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# COMPOSITION OPERATORS ON LORENTZ SPACES

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Fredholm, injective, isometric and surjective composition operators on Lorentz spaces $L(p,q)$ are characterised in this paper.

## 1. INTRODUCTION

Let $f$ be a complex-valued measurable function defined on a $\sigma$-finite measure space $(X,\mathcal{A},\mu)$. For $s \geqslant 0$, define $\mu_f$ the *distribution function* of $f$ as

$$\mu_f(s) = \mu\Big\{x \in X : \big|f(x)\big| > s\Big\}.$$

By $f^*$ we mean the *non-increasing rearrangement* of $f$ given as

$$f^*(t) = \inf\big\{s > 0 : \mu_f(s) \leqslant t\big\},\ t \geqslant 0.$$

We also denote the rearrangement of $f$ with respect to the measure $\mu$ by $f^{*,\mu}$. For $t > 0$, let

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds.$$

For $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, and for measurable function $f$ on $X$ define $\|f\|_{pq}$ as

$$\|f\|_{pq} = \begin{cases} \left\{\dfrac{q}{p}\displaystyle\int_0^\infty \left(t^{1/p}f^{**}(t)\right)^q \dfrac{dt}{t}\right\}^{1/q}, & 1 < p < \infty,\ 1 \leqslant q < \infty \\ \sup\limits_{t>0} t^{1/p}f^{**}(t), & 1 < p \leqslant \infty,\ q = \infty \end{cases}$$

The *Lorentz space* denoted by $L(p,q)(X,\mathcal{A},\mu)$ (or shortly $L(p,q)$) is defined to be the vector space of all (equivalence classes of) measurable functions $f$ on $X$ such that $\|f\|_{pq} < \infty$. Also $\|\cdot\|_{pq}$ is a norm and $L(p,q)$ is a Banach space with respect to this norm. The $L^p$- spaces for $1 < p \leqslant \infty$ are equivalent to the spaces $L(p,p)$. For more on Lorentz spaces one can refer to [**1, 2, 3, 7, 10, 13, 14, 15, 17**].

On the measure space $(X,\mathcal{A},\mu)$, let $T : X \to X$ be a measurable transformation. Then we define a linear transformation $C_T$ on the Lorentz space $L(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$ into the linear space of all complex-valued measurable functions on $X$ by

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$C_T f = f \circ T$. If $C_T$ is bounded with range in $L(p,q)$, then it is called a *composition operator* on $L(p,q)$ induced by $T$. There is a vast literature for composition operators on measurable function spaces and their applications, one can refer to [**4, 5, 6, 8, 9, 11, 12, 16, 18, 19, 20**] and references therein.

For a complex-valued measurable function $u$ on $X$, we define a linear transformation $M_u$ on the Lorentz space $L(p,q)$ as $M_u f = u \cdot f$, where the product of functions is pointwise. If $M_u$ is bounded with range in $L(p,q)$, then it is called a *multiplication operator* on $L(p,q)$ induced by $u$.

For a bounded linear operator $A$ on a Banach space; we use the symbols $N(A)$ and $R(A)$ to denote the kernel and the range of $A$, respectively. We recall that $A$ is called compact if the closure of the image of the unit ball is compact; and *Fredholm* if $R(A)$ is closed, $\dim N(A) < \infty$ and $\operatorname{codim} R(A) < \infty$, where $\dim N(A)$ is the dimension of $N(A)$ and $\operatorname{codim} R(A)$ is the codimension of $R(A)$, namely the dimension of any subspace complimentary to $R(A)$.

The main aim of this paper is to study Fredholmn property, isometry, invertibility of composition operators on Lorentz spaces $L(p,q)$. In Section 2, we study the boundedness of composition operators between Lorentz spaces with different measure spaces. In Section 3, we discuss the closedness of the range $R(C_T)$, denseness and surjectiveness of composition operator. In Section 4, adjoint of a composition operator is obtained and Fredholm, isometric and invertible composition operators are characterised.

## 2. Boundedness

In this section we characterise those measurable transformations $T : Y \to X$, where $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are two $\sigma$-finite measure spaces, for which

$$C_T : L(p,q)(X, \mathcal{A}, \mu) \to L(p,q)(Y, \mathcal{B}, \nu) \ (f \mapsto f \circ T)$$

is bounded.

