12\lambda_1)$
(b) $\mathrm{CH}(\mathcal{M}_{1,2}) \cong \mathbb{Z}[\lambda_1]/(12\lambda_1)$.

Proof. From Corollaries 2.4 and 2.5, we know that $\mathrm{CH}(\mathcal{M}_{1,1})$ and $\mathrm{CH}(\mathcal{M}_{1,2})$ are both quotients of $\mathbb{Z}[\lambda_1]/(12\lambda_1)$. Now consider any two-pointed elliptic curve with μ_3 automorphisms, such as $(C_{(0,1)}, \infty, [0:1:1])$, and with μ_4 automorphisms, such as $(C_{(1,0)}, \infty, [0:0:1])$. These induce residual gerbes

$$\begin{tikzcd} B\mu_n \arrow[rr] \arrow[dr] & & \mathcal{M}_{1,2} \arrow[dl] \\ & B\mathbb{G}_m & \end{tikzcd}$$

for $n = 3, 4$. The previous diagram induces the following diagram

$$\begin{tikzcd} \mathrm{CH}(B\mu_n) & & \mathrm{CH}(\mathcal{M}_{1,2}) \arrow[ll] \\ & \mathrm{CH}(B\mathbb{G}_m) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

at the level of Chow rings.

Since $\mathrm{CH}(B\mathbb{G}_m) \to \mathrm{CH}(B\mu_n)$ is surjective, we see that $\mathrm{CH}(\mathcal{M}_{1,2})$ surjects onto $\mathbb{Z}[x]/(nx)$ for $n = 3, 4$. Therefore, $\mathrm{CH}(\mathcal{M}_{1,2}) = \mathbb{Z}[\lambda_1]/(12\lambda_1)$. Considering these curves as one-pointed elliptic curves shows that $\mathrm{CH}(\mathcal{M}_{1,1}) \cong \mathbb{Z}[\lambda_1]/(12\lambda_1)$ as well. ◻

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3. Aside 1: Higher Chow groups with ℓ-adic coefficients

In [4], Bloch introduced higher Chow groups, which complete the excision exact sequence into a long exact sequence. They are defined as the homology of a certain complex named $z^*(X, \bullet)$ and are, unfortunately, usually rather difficult to compute. In [15], Larson used higher Chow groups with ℓ-adic coefficients to remedy this. Without getting into too much detail, we list here some important properties that higher Chow groups with ℓ-adic coefficients possess, along with some important computations.

Definition 3.1. Define the n^{th} higher Chow group with ℓ-adic coefficients to be

$$\mathrm{CH}(X, n; \mathbb{Z}_\ell) = H_n\left(\lim z^*(X_{\bar{\Bbbk}}, \bullet) \otimes^L \mathbb{Z}/\ell^m\mathbb{Z}\right).$$

In the case where each $\mathrm{CH}(X, n; \mathbb{Z}/\ell^m\mathbb{Z}) := H_n(z^*(X_{\bar{\Bbbk}}, \bullet) \otimes \mathbb{Z}/\ell^m\mathbb{Z})$ is finitely generated, we have

$$\mathrm{CH}(X, n; \mathbb{Z}_\ell) = \lim \mathrm{CH}(X, n; \mathbb{Z}/\ell^m\mathbb{Z}).$$

Proposition 3.2. *If $Z \to X$ is closed with complement U and*

- $\mathrm{CH}(Z)$ *and* $\mathrm{CH}(U)$ *are finitely generated,*
- $\mathrm{CH}(Z) \to \mathrm{CH}(Z_{\bar{\Bbbk}})$ *is injective,*
- *there exists at least one ℓ for which* $\mathrm{CH}(U, 1; \mathbb{Z}_\ell) = 0$,
- *and* $\mathrm{CH}(U, 1; \mathbb{Z}_l) = 0$ *whenever* $\mathrm{CH}(Z)$ *has ℓ-torsion,*

then the excision sequence is exact on the left.

Proof. Notice that, at first glance, ℓ-adic higher Chow groups tell us about the injectivity of the excision sequence with all spaces base-changed to $\bar{\Bbbk}$. However, we can infer the injectivity of $\mathrm{CH}(Z) \to \mathrm{CH}(X)$ via the following diagram:

$$\begin{tikzcd} \mathrm{CH}(Z) \arrow[r] \arrow[d, hook] & \mathrm{CH}(X) \arrow[d] \\ \mathrm{CH}(Z_{\bar{\Bbbk}}) \arrow[r] & \mathrm{CH}(X_{\bar{\Bbbk}}). \end{tikzcd}$$

Let $\alpha \in \mathrm{CH}(Z)$, and, abusing notation, refer to its image in $\mathrm{CH}(Z_{\bar{\Bbbk}})$ as α as well. Pick an ℓ such that α is ℓ-torsion (if α is not torsion, then pick any ℓ where U's first ℓ-adic higher Chow group vanishes). This gives

$$\mathrm{CH}(Z) \otimes \mathbb{Z}_\ell \hookrightarrow \mathrm{CH}(Z_{\bar{\Bbbk}}) \otimes \mathbb{Z}_\ell \hookrightarrow \mathrm{CH}(X_{\bar{\Bbbk}}) \otimes \mathbb{Z}_\ell,$$

and so the image of α under $\mathrm{CH}(Z) \to \mathrm{CH}(X)$ cannot vanish. □

Proposition 3.3. *Suppose ℓ is coprime to* $\operatorname{char} \Bbbk$ *(later we will always have $\ell = 2$ or 3). Then*

(a) $\mathrm{CH}(\operatorname{Spec} k, 1; \mathbb{Z}_l) = 0.$
(b) $\mathrm{CH}(\mathbb{A}^n, 1; \mathbb{Z}_l) = 0.$
(c) $\mathrm{CH}(\mathbb{P}^n, 1; \mathbb{Z}_l) = 0.$
(d) $\mathrm{CH}(B\mathbb{G}_m, 1; \mathbb{Z}_l) = 0.$
(e) $\mathrm{CH}(B\mu_n, 1; \mathbb{Z}_l) = 0.$

Proof. As noted in [15], (a) is a consequence of motivic cohomology. Then (b) and (c) follow from the vector and projective bundle formulas, and (d) follows from computing equivariantly. The last follows from the excision sequence

$$0 \to \mathrm{CH}(B\mathbb{G}_m) \to \mathrm{CH}([\mathbb{A}^1/\mathbb{G}_m]) \to \mathrm{CH}(B\mu_n) \to 0.$$ □

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4. Aside 2: A_r-singularities

Definition 4.1. A (proper and reduced) n-pointed connected curve C over an algebraically closed field K is said to be *A_r-stable* if

1. C has at worst A_r singularities; that is, each closed point $p \in C$ has

$$\widehat{\mathcal{O}}_{C,p} \cong \frac{K[[x,y]]}{(y^2 - x^{h+1})}$$

for $0 \leq h \leq r$,
2. the p_i are distinct and lie in the smooth locus of C, and
3. $\omega_C(p_1 + \cdots + p_n)$ is ample.