**Theorem 2.1.** *A measurable transformation $T : Y \to X$ induces a composition operator*

$$C_T : L(p,q)(X, \mathcal{A}, \mu) \to L(p,q)(Y, \mathcal{B}, \nu), 1 < p \leqslant \infty, 1 \leqslant q \leqslant \infty$$

*if and only if*

$$(\nu \circ T^{-1})(E) \leqslant b\, \mu(E), \text{ for all } E \in \mathcal{A}, \text{ for some } b > 0.$$

*Moreover*

$$\|C_T\| = k^{1/p}, \text{ where } \quad k = \inf\left\{b_0 > 0 : (\nu \circ T^{-1})(E) \leqslant b_0\, \mu(E), \text{ for all } E \in \mathcal{A}\right\}.$$

PROOF: First assume that $1 < p < \infty$, $1 \leqslant q < \infty$ and suppose $C_T$ is a composition operator induced by $T$. Let $E \in \mathcal{A}$, $\mu(E) < \infty$. Then the non-increasing rearrangement of the characteristic function $\chi_E$ is given by

$$\chi_E^*(t) = \chi_{[0,\ \mu(E))}(t).$$

Thus

$$\begin{aligned}\chi_E^{**}(t) &= \frac{1}{t}\int_0^t \chi_E^*(s)ds \\ &= \begin{cases} 1, & \text{if } 0 \leqslant t < \mu(E) \\ \dfrac{1}{t}\ \mu(E), & \text{if } t \geqslant \mu(E). \end{cases}\end{aligned}$$

Therefore

$$\begin{aligned}\|\chi_E\|_{pq}^q &= \frac{q}{p}\int_0^\infty \left(t^{1/p}\chi_E^{**}(t)\right)^q \frac{dt}{t} \\ &= \mu(E)^{q/p} + \frac{1}{p-1}\ \mu(E)^{q/p} = p'\ \left(\mu(E)\right)^{q/p},\end{aligned}$$

where $1/p + 1/p' = 1$. This implies that $\chi_E \in L(p,q)(X,\mathcal{A},\mu)$ and

$$\begin{aligned}(\nu \circ T^{-1})(E) = \nu\left(T^{-1}(E)\right) &= (p')^{-p/q}\|\chi_{T^{-1}(E)}\|_{pq}^p \\ &= (p')^{-p/q}\ \|\chi_E \circ T\|_{pq}^p = (p')^{-p/q}\|C_T\chi_E\|_{pq}^p \\ &\leqslant (p')^{-p/q}\|C_T\|^p\ \|\chi_E\|_{pq}^p = \|C_T\|^p\ \mu(E).\end{aligned}$$

Hence

$$\left(\nu \circ T^{-1}\right)(E) \leqslant b\ \mu(E),$$

where $b = \|C_T\|^p$. If $\mu(E) = \infty$, then the inequality is trivial. For $q = \infty$, $1 < p \leqslant \infty$, we have

$$\|\chi_E\|_{p\infty} = \sup_{t>0} t^{1/p}\chi_E^{**}(t) = \left(\mu(E)\right)^{1/p}.$$

Therefore

$$(\nu \circ T^{-1})(E) = \|C_T\chi_E\|_{p\infty}^p \leqslant \|C_T\|^p\ \mu(E).$$

Conversely, suppose there is a constant $b > 0$ such that for all $E \in \mathcal{A}$,

$$(\nu \circ T^{-1})(E) \leqslant b\ \mu(E).$$

For $f$ in $L(p,q)(X,\mathcal{A},\mu)$, the distribution of $f \circ T$ satisfies

$$\begin{aligned}\nu_{(f\circ T)}(s) &= \nu\left\{y \in Y : \left|f\left(T(y)\right)\right| > s\right\} \\ &= (\nu \circ T^{-1})\left\{x \in X : \left|f(x)\right| > s\right\} \\ &\leqslant b\ \mu\left\{x \in X : \left|f(x)\right| > s\right\} = b\ \mu_f(s).\end{aligned}$$

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Therefore

$$\{s>0:\mu_f(s)\leqslant t\}\subseteq\{s>0:\nu_{f\circ T}(s)\leqslant bt\}.$$

This gives

$$(f\circ T)^{*,\nu}(bt)\leqslant f^{*,\mu}(t)$$

and consequently

$$(f\circ T)^{**,\nu}(bt)\leqslant f^{**,\mu}(t),\quad t>0.$$

Now for $f$ in $L(p,q)$, $1<p<\infty$, $1\leqslant q<\infty$,

$$\begin{aligned}\|C_Tf\|_{pq}^q&=\frac{q}{p}\int_0^\infty\left(t^{1/p}(f\circ T)^{**,\nu}(t)\right)^q\frac{dt}{t}\\&=(b^{q/p})\,\frac{q}{p}\int_0^\infty\left(t^{1/p}(f\circ T)^{**,\nu}(bt)\right)^q\frac{dt}{t}\\&\leqslant(b^{q/p})\,\frac{q}{p}\int_0^\infty\left(t^{1/p}f^{**,\mu}(t)\right)^q\frac{dt}{t}=(b^{q/p})\,\|f\|_{pq}^q.\end{aligned}$$

This proves that $C_T$ is bounded. For $q=\infty$, $1<p\leqslant\infty$, we have

$$\begin{aligned}\|C_Tf\|_{p\infty}&=\sup_{t>0}t^{1/p}(f\circ T)^{**,\nu}(t)\\&=b^{1/p}\sup_{t>0}t^{1/p}(f\circ T)^{**,\nu}(bt)\\&\leqslant b^{1/p}\sup_{t>0}t^{1/p}f^{**,\mu}(t)=b^{1/p}\,\|f\|_{p\infty}.\end{aligned}$$

Hence the result. Moreover, we have

$$\|C_T\|=k^{1/p},$$

where $k=\inf\{b_0>0:(\nu\circ T^{-1})(E)\leqslant b_0\,\mu(E)\}$. ∎

**Corollary 2.2.** ([11]) *Let $T:X\to X$ be a non-singular measurable transformation. Then $T$ induces a composition operator $C_T$ on $L(p,q)$, $1<p<\infty$, $1\leqslant q\leqslant\infty$, if and only if there exists some constant $b>0$ such that*

$$(\mu\circ T^{-1})(E)\leqslant b\,\mu(E),\ \text{for all } E\in\mathcal{A}.$$

## 3. Ranges of Composition operators

In this section, we establish conditions for a composition operator to have a closed range or dense range and then we present a characterisation of surjective composition operators.

**Theorem 3.1.** *If $C_T$ is a bounded composition operator on $L(p,q)$, $1<p\leqslant\infty$, $1\leqslant q\leqslant\infty$. Then $C_T$ has closed range if and only if there exists $\varepsilon>0$ such that $f_T(x)\geqslant\varepsilon$ for almost all $x\in S$, where $S=\{x\in X:f_T(x)\neq0\}$ and $f_T$ is the Radon Nikodym derivative of $\mu T^{-1}$ with respect to $\mu$.*

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PROOF: Suppose $f_T(x) \geqslant \varepsilon$ for almost all $x \in S$. Then for $f \in L_\mu^{p,q}(S)$, where

$$L_\mu^{p,q}(S) = \{f \in L(p,q) : f \text{ vanishes outside } S\},$$
$$(f \circ T)^*(t) = \inf\Big\{s > 0 : \mu\Big\{x \in X : \big|f(T(x))\big| > s\Big\} \leqslant t\Big\}$$
$$= \inf\Big\{s > 0 : \mu T^{-1}\Big\{x \in S : \big|f(x)\big| > s\Big\} \leqslant t\Big\}.$$

Now

$$\mu T^{-1}(E) = \int_E f_T(x)d\mu \geqslant \varepsilon\, \mu(E),$$

where $E = \Big\{x \in S : \big|f(x)\big| > s\Big\}$. Hence

$$(f \circ T)^*(\varepsilon t) \geqslant \inf\Big\{s > 0 : \mu\Big\{x \in S : \big|f(x)\big| > s\Big\} \leqslant t\Big\}$$
$$= \inf\Big\{s > 0 : \mu\Big\{x \in X : \big|f(x)\big| > s\Big\} \leqslant t\Big\} = f^*(t), \text{ for all } t > 0,$$

and so

$$(f \circ T)^{**}(\varepsilon t) \geqslant f^{**}(t), \text{ for all } t > 0.$$

Therefore

$$\|C_T\|_{pq} \geqslant \varepsilon^{1/p}\|f\|_{pq}, \text{for all} f \in L_\mu^{p,q}(S).$$

As $N(C_T) = L_\mu^{p,q}(X \setminus S)$, we get that $C_T$ has closed range.