Definition 4.2. A morphism $\mathcal{C} \to S$ with n sections $p_i : S \to \mathcal{C}$ is a *family of n-pointed A_r-stable genus g curves* if $\mathcal{C} \to S$ is proper, flat and finitely presented, and each geometric fiber is an n-pointed A_r-stable genus g curve.

Definition 4.3. Denote by $\widetilde{\mathcal{M}}^r_{g,n}$ the stack whose objects over a scheme S are families of n-pointed A_r-stable genus g curves and whose morphisms are defined in the natural way.

For more about A_r-stable curves, see [19].

Definition 4.4. Denote by $\widetilde{\mathcal{M}}^{r,\mathrm{irr}}_{g,n}$ the open substack of $\widetilde{\mathcal{M}}^r_{g,n}$ consisting of irreducible curves.

In the next section, while computing the Chow ring of $\mathcal{M}_{1,3}$, we will work with an enlargement of $\widetilde{\mathcal{M}}^{2,\mathrm{irr}}_{1,3}$. More specifically, we will allow the second and third marked points to overlap with the node/cusp of a nodal/cuspidal rational curve, but we will not allow both the second and third marked points to overlap with the node/cusp (as we still insist on the marked points being distinct).

Definition 4.5. Let $\mathcal{X}$ be the stack whose objects over a scheme S are proper, flat and finitely presented morphisms $\mathcal{C} \to S$ with three sections $p_i : S \to \mathcal{C}$ where the geometric fibers over each $s \to S$ satisfy:

- $(\mathcal{C}_{\bar{s}}, p_1)$ is an irreducible A_2-stable elliptic curve and each p_i is distinct.

We then see $\mathcal{M}_{1,3}$ inside of $\mathcal{X}$ as the complement of the locus of singular curves. Before we can move on and perform computations with $\mathcal{X}$ using the equivariant intersection theory of [8], we must see that it is a smooth quotient stack.

Proposition 4.6. *The stack $\mathcal{X}$ is smooth.*

Proof. Observe that $\mathcal{X}$ is a union of two opens

$$\mathcal{X} = \widetilde{\mathcal{M}}^{2,\mathrm{irr}}_{1,3} \cup \mathcal{U}_3,$$

with $\mathcal{U}_3$ as defined in Lemma 5.8. Since both of these are smooth ($\widetilde{\mathcal{M}}^{2,\mathrm{irr}}_{1,1}$ by [19] and $\mathcal{U}_3$ by the quotient description of Lemma 5.8), we see that $\mathcal{X}$ is smooth as well. □

Proposition 4.7. *The stack $\mathcal{X}$ is a quotient stack.*

We will first include a quick lemma.

Lemma 4.8. *Suppose $\mathcal{W} \cong [W/G]$ is a quotient stack in the sense of [8], and consider the following diagram of algebraic stacks:*

$$\begin{tikzcd} \mathcal{W} \times_{\mathcal{Z}} \mathcal{Y} \arrow[r] \arrow[d] & \mathcal{W} \arrow[d] \\ \mathcal{Y} \arrow[r] & \mathcal{Z}. \end{tikzcd}$$

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If the morphism $\mathcal{Y} \to \mathcal{Z}$ is representable by algebraic spaces, then the fiber product $\mathcal{W} \times_{\mathcal{Z}} \mathcal{Y}$ is also a quotient stack.

Proof of Lemma 4.8. Consider the following diagram:

$$\begin{tikzcd} V \arrow[r] \arrow[d] & W \arrow[d] \\ \mathcal{W} \times_{\mathcal{Z}} \mathcal{Y} \arrow[r] \arrow[d] & \mathcal{W} \arrow[d] \\ \mathcal{Y} \arrow[r] & \mathcal{Z}. \end{tikzcd}$$

Since $W \to \mathcal{W}$ is a principal G-bundle, so is $V \to \mathcal{W} \times_{\mathcal{Z}} \mathcal{Y}$, and since $\mathcal{Y} \to \mathcal{Z}$ is representable by algebraic spaces, so is $V \to W$. Therefore, since W is an algebraic space, V must also be an algebraic space. Hence, $\mathcal{W} \times_{\mathcal{Z}} \mathcal{Y} \cong [V/G]$ is a quotient stack. □

Proof of Proposition 4.7. Observe that $\mathcal{X}$ is naturally open inside of

$$\widetilde{\mathcal{C}}^2_{1,1} \times_{\widetilde{\mathcal{M}}^2_{1,1}} \widetilde{\mathcal{C}}^2_{1,1}.$$

Since $\widetilde{\mathcal{C}}^2_{1,1}$ is a quotient stack and $\widetilde{\mathcal{C}}^2_{1,1} \to \widetilde{\mathcal{M}}^2_{1,1}$ is representable by algebraic spaces (by schemes, in fact), the above fiber product is a quotient stack by Lemma 4.8. □

5. The $\mathcal{M}_{1,n}$ case for $n = 3, \dots, 10$

We will now compute the integral Chow rings of $\mathcal{M}_{1,n}$ for $n = 3, \dots, 10$. The overall structure of the computation is to stratify $\mathcal{M}_{1,n}$ into an open whose complement stratifies into closed substacks which are isomorphic to opens inside of $\mathcal{M}_{1,n-1}$.

5.1. *The integral Chow ring of $\mathcal{M}_{1,3}$*

Definition 5.1. For an elliptic curve (E, ∞), we denote by $\iota : E \to E$ the unique hyperelliptic involution that fixes ∞. Note that the involution extends uniquely to families of elliptic curves.

We first stratify $\mathcal{M}_{1,n}$ into the open locus where $p_2 \neq \iota(p_3)$ and the divisor where $p_2 = \iota(p_3)$.

Definition 5.2. For $n \geq 2$,

(a) Let $U_n \subseteq \mathcal{M}_{1,n}$ be the locus where $p_2 \neq \iota(p_3)$.
(b) Let $U'_n \subseteq \mathcal{M}_{1,n}$ be the locus where $p_2 \neq \iota(p_i)$ for any i.

Note 5.3. Note that the condition for U_2 is ill-defined. We use the convention that this is an empty condition, so that $U_2 = \mathcal{M}_{1,2}$.

Observation 5.4. If π is, as usual, the map $\pi : \mathcal{M}_{1,n} \to \mathcal{M}_{1,n-1}$ forgetting the last marked point, we always have $\pi(U_{n+1}) \subseteq U_n$ and $\pi(U'_{n+1}) \subseteq U'_n$, and for $n \geq 4$, we have $\pi^{-1}(U_{n-1}) = U_n$.

Therefore, we have induced pullback maps on Chow rings given by π^*. Since $\pi^*(\lambda_1) = \lambda_1$, we see that π pulls relations back to relations: if $a\lambda_1 = 0$ in $\mathrm{CH}(U_m)$ or $\mathrm{CH}(U'_m)$ for some m, then $a\lambda_1 = 0$ on that same locus for all $n \geq m$.

Definition 5.5. For $n \geq 3$, define the morphism of stacks $\sigma_{n-1} : U'_{n-1} \to \mathcal{M}_{1,n}$ by

$$(C, p_i) \mapsto (C, p_1, p_2, \iota(p_2), p_3, \dots, p_{n-1}).$$

Notice that while we defined this morphism on points, it extends to families by extending the involution.

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This map sheds light on why the loci in Definition 5.2 were defined: the defining conditions for U'_n are precisely the conditions needed to insure that this map exists.

Proposition 5.6. *For $n \geq 3$, the map $\sigma_{n-1} : U'_{n-1} \to \mathcal{M}_{1,n}$ is a closed immersion.*

Proof. Let $\pi_3 : \mathcal{M}_{1,n} \to \mathcal{M}_{1,n-1}$ be the morphism which forgets the third marked point, and consider the following Cartesian diagram:

$$\begin{tikzcd} \pi_3^{-1}(U'_{n-1}) \arrow[d, "\pi_3"] \arrow[r, phantom, "\subseteq"] & \mathcal{M}_{1,n} \arrow[d, "\pi_3"] \\ U'_{n-1} \arrow[u, bend left, "\sigma_{n-1}"] \arrow[r, phantom, "\subseteq"] & \mathcal{M}_{1,n-1}, \end{tikzcd}$$

where $\pi_3 \circ \sigma_{n-1} = \mathrm{id}$. Therefore, $\sigma_{n-1} : U'_{n-1} \to \mathcal{M}_{1,n}$ factors as a closed immersion followed by an open immersion into $\mathcal{M}_{1,n}$. Since its image in $\mathcal{M}_{1,n}$ is the closed locus of curves with $p_3 = \iota(p_2)$, we see that $\sigma_{n-1} : U'_{n-1} \to \mathcal{M}_{1,n}$ is a closed immersion. □

Corollary 5.7. *For $n \geq 3$, the stack $\mathcal{M}_{1,n}$ stratifies into the disjoint union*

$$\mathcal{M}_{1,n} = U_n \sqcup \operatorname{im} \sigma_{n-1} \cong U_n \sqcup U'_{n-1}.$$

Lemma 5.8. *The stack U_3 is isomorphic to an open substack of a vector bundle $\mathcal{U}_3$ over $B\mu_2$.*

Proof. A smooth three-pointed elliptic curve is determined, up to scaling, by a choice of (a, b), $p_2 = (x_2, y_2)$, and $p_3 = (x_3, y_3)$ such that

$$y_i^2 = x_i^3 + ax_i + b \quad \text{and} \quad D \neq 0.$$

Solving for b and then a gives

$$a = \frac{(y_3^2 - x_3^3) - (y_2^2 - x_2^3)}{x_3 - x_2}.$$

Therefore, we see that x_2, x_3, y_2, y_3 may vary freely, provided $x_2 \neq x_3$ and $D \neq 0$. But the condition that $x_2 \neq x_3$ is precisely the condition that p_2 and p_3 do not overlap and are not exchanged by the hyperelliptic involution (the defining condition for U_3), and so U_3 is open inside of

$$\mathcal{U}_3 := \left[\frac{\mathbb{A}^2_{y_i} \times (\mathbb{A}^2_{x_i} \setminus \Delta)}{\mathbb{G}_m}\right],$$

where Δ is the diagonal and $\mathbb{G}_m$ acts with weight -2 on x_i and -3 on y_i. This is a vector bundle over

$$\left[\frac{\mathbb{A}^2_{x_i} \setminus \Delta}{\mathbb{G}_m}\right]$$

which is a vector bundle over

$$\left[\frac{\mathbb{A}^1 \setminus 0}{\mathbb{G}_m}\right] \cong B\mu_2$$

since $\mathbb{G}_m$ acts with weight -2 on x_i. □

Lemma 5.9. *The stack $\operatorname{im} \sigma_2$ is isomorphic to an open substack of a vector bundle $\mathcal{U}'_2$ over $B\mu_3$.*

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Proof. Since $\operatorname{im}\sigma_2$ is isomorphic to the locus U_2' in $\mathcal{M}_{1,2}$, we just need to analyze two-pointed elliptic curves where $\iota(p_2) \neq p_2$. Recall from Corollary 2.3 that $\mathcal{M}_{1,2}$ is open inside of a vector bundle over $B\mathbb{G}_m$. More specifically, $\mathcal{M}_{1,2}$ is open inside of $[\mathbb{A}^3/\mathbb{G}_m]$ with coordinates a, x, y. The condition that $\iota(p_2) \neq p_2$ is equivalent to the condition $y \neq 0$, since $\iota(p_2) = \iota([x : y : z]) = [x : -y : z]$. Therefore, U_2' is open inside of

$$\mathcal{U}_2' := \left[\frac{\mathbb{A}^3 \setminus \{y = 0\}}{\mathbb{G}_m}\right] \cong \left[\frac{\mathbb{A}^2_{a,x} \times (\mathbb{A}^1_y \setminus 0)}{\mathbb{G}_m}\right]$$

which is a vector bundle over

$$\left[\frac{\mathbb{A}^1_y \setminus 0}{\mathbb{G}_m}\right] \cong B\mu_3,$$

since $\mathbb{G}_m$ acts with weight -3 on y. □

Now we compute the integral Chow ring of $\mathcal{M}_{1,3}$ by first observing that the vector bundles $\mathcal{U}_3$ and $\mathcal{U}_2'$ of the previous section naturally live inside of $\mathcal{X}$, the enlargement of $\mathcal{M}_{1,3}$ from Definition 4.5. In fact, we have $\mathcal{X} = \mathcal{U}_3 \sqcup \mathcal{U}_2'$, since $\mathcal{U}_3$ contains curves where the second and third marked points are not exchanged by the involution, while $\mathcal{U}_2'$ contains curves where the second and third marked points *are* exchanged by the involution. We patch these vector bundles together inside of $\mathcal{X}$ using higher Chow groups with ℓ-adic coefficients, and from there deduce $\mathrm{CH}(\mathcal{M}_{1,3})$.