Conversely, suppose that $C_T$ has closed range. Then there exists $\varepsilon > 0$ such that

$$\|C_T\|_{pq} \geqslant \varepsilon\|f\|_{pq}, \text{ for all } f \in L_\mu^{p,q}(S). \tag{3.1}$$

Choose positive integer $n$ such that $1/n < \varepsilon$. Let $E = \Big\{x \in S : f_T(x) < 1/n^p\Big\}$.

If possible $\mu(E) > 0$, then $\mu(E) < (1/n^p)\mu(E)$ and

$$(\chi_E \circ T)^*(t) \leqslant \chi_E^*(n^p t), \text{ for all } t > 0.$$

This gives

$$\|C_T\chi_E\|_{pq}^q \leqslant \frac{1}{n^q}\|\chi_E\|_{pq}^q < \varepsilon^q\|\chi_E\|_{pq}^q.$$

This contradicts (3.1). Hence $f_T$ is bounded away from zero. ∎

For a measurable transformation $T$ on measure space $(X, \mathcal{A}, \mu)$, $T^{-1}(\mathcal{A})$ is a $\sigma$–subalgebra of $\mathcal{A}$. Then $L(p,q)(X, T^{-1}(\mathcal{A}), \mu)$ is a subspace of $L(p,q)$. Now we study the range of composition operators in terms of $L(p,q)(X, T^{-1}(\mathcal{A}), \mu)$.

**THEOREM 3.2.** *If $C_T$ is a composition operator on $L(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q < \infty$, then the range of $C_T$ is dense in $L(p,q)(X, T^{-1}(\mathcal{A}), \mu)$.*

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PROOF: In case $X$ is of finite measure, then $\chi_S$ in $L(p,q)(X,T^{-1}(\mathcal{A}),\mu)$ implies that $\chi_S = C_T\chi_{S'}$, for some $S' \in \mathcal{A}$. Thus all simple functions of $L(p,q)(X,T^{-1}(\mathcal{A}),\mu)$ belong to the range of $C_T$ and hence using (2.4, Hunt [7]), we find that range of $C_T$ is dense in $L(p,q)(X,T^{-1}(\mathcal{A}),\mu)$. In case $X$ is a $\sigma$–finite measure space, the proof follows from Lebesgue's theorem on dominated convergence. □

**THEOREM 3.3.** *A composition operator $C_T$ on $L(p,q)$ is surjective if and only if $f_T$ is bounded away from zero on its support and $T^{-1}(\mathcal{A}) = \mathcal{A}$.*

PROOF: In case $C_T$ is surjective then by using Theorem 3.1, $f_T$ is bounded away from zero on its support. Let $E \in \mathcal{A}$ be of finite measure. Since $C_T$ is surjective, there exists $f \in L(p,q)$ such that $\chi_E = C_T f$. Then we find $\chi_E = \chi_{T^{-1}(E_0)}$, where $E_0 = \{x \in X : f(x) = 1\}$. Hence $E = T^{-1}(E_0)$. This proves $\mathcal{A} \subseteq T^{-1}(\mathcal{A})$ and therefore equality. The converse follows by using the Theorems 3.1 and 3.2. □

**THEOREM 3.4.** *If $C_T$ is a composition operator on $L(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q < \infty$. Then $C_T$ has dense range if and only if $T^{-1}(\mathcal{A}) = \mathcal{A}$.*

PROOF: Suppose $C_T$ has dense range. Let $E \in \mathcal{A}$ be such that $\chi_E \in L(p,q)$. Then there exists a sequence $\langle f_n \rangle$ in $L(p,q)$ such that $C_T f_n \to \chi_E$ in $\|\cdot\|_{pq}$ and so $C_T f_n \to \chi_E$ almost everywhere. Since each $C_T f_n$ is measurable with respect to $T^{-1}(\mathcal{A})$, therefore $\chi_E$ is measurable with respect $T^{-1}(\mathcal{A})$ so that $\chi_E = \chi_{T^{-1}(F)}$ for some $F \in \mathcal{A}$. Thus $T^{-1}(\mathcal{A}) = \mathcal{A}$.

Conversely suppose $T^{-1}(\mathcal{A}) = \mathcal{A}$. Let $E \in \mathcal{A}$ be such that $\mu(E) < \infty$, then we can find $F \in \mathcal{A}$ such that $\mu\big(E \Delta T^{-1}(F)\big) = 0$. Since $X$ is $\sigma$-finite, we have an increasing sequence $\langle F_n \rangle$ of measurable set of finite measure such that $F_n \uparrow F$ or $T^{-1}(F \sim F_n) \downarrow \phi$. Hence for $\varepsilon > 0$, there exists a positive integer $n_0$ such that

$$\mu T^{-1}(F \sim F_n) < \left(\frac{\varepsilon}{p'}\right)^p, \ \forall\, n \geqslant n_0.$$

Hence

$$\begin{aligned} \|C_T\chi_F - C_T\chi_{F_n}\|_{pq} &= \|C_T\chi_{(F\sim F_n)}\|_{pq} = \|\chi_{T^{-1}(F\sim F_n)}\|_{pq} \\ &= p'\big(\mu T^{-1}(F \sim F_n)\big)^{1/p} < \varepsilon, \quad \forall\, n \geqslant n_0. \end{aligned}$$

This implies that $\chi_E \in \overline{R(C_T)}$. Now the result follows by using [7]. □

## 4. Fredholm and isometric composition operators

In this section we have made an attempt to study the adjoint of the composition operator on $L(p,q)$, $1 < p < \infty$, $1 \leqslant q < \infty$. Fredholm, isometric and invertible composition operators are characterised. By using (2.7, Hunt [7, p. 262]) for every

$g \in L(p', q')$, we can find a bounded linear functional $F_g \in \left(L(p,q)\right)^* = L(p', q')$, where $1/p + 1/p' = 1 = 1/q + 1/q'$, defined as

$$F_g(f) = \int fg d\mu, \quad \text{for all } f \in L(p,q).$$

For each $g \in L(p', q')$, there exists a unique $T^{-1}(\mathcal{A})$ measurable function $E(g)$ such that

$$\int fg d\mu = \int f E(g) d\mu,$$

for each $T^{-1}(\mathcal{A})$ measurable function $f$ for which the left integral exists. $E(g)$ is called the *conditional expectation* of $g$ with respect to $\sigma$–algebra $T^{-1}(\mathcal{A})$. The Frobenius Perron operator $P_T$ on $L(p', q')$ is defined as

$$P_T g = f_T \cdot E(g) \circ T^{-1},$$

where $E(g) \circ T^{-1} = f$ if and only if $E(g) = f \circ T$.

**THEOREM 4.1.** *If $C_T$ is a composition operator on $L(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q < \infty$, then $C_T^*$, the adjoint of the composition operator $C_T$, is $P_T$.*

PROOF: Let $E \in \mathcal{A}$ be such that $\mu(E) < \infty$. Then for $g \in L(p', q')$

$$\begin{aligned}(C_T^* F_g)(\chi_E) &= F_g(C_T \chi_E) = \int C_T \chi_E \cdot g \, d\mu \\ &= \int (\chi_E \circ T) \cdot g \, d\mu = \int E(g) \cdot \chi_E \circ T \, d\mu \\ &= \int E(g) \circ T^{-1} \cdot \chi_E \, d\mu T^{-1} = \int E(g) \circ T^{-1} \cdot \chi_E f_T \, d\mu \\ &= F_{(E(g)\circ T^{-1})\cdot f_T}(\chi_E)\end{aligned}$$

Thus $C_T^* F_g = F_{(E(g)\circ T^{-1})\cdot f_T}$. By identifying $g \in L(p', q')$ with $F_g \in \left(L(p,q)\right)^*$, we can write

$$C_T^* g = (E(g) \circ T^{-1}) \cdot f_T = P_T g \qquad \square$$

**THEOREM 4.2.** *If $C_T$ is a composition operator on $L(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q < \infty$, then $N(C_T^*)$ is either zero dimensional or infinite dimensional.*

PROOF: Suppose $0 \neq g \in N(C_T^*)$. Let $E = \{x \in X : g(x) \neq 0\}$, then $\mu(E) \neq 0$. Let $\langle E_n \rangle$ be a sequence of disjoint measurable subsets of $E$ such that

$$E = \bigcup_{n=1}^{\infty} E_n, \; 0 < \mu(E_n) < \infty.$$

For each natural number $n$, let $g_n = g \cdot \chi_E \circ T$. For each $n$,

$$\begin{aligned}C_T^*(g_n) f &= \int (g \cdot \chi_E \circ T)(f \circ T) \, d\mu \\ &= \int g \cdot (\chi_E f \circ T) \, d\mu \\ &= C_T^*(g)(\chi_E f) = 0.\end{aligned}$$