Lemma 5.10. *With $\mathcal{X}$ defined as in Definition 4.5, we have*

$$\mathrm{CH}(\mathcal{X}) = \frac{\mathbb{Z}[x]}{(6x^2)} \quad \textit{and} \quad \mathrm{CH}(\mathcal{X}, 1; \mathbb{Z}_\ell) = 0$$

for ℓ coprime to $\operatorname{char} \Bbbk$.

To show this, we will use the following Theorem.

Theorem 5.11 [11]. *The Picard group of $\mathcal{M}_{1,n}$ is isomorphic to $\mathbb{Z}/12$ for all n, generated by the Hodge bundle.*

Proof of Lemma 5.10. Recall that because $\mathcal{U}_3$ and $\mathcal{U}_2'$ are both quotients by $\mathbb{G}_m$ and vector bundles over $B\mu_2$ and $B\mu_3$, respectively, that their first higher Chow groups with ℓ-adic coefficients vanish for ℓ co-prime to $\operatorname{char} \Bbbk$ (by Proposition 3.3) and that their Chow rings are

$$\mathrm{CH}(\mathcal{U}_3) = \frac{\mathbb{Z}[x]}{(2x)} \quad \text{and} \quad \mathrm{CH}(\mathcal{U}_2') = \frac{\mathbb{Z}[x]}{(3x)},$$

where in both rings x, denotes the pullback of the generator $x \in \mathrm{CH}(B\mathbb{G}_m) = \mathbb{Z}[x]$.

Consider the following diagram

$$\begin{tikzcd}
\mathcal{U}_2' \arrow[r, "\sigma_2"] \arrow[dr, "\pi_2"'] & \mathcal{X} \arrow[d, "\pi"] & \mathcal{U}_3 \arrow[l, "j"'] \arrow[dl, "\pi_3"] \\
& B\mathbb{G}_m, &
\end{tikzcd}$$

where $\pi : \mathcal{X} \to B\mathbb{G}_m$ is defined by the Hodge bundle. Denote the pullback of $x \in \mathrm{CH}(B\mathbb{G}_m)$ to $\mathrm{CH}(\mathcal{X})$ by x as well, so that the pullback of x along any map is again x. We make the important note here that since each morphism to $B\mathbb{G}_m$ is determined by the Hodge bundle, the generator x is really λ_1 (see Corollary 2.6).

Since $\mathrm{CH}(\mathcal{U}_3, 1; \mathbb{Z}_\ell)$ vanishes for ℓ co-prime to char $\Bbbk$, the excision sequence for $\mathcal{U}_3$ and $\mathcal{U}_2'$ gives

$$0 \to \mathrm{CH}(\mathcal{U}_2') \xrightarrow{\sigma_{2*}} \mathrm{CH}(\mathcal{X}) \to \mathrm{CH}(\mathcal{U}_3) \to 0$$

$$0 \to \mathbb{Z}[x]/(3x) \xrightarrow{\sigma_{2*}} \mathrm{CH}(\mathcal{X}) \to \mathbb{Z}[x]/(2x) \to 0.$$

Moreover, since $\mathrm{CH}(\mathcal{U}_2', 1; \mathbb{Z}_\ell) = 0$ for ℓ coprime to char $\Bbbk$, we see that $\mathrm{CH}(\mathcal{X}, 1; \mathbb{Z}_\ell) = 0$.

In all degrees $k \geq 2$, the above sequence looks like

$$0 \to \mathbb{Z}/3 \to \mathrm{CH}^k(\mathcal{X}) \to \mathbb{Z}/2 \to 0,$$

and so $\mathrm{CH}^k(\mathcal{X}) \cong \mathbb{Z}/6$ for $k \geq 2$. Note that x^k has order 6 in $\mathrm{CH}^k(\mathcal{X})$ since it pulls back to x^k in each of $\mathrm{CH}(\mathcal{U}_2') \cong \mathbb{Z}/3$ and $\mathrm{CH}(\mathcal{U}_3) \cong \mathbb{Z}/2$, and so we may take x^k as the generator of $\mathrm{CH}^k(\mathcal{X})$ for $k \geq 2$.

In degree one, the sequence looks like

$$0 \to \mathbb{Z} \xrightarrow{\sigma_{2*}} \mathrm{CH}^1(\mathcal{X}) \xrightarrow{j^*} \mathbb{Z}/2 \to 0.$$

We now have either $\mathrm{CH}^1(\mathcal{X}) \cong \mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}/2$, and we seek to show that $\mathrm{CH}^1(\mathcal{X}) \cong \mathbb{Z}$.

We know that $\mathcal{M}_{1,3} \subseteq \mathcal{X}$ is the complement of the locus of singular curves. From the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[dr] & \widetilde{\mathcal{M}}^2_{1,1} \arrow[d] \\ & B\mathbb{G}_m, \end{tikzcd}$$

the fundamental class of this locus in $\mathcal{X}$ is the pullback of the fundamental class of this locus in $\widetilde{\mathcal{M}}^2_{1,1}$ – that is, $12x$. Therefore,

$$\mathbb{Z}/12 \cong \langle \lambda_1 \rangle = \mathrm{CH}^1(\mathcal{M}_{1,3}) \cong \mathrm{CH}^1(\mathcal{X})/\langle 12x \rangle,$$

and so $\mathrm{CH}^1(\mathcal{X}) = \langle x \rangle \cong \mathbb{Z}$. We conclude that, as groups,

$$\mathrm{CH}(\mathcal{X}) \cong \frac{\mathbb{Z}[x]}{(6x^2)}.$$

To see that this holds on the level of rings, observe that there is a homomorphism $\varphi : \mathbb{Z}[y] \to \mathrm{CH}(\mathcal{X})$ given by $y \mapsto x$. Since $\mathrm{CH}^k(\mathcal{X}) = \left\langle x^k \right\rangle$ for all $k \geq 0$, φ is surjective. Moreover, the above group isomorphism of $\mathrm{CH}(\mathcal{X})$ with $\mathbb{Z}[x]/(6x^2)$ shows that the kernel of φ must be $(6y^2)$, which establishes the isomorphism on the level of rings. □

Note 5.12.

(a) This argument has a very by-hand feel. There are alternate arguments, similar to those in [3], which are less piecewise. We, however, choose to use this slightly clunkier argument simply because it is possible and shows a low-information way of computing Chow rings.

(b) Our argument can also be modified to not use higher Chow groups, in a similar fashion as the argument for $\mathcal{M}_{1,2}$ in Theorem 2.7. However, the argument presented here allows us to conclude that the first higher Chow group of the stack $\mathcal{X}$ with ℓ-adic coefficients vanishes, a fact which is important to later computations in [3].

Corollary 5.13. *The Chow ring of $\mathcal{M}_{1,3}$ is a quotient of $\mathbb{Z}[\lambda_1]/(6\lambda_1^2)$.*

Proof. This follows from the excision sequence, since $\mathcal{M}_{1,3}$ is open in $\mathcal{X}$ – namely, it is the complement of the locus of singular curves. The fact that it is generated by λ_1 is a consequence of Corollary 2.6. □

Theorem 5.14. *The integral Chow ring of $\mathcal{M}_{1,3}$ is*

$$\mathrm{CH}(\mathcal{M}_{1,3}) = \frac{\mathbb{Z}[\lambda_1]}{(12\lambda_1, 6\lambda_1^2)}.$$

Proof. The inclusion of any three-pointed curve with μ_2 automorphisms, such as $(C_{(-1,0)}, \infty, [1:0:1], [0:0:1])$, or μ_3 automorphisms, such as $(C_{(0,1)}, \infty, [0:1:1], [0:-1:1])$, shows that $\mathrm{CH}(\mathcal{M}_{1,3})$ surjects onto $\mathbb{Z}[x]/(2x)$ and $\mathbb{Z}[x]/(3x)$, respectively. Since $\mathrm{Pic}(\mathcal{M}_{1,3}) = \mathbb{Z}/12$, generated by λ_1, the theorem is proven. □

5.2. *The case* $4 \leq n \leq 10$

We first make an analogous definition of the *tautological ring* in the integral case.

Definition 5.15. The *integral tautological ring* of $\mathcal{M}_{1,n}$, written $\mathcal{R}(\mathcal{M}_{1,n})$, is the subring of the Chow ring generated by λ_1.

The remainder of this section has the following structure: first, we compute the integral tautological ring of $\mathcal{M}_{1,n}$ for $n \geq 4$, and then we show that the full Chow ring is indeed generated by λ_1 for $4 \leq n \leq 10$.

Corollary 5.16. *For $n \geq 3$:*

(a) *the integral tautological ring $\mathcal{R}(U_n)$ is a quotient of $\mathbb{Z}[\lambda_1]/(2\lambda_1)$,*
(b) $\mathrm{CH}(U_3) = \mathbb{Z}[\lambda_1]/(2\lambda_1)$,
(c) *the integral tautological ring $\mathcal{R}(U_n')$ is equal to $\mathbb{Z}$, and*
(d) *the element $[\sigma_{n-1}(U_{n-1}')]$ is tautological.*

Proof. We showed in Lemma 5.8 that $2\lambda_1 = 0$ on U_3, and so this relation holds on U_n for all $n \geq 3$, showing (a). Considering the three-pointed elliptic curve $(C_{(-1,0)}, \infty, [1:0:1], [0:0:1])$ and its induced residual gerbe, following the proofs of Theorems 2.7 and 5.14 shows (b).

We also showed in Lemma 5.9 that $3\lambda_1 = 0$ on U_2', and so this relation holds on U_3' and hence on U_n' for all $n \geq 3$. Since $U_n' \subseteq U_n$ for all n, we see that for all $n \geq 3$, both relations $2\lambda_1 = 0$ and $3\lambda_1 = 0$ hold on U_n'. Therefore, $\lambda_1 = 0$ on U_n' for $n \geq 3$, which proves (c).

To see that $[\sigma_{n-1}(U_{n-1}')]$ is tautological, just observe that it is a divisor and hence tautological by Theorem 5.11. □

Lemma 5.17. *The excision sequence for $U_{n-1}' \to \mathcal{M}_{1,n}$ and the later-defined (see Definition 5.22) $V_{n-1}' \to \mathcal{M}_{1,n}$ restricts to integral tautological rings as well. That is, we get exact sequences*

$$\mathcal{R}(U_{n-1}') \to \mathcal{R}(\mathcal{M}_{1,n}) \to \mathcal{R}(U_n) \to 0$$

and

$$\mathcal{R}(V_{n-1}') \to \mathcal{R}(\mathcal{M}_{1,n}) \to \mathcal{R}(V_n) \to 0.$$

Proof. We prove this in the U_{n-1}' case, since the case for V_{n-1}' is identical. Note that it suffices to show that any tautological element pushes forward to a tautological element.

The structure morphism to $B\mathbb{G}_m$ exhibits the pushforward

$$\mathrm{CH}(U_{n-1}') \xrightarrow{\sigma_{n-1,*}} \mathrm{CH}(\mathcal{M}_{1,n})$$

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as a $\mathrm{CH}(B\mathbb{G}_m)$-algebra homomorphism (that is, a $\mathbb{Z}[\lambda_1]$-algebra homomorphism). By Corollary 5.16, the tautological ring of U'_{n-1} is generated by λ_1. Therefore, the pushforward of any monomial is given by

$$\sigma_{n-1,*}(\lambda_1^k) = \lambda_1^k \sigma_{n-1,*}(1) = \lambda_1^k[\sigma_{n-1}(U'_{n-1})].$$

By Corollary 5.16(c), $[\sigma_{n-1}(U'_{n-1})]$ is tautological, and so we see that the pushforward of any tautological element is itself tautological. □

Lemma 5.18. *For all $n \geq 4$, the integral tautological ring of $\mathcal{M}_{1,n}$ is a quotient of $\mathbb{Z}[\lambda_1]/(12\lambda_1, 2\lambda_1^2)$.*

Proof. Since by Corollary 5.16 the tautological ring of U_n is a quotient of $\mathbb{Z}[\lambda_1]/(2\lambda_1)$, we can write

$$\mathcal{R}(U_n) = \frac{\mathbb{Z}[\lambda_1]/(2\lambda_1)}{I}$$

for some ideal I. The excision sequence for $\operatorname{im}\sigma_{n-1} \cong U'_{n-1}$ restricts to integral tautological rings by Lemma 5.17 and hence gives

$$\mathcal{R}(U'_{n-1}) \to \mathcal{R}(\mathcal{M}_{1,n}) \to \mathcal{R}(U_n) \to 0$$

$$\mathbb{Z} \to \mathcal{R}(\mathcal{M}_{1,n}) \to \frac{\mathbb{Z}[\lambda_1]/(2\lambda_1)}{I} \to 0.$$