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Therefore $\{g_n : n \geqslant 1\}$ is a linearly independent subset of $N(C_T^*)$. Hence, if $N(C_T^*)$ is not zero dimensional, it is infinite dimensional. ∎

**Corollary 4.3.** *If $C_T$ is a composition operator on $L(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q < \infty$. Then $C_T$ is injective if and only if $T$ is surjective.*

**Theorem 4.4.** *If $C_T$ is a composition operator on $L(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q < \infty$. Then $C_T$ is Fredholm if and only if $C_T$ is invertible.*

Proof: If $C_T$ is Fredholm, then $N(C_T^*)$ and $N(C_T)$ both are finite dimensional and are of zero dimension. Therefore $C_T$ is injective and $R(C_T)$ is dense in $L(p,q)$. Since $R(C_T)$ is closed, therefore $C_T$ is surjective. This proves the invertibility of $C_T$. The proof of the converse is obvious. ∎

**Theorem 4.5.** *If $C_T$ is a composition operator on $L(p',q')$, then $C_T^*C_T = M_{f_T}$.*

Proof: On replacing $g$ by $C_T g$ in the Theorem 4.1, we find that for every $g \in L(p',q')$

$$C_T^*C_T g = C_T^*(g \circ T) = E(g \circ T) \circ T^{-1} \cdot f_T = g \cdot f_T = M_{f_T} g.$$

Hence $C_T^*C_T = M_{f_T}$. ∎

**Corollary 4.6.** *If $C_T$ is a composition operator on $L(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q < \infty$, then $C_T$ is an isometry if and only if $T$ is measure preserving.*

Definition 4.7: ([**16**]) The essential range of a complex-valued measurable function $f$ defined on the measure space $(X, \mathcal{A}, \mu)$ is given by the set

$$\Big\{\lambda \in \mathcal{C} : \mu\big(\{x \in X : \left|f(x) - \lambda\right| < \varepsilon\}\big) > 0, \text{ for each } \varepsilon > 0\Big\}.$$

By the theory developed so far, we have the following

**Theorem 4.8.** *If $C_T$ is a composition operator on $L(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q < \infty$, then the following are equivalent:*

(1) *$C_T$ is injective.*

(2) *$f$ and $f \circ T$ have the same essential ranges for every $f \in L(p,q)$.*

(3) *$\mu \ll \mu \circ T^{-1}$.*

(4) *$f_T$ is different from zero almost everywhere.*

(5) *$M_{f_T}$ is injective.*

**Theorem 4.9.** *If $C_T$ is a composition operator on $L(p,q)$, $1 < p \leqslant \infty$, $1 \leqslant q < \infty$, then $C_T$ is invertible if and only if $f_T$ is bounded away from zero almost everywhere on $X$ and $T^{-1}(\mathcal{A}) = \mathcal{A}$.*

Proof: If $f_T$ is bounded away from zero almost everywhere on $X$ and $T^{-1}(\mathcal{A}) = \mathcal{A}$, then $M_{f_T}$ is injective and hence $C_T$ is injective. In view of Theorem 3.3, $C_T$ is surjective. Therefore $C_T$ is invertible. Converse follows by the Theorems 3.3 and 4.8. ∎

Invertibility of $T$ does not imply invertibility of $C_T$. This is proved by the following examples.

**EXAMPLE 4.10.** *Let $X = [0,1]$, with the Lebesgue measure $\mu$ on the Borel subsets. Let $T(x) = \sqrt{x}$, $\forall\ x \in X$. Then $C_T$ is a composition operator on $L(p,q)$ (*[**11**, Example 5.1]*). $U(x) = x^2$ is the inverse of $T$. But $C_T$ is not invertible as*

$$\frac{\|C_T \chi_{[0,1/n]}\|_{pq}^q}{\|\chi_{[0,1/n]}\|_{pq}^q} = \frac{1}{n^p},$$

*for each natural number $n$. So $C_T$ is not bounded away from zero.*

**EXAMPLE 4.11.** *Let $X = \mathbf{R}$ with Lebesgue measure and let $T(x) = ax + b$, $a \neq 0, 1$. Then $T$ is not measure preserving and $C_T$ is a composition operator on $L(p,q)$ but $C_T$ is not an isometry.*

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