Since the image of the morphism lands in degree one and the Picard group of $\mathcal{M}_{1,n}$ is known to be $\mathbb{Z}/12$, the lemma follows. □

Proposition 5.19. *The integral tautological ring of $\mathcal{M}_{1,4}$ is*

$$\mathcal{R}(\mathcal{M}_{1,4}) = \frac{\mathbb{Z}[\lambda_1]}{(12\lambda_1, 2\lambda_1^2)}.$$

Proof. Observe that by Appendix A, there still exists four-pointed smooth elliptic curves with μ_2-automorphisms: $n = 4$ is the largest n for which such a curve exists, and all such curves have μ_2-automorphisms, generated by the involution. Moreover, such a curve is necessarily contained inside of U_4, the locus where the second and third points are not involutions of each other, since each marked point is fixed by the involution. Therefore, we get a surjection

$$\mathrm{CH}(U_4) \twoheadrightarrow \mathbb{Z}[x]/(2x).$$

However, since the degree one generator of $\mathrm{CH}(U_4)$ is λ_1, this morphism in fact factors as

$$\begin{tikzcd} \mathcal{R}(U_4) \arrow[r, hook] \arrow[rr, bend left, two heads] & \mathrm{CH}(U_4) \arrow[r, two heads] & \mathbb{Z}[x]/(2x), \end{tikzcd}$$

and so $\mathcal{R}(U_4) = \mathbb{Z}[\lambda_1]/(2\lambda_1)$. From the following facts,

- $\mathcal{R}(\mathcal{M}_{1,4}) \to \mathcal{R}(U_4) = \mathbb{Z}[\lambda_1]/(2\lambda_1)$ is surjective;
- $\mathcal{R}(\mathcal{M}_{1,4})$ is a quotient of $\mathbb{Z}[\lambda_1]/(12\lambda_1, 2\lambda_1^2)$ (Lemma 5.18);
- the Picard group of $\mathcal{M}_{1,4}$ is isomorphic to $\mathbb{Z}/12$, generated by λ_1 (Theorem 5.11),

we conclude that $\mathcal{R}(\mathcal{M}_{1,4}) = \mathbb{Z}[\lambda_1]/(12\lambda_1, 2\lambda_1^2)$. □

Before we can compute the integral tautological ring for $n \geq 5$, we must analyze $\mathcal{M}_{1,4}$ more thoroughly.

Definition 5.20. Let $Z_n \subseteq \mathcal{M}_{1,n}$ be the locus of curves with nontrivial automorphisms.

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Observation 5.21. Since $\mathcal{M}_{1,4} \setminus Z_4$ is a four-dimensional variety, we must have $\lambda_1^5 = 0$ on this locus, and hence on any locus inside of it. Moreover, observe that every curve in Z_4 must have p_2, p_3 and p_4 colinear: the only four-pointed smooth elliptic curves with automorphisms are the ones where $y_i = 0$ for $i = 2, 3, 4$, and hence, p_2, p_3 and p_4 lie on the line $y = 0$ (see Appendix A).

We now give a second stratification of $\mathcal{M}_{1,n}$ for $n \geq 4$ as follows:

- the open locus where p_2, p_3 and p_4 are not colinear under the Weierstrass embedding.
- and the divisor where p_2, p_3 and p_4 *are* colinear.

Definition 5.22. For $n \geq 3$,

(a) Let $V_n \subseteq \mathcal{M}_{1,n}$ be the locus where $p_2 + p_3 \neq \iota(p_4)$.
(b) Let $V_n' \subseteq \mathcal{M}_{1,n}$ be the locus where $p_2 + p_3 \neq \iota(p_i)$ for any $i = 1, \ldots, n$.

Note 5.23. Note that the condition for V_3 is ill-defined. We use the convention that this is an empty condition, so that $V_3 = \mathcal{M}_{1,3}$.

Observation 5.24. As before, in Observation 5.4, π pulls relations back to relations.

Definition 5.25. Define the morphism of stacks $\tau_{n-1} : V_{n-1}' \to \mathcal{M}_{1,n}$ by

$$(C, p_i) \mapsto (C, p_1, p_2, p_3, \iota(p_2 + p_3), \ldots, p_{n-1}).$$

Note that, as in Definition 5.5, the additive structure also extends to families of elliptic curves.

Proposition 5.26. *The map $\tau_{n-1} : V_{n-1}' \to \mathcal{M}_{1,n}$ is a closed immersion.*

Proof. Similar to Proposition 5.6. □

Corollary 5.27. *For $n \geq 4$, the stack $\mathcal{M}_{1,n}$ stratifies into the disjoint union*

$$\mathcal{M}_{1,n} = V_n \sqcup \operatorname{im} \tau_{n-1} \cong V_n \sqcup V_{n-1}'.$$

Proposition 5.28. *For $n = 2, \ldots, 10$, the stacks $\mathcal{M}_{1,n}$ are rational. Moreover, for $n = 4, \ldots, 10$, the open in $\mathcal{M}_{1,n}$ which exhibits this rationality is $U_n \cap V_n$.*

Proof. This was proven by Belorousski in the case where $\Bbbk$ is algebraically closed and characteristic zero in [2] by constructing a bijective morphism between $U_n \cap V_n$ and an open subset of $\mathbb{P}^n$. He concludes that it is an isomorphism since $\mathbb{P}^n$ is normal. This proof does not work in arbitrary characteristic (for example, the Frobenius morphism on $\mathbb{P}^1$ is a bijective morphism between normal varieties which is not an isomorphism). However, Belorousski's argument showing that the morphism is bijective is, in fact, functorial and works in families, therefore directly establishing that the moduli stacks are isomorphic. □

Lemma 5.29. *The element $[\tau_{n-1}(V_{n-1}')]$ is tautological.*

Proof. Similar to Corollary 5.16(c). □

Proposition 5.30. *The Chow ring of $\mathcal{M}_{1,n}$ is tautological for $n = 1, \ldots, 10$.*

Proof. We have already shown this for $n = 1, 2, 3$. For $n \geq 4$, observe that $\operatorname{im} \sigma_{n-1}$ and $\operatorname{im} \tau_{n-1}$ are disjoint, since the image of σ_{n-1} consists of curves where $p_2 = \iota(p_3)$ and the image of τ_{n-1} consists of curves where $p_2 + p_3 = p_4$. Any curve in the intersection of these loci would then satisfy $p_4 = p_2 + p_3 = p_2 + \iota(p_2) = \infty = p_1$, a contradiction. Therefore, we may stratify $\mathcal{M}_{1,n}$ into $\mathcal{M}_{1,n} = (U_n \cap V_n) \sqcup \operatorname{im} \sigma_{n-1} \sqcup \operatorname{im} \tau_{n-1}$. That is, $\mathcal{M}_{1,n}$ is the union of the open locus where p_2 and p_3 are not involutions and p_2, p_3 and p_4 are not colinear along with the divisors where these conditions do hold. But $U_n \cap V_n$ is isomorphic to an open in $\mathbb{P}^n$ by the above Proposition and hence generated in degree one, hence generated by λ_1, hence tautological. Since $\operatorname{im} \sigma_{n-1}$ and $\operatorname{im} \tau_{n-1}$ are isomorphic to opens in $\mathcal{M}_{1,n-1}$ and have tautological fundamental classes by Corollary 5.16(c) and Lemma 5.29, $\mathcal{M}_{1,n}$ is inductively built out of tautological pieces, and hence itself tautological. This breaks at $n = 11$ since $U_{11} \cap V_{11}$ is not birational to an open in $\mathbb{P}^{11}$ by [2]. □

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Theorem 5.31. *The integral Chow ring of $\mathcal{M}_{1,4}$ is given by*

$$\mathrm{CH}(\mathcal{M}_{1,4}) = \frac{\mathbb{Z}[\lambda_1]}{(12\lambda_1, 2\lambda_1^2)}.$$

Proof. Since $\mathcal{M}_{1,4}$ is tautological by the above Proposition, we have

$$\mathrm{CH}(\mathcal{M}_{1,4}) = \mathcal{R}(\mathcal{M}_{1,4}) = \frac{\mathbb{Z}[\lambda_1]}{(12\lambda_1, 2\lambda_1^2)}$$

by Proposition 5.19. □

Proposition 5.32. *The Chow ring of U_4 is* $\mathrm{CH}(U_4) = \mathbb{Z}[\lambda_1]/(2\lambda_1)$.

Proof. By Theorem 5.31, the Chow ring of U_4 is generated by λ_1, and so by Corollary 5.16(a), $\mathrm{CH}(U_4)$ is a quotient of $\mathbb{Z}[\lambda_1]/(2\lambda_1)$. Similarly to the proof of Corollary 5.16(b), the four-pointed elliptic curve $(C_{(-1,0)}, \infty, [1 : 0 : 1], [0 : 0 : 1], [-1 : 0 : 1])$ induces a residual gerbe which shows that $\mathrm{CH}(U_4) = \mathbb{Z}[\lambda_1]/(2\lambda_1)$. □

Proposition 5.33. *The Chow ring of $U_4 \cap V_4$ is $\mathrm{CH}(U_4 \cap V_4) = \mathbb{Z}$, and the Chow ring of V_4' is $\mathrm{CH}(V_4') = \mathbb{Z}$.*

Proof. Note that the image of τ_3 is contained inside of U_4, as points in $\mathrm{im}\,\tau_3$ are of the form $(C, p_1, p_2, p_3, \iota(p_2 + p_3))$ where $p_2 + p_3 \neq \iota(p_i)$ for any i. In particular, $p_2 + p_3 \neq \iota(p_1) = \infty$, and so $p_2 \neq \iota(p_3)$, which is the defining property of U_4. Therefore, we may consider the following excision sequence:

$$\mathrm{CH}(\mathrm{im}\,\tau_3) \to \mathrm{CH}(U_4) \to \mathrm{CH}(U_4 \cap V_4) \to 0.$$

Since $\mathrm{im}\,\tau_3 \cong V_3' \subseteq U_3$ and V_3' contains three-pointed curves inducing a residual gerbe (as in Theorem 5.14), this sequence is really

$$\frac{\mathbb{Z}[\lambda_1]}{(2\lambda_1)} \xrightarrow{\tau_{3*}} \frac{\mathbb{Z}[\lambda_1]}{(2\lambda_1)} \to \mathrm{CH}(U_4 \cap V_4) \to 0.$$

Since by Observation 5.21 $\lambda_1^5 = 0$ on $U_4 \cap V_4$ and $\lambda_1^5 \neq 0$ on U_4, we see that λ_1^5 must be in the image of τ_{3*}. Hence, $\tau_{3*}(\lambda_1^4) = \lambda_1^5$ in $\mathrm{CH}(U_4)$. But we also have $\tau_{3*}(\lambda_1^4) = \tau_{3*}(\tau_3^*(\lambda_1^4)) = \lambda_1^4\tau_{3*}(1)$. Therefore, we must have $\tau_{3*}(1) = \lambda_1$, and so $\mathrm{CH}(U_4 \cap V_4) = \mathbb{Z}$.

Note that, in particular, V_4' describes curves where p_2 and p_3 are not exchanged by the involution, and so $V_4' \subseteq U_4$; hence, $V_4' \subseteq U_4 \cap V_4$. Therefore, $\mathrm{CH}(V_4') = \mathbb{Z}$ as well. □

Corollary 5.34. *For $n \geq 4$,*

(a) $\mathcal{R}(U_n \cap V_n) = \mathbb{Z}$
(b) $\mathcal{R}(V_n') = \mathbb{Z}$.

Proof. From Proposition 5.33, we see that we have the relation $\lambda_1 = 0$ on $U_4 \cap V_4$ and V_4', and hence on $U_n \cap V_n$ and V_n' for all $n \geq 4$. Therefore, $\mathcal{R}(U_n \cap V_n) \cong \mathcal{R}(V_n') = \mathbb{Z}$. □

Proposition 5.35. *For $n \geq 5$, the integral tautological ring of $\mathcal{M}_{1,n}$ is*

$$\mathcal{R}(\mathcal{M}_{1,n}) = \frac{\mathbb{Z}[\lambda_1]}{(12\lambda_1, \lambda_1^2)}.$$

Proof. We use the stratification

$$\mathcal{M}_{1,n} = (U_n \cap V_n) \sqcup \mathrm{im}\,\sigma_{n-1} \sqcup \mathrm{im}\,\tau_{n-1}$$

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from Proposition 5.30. Notice that $\operatorname{im}\sigma_{n-1} \sqcup \operatorname{im}\tau_{n-1}$ has tautological ring $\mathbb{Z}\oplus\mathbb{Z}$, with both components in degree zero, since it is isomorphic to the abstract disjoint union $U'_{n-1} \sqcup V'_{n-1}$ and $\mathcal{R}(U'_{n-1}) \cong \mathcal{R}(V'_{n-1}) = \mathbb{Z}$ by Corollaries 5.16 and 5.34.

The excision sequence restricts to integral tautological rings by Lemma 5.17 and hence gives

$$\mathcal{R}(\operatorname{im}\sigma_{n-1} \sqcup \operatorname{im}\tau_{n-1}) \to \mathcal{R}(\mathcal{M}_{1,n}) \to \mathcal{R}(U_n \cap V_n) \to 0$$

$$\mathbb{Z}\oplus\mathbb{Z} \to \mathcal{R}(\mathcal{M}_{1,n}) \to \mathbb{Z} \to 0.$$

Since the $\mathbb{Z}\oplus\mathbb{Z}$ has both components in degree zero, its image in the tautological ring of $\mathcal{M}_{1,n}$ lands in degree one. Hence, we see that the integral tautological ring of $\mathcal{M}_{1,n}$ is concentrated in degrees 0 and 1, and so $\mathcal{R}(\mathcal{M}_{1,n}) = \mathbb{Z}[\lambda_1]/(12\lambda_1, \lambda_1^2)$. □

Theorem 5.36. *For $5 \le n \le 10$, the integral Chow ring of $\mathcal{M}_{1,n}$ is*

$$\mathrm{CH}(\mathcal{M}_{1,n}) = \frac{\mathbb{Z}[\lambda_1]}{(12\lambda_1, \lambda_1^2)}.$$

Proof. The Chow ring of $\mathcal{M}_{1,n}$ is tautological for $5 \le n \le 10$ by Proposition 5.30, and the tautological ring was computed in the above Proposition. □

A. Automorphisms of marked elliptic curves

In this Appendix, we note the following facts about automorphisms of marked elliptic curves.

Proposition A.1. *Over a field $\Bbbk$ of characteristic not equal to 2 or 3, there exists*

- *one-pointed elliptic curves with automorphism groups μ_2, μ_4 and μ_6;*
- *two-pointed elliptic curves with automorphism groups μ_2, μ_3 and μ_4;*
- *three-pointed elliptic curves with automorphism groups μ_2 and μ_3;*
- *and four-pointed elliptic curves with automorphism group μ_2.*

Every four-pointed elliptic curve with μ_2 automorphisms has p_2, p_3, p_4 colinear, and every n-pointed elliptic curve with $n \ge 5$ has no (nontrivial) automorphisms.

Proof. Recall the Weierstrass form for elliptic curves:

Theorem A.2 (Weierstrass). *Any one-pointed smooth elliptic curve over a field $\Bbbk$ of characteristic not equal to 2 or 3 can be written in the form $y^2z = x^3 + axz^2 + bz^3$, where the marked point is the point at infinity $[0 : 1 : 0]$. Moreover, if we denote such a curve by $C_{(a,b)}$, then $C_{(a,b)} \cong C_{(a',b')}$ if and only if $(a', b') = (t^4a, t^6b)$. The isomorphism between these curves is given by $[x : y : z] \mapsto [t^2x : t^3y : z]$. Lastly, an elliptic curve is smooth if and only if $D = 4a^3 + 27b^2 = 0$, nodal if and only if $D = 0$ and $(a, b) \ne (0, 0)$, and cuspidal if and only if $(a, b) = (0, 0)$.*

From this, we see that an elliptic curve with n marked points over $\Bbbk$ is determined by a choice of (a, b) and $p_2, \ldots, p_n$, $p_i = (x_i, y_i)$, and that the automorphisms of this curve are given by the $t \in \mathbb{G}_m$ such that $t \cdot (a, b) = (t^4a, t^6b) = (a, b)$ and $t \cdot p_i = (t^2x_i, t^3y_i) = (x_i, y_i)$.

Now for each $m > 1$, let ζ_m denote a primitive m^{th} root of unity. From $(t^4a, t^6b) = (a, b)$, we see that the automorphism group of every one-pointed elliptic curve contains a copy of μ_2 corresponding to $t = \zeta_2 = -1$, the involution. Additionally, the curves $C_{(1,0)}$ and $C_{(0,1)}$ are fixed by $\mu_4 = \langle\zeta_4\rangle$ and $\mu_6 = \langle\zeta_6\rangle$. Since any automorphism of an n-pointed elliptic elliptic curve $(C, p_1, \ldots, p_n)$ is in particular an automorphism of (C, p_1), they must all correspond to elements of μ_2, μ_4, or μ_6.

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The element ζ_2 is an automorphism of every elliptic curve and induces the map $\zeta_2 : [x : y : z] \mapsto [x : -y : z]$, and so for a point $p_i \neq \infty$ to be fixed by this, we must have $p_i = [x : 0 : 1]$. Then we have

$$y^2 = x^3 + ax + b$$

$$0 = x^3 + ax + b,$$

which has at most three solutions. Therefore, the involution $\iota = \zeta_2$ fixes at most four points in total. An example of a four-pointed elliptic curve with automorphism group μ_2 is $(C_{(-1,0)}, \infty, [1 : 0 : 1], [0 : 0 : 1], [-1 : 0 : 1])$. Notice that any four-pointed elliptic curve fixed by the involution must have p_2, p_3, p_4 colinear, as each point lies on the line $y = 0$.

The element ζ_4 is an automorphism of the curve corresponding to $(1, 0)$ and induces the map $\zeta_4 : [x : y : z] \mapsto [-x : \zeta_4^3 y : z]$, and so for a point $p_i \neq \infty$ to be fixed by this, we must have $p_i = [0 : 0 : 1]$, which is indeed a point on the curve $C_{(1,0)}$. Therefore, there is exactly one two-pointed elliptic curve with automorphism group μ_4, the curve $(C_{(1,0)}, \infty, [0 : 0 : 1])$.

The element ζ_6 is an automorphism of the curve corresponding to $(0, 1)$ and induces the map $\zeta_6 : [x : y : z] \mapsto [\zeta_3 x : -y : z]$, and so for a point $p_i \neq \infty$ to be fixed by this, we must have $p_i = [0 : 0 : 1]$, which is *not* a point on the curve $C_{(0,1)}$. Therefore, there is no n-pointed elliptic curve with automorphism group μ_6 for $n \geq 2$.

Lastly, the element $\zeta_6^2 = \zeta_3$ is an automorphism of the curve corresponding to $(0, 1)$ and induces the map $\zeta_3 : [x : y : z] \mapsto [\zeta_3^2 x : y : z]$, and so for a point $p_i \neq \infty$ to be fixed by this, we must have $p_i = [0 : y : z]$. Then we have

$$y^2 = x^3 + ax + b$$

$$y^2 = 1.$$

Therefore, an example of a three pointed elliptic curve with automorphism group μ_3 is $(C_{(0,1)}, \infty, [0 : 1 : 1], [0 : -1 : 1])$.

This exhausts all possible automorphisms, and so there are no n-pointed elliptic curves with nontrivial automorphisms for $n \geq 5$. □

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