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The theorem of Marggraff on primitive permutation groups which contain a cycle

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A short elementary proof is given of the theorem of Marggraff which states that a primitive permutation group which contains a cycle fixing k points is $(k+1)$-fold transitive. It is then shown that the method of proof actually yields a generalization of Marggraff's theorem.

The following theorem is quoted by Wielandt [5, p. 38], who refers to the thesis of Marggraff [3] for a proof. [The authors have not seen this thesis, which Kantor [2, p. 64] describes as "inaccessible".] The theorem can also be deduced from a more general result of Kantor [2, 7D(4)] on Jordan groups.

THEOREM A. *Let G be a primitive permutation group on a set Ω of n points and suppose that G contains a non-trivial subgroup X which fixes k points of Ω and which is transitive on the remaining points. If X is cyclic, then G is $(k+1)$-fold transitive.*

In this note we give a proof of this theorem based on results from Wielandt's book together with elementary considerations of 2-designs. The same method of proof combined with the Hall-Bruck theorem [1] yields the following extension of Theorem A.

THEOREM B. *Let G, Ω, and X satisfy the hypotheses of the first sentence of Theorem* A. *If X contains a cyclic subgroup of index* 2 ,

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then one of the following holds:

(i) G *is* $(k+1)$*-fold transitive;*

(ii) $n = 7$, $k = 3$, X *is a Klein four-group, and* $G \simeq \mathrm{PSL}(3, 2)$;

(iii) $n = 8$, $k = 4$, X *is a Klein four-group, and* G *is the holomorph of an elementary abelian group of order* 8 ;

(iv) $n = 9$, $k = 3$, X *is a non-abelian group of order* 6 , *and* G *is the holomorph of an elementary abelian group of order* 9 .

If Δ is a subset of Ω we shall use $G(\Delta)$ and G_Δ to denote the pointwise and setwise stabilizers of Δ , respectively. The group G_Δ induces the permutation group $G_\Delta^\Delta \simeq G_\Delta/G(\Delta)$ on Δ .

Suppose that G, Ω , and X satisfy the hypotheses of Theorem A or B, but G is not $(k+1)$-fold transitive. Let Δ be the set of fixed points of X and set $\Gamma = \Omega - \Delta$. By a theorem of Jordan (see [5, 13.1]), G is doubly transitive and hence the sets Δ^x , $x \in G$, are the blocks of a 2-design $\mathcal{D}$ on Ω (*cf.* [2, Section 7]). Let λ be the number of blocks containing two distinct points. Our aim is to show that $\lambda = 1$.

For Theorem B, a transitive extension of the group occurring in conclusion *(ii)* is isomorphic to the group of conclusion *(iii)*, while the groups of conclusions *(iii)* and *(iv)* admit no transitive extensions. Thus we may suppose, by induction, that G is not doubly primitive. From [5, 13.4 and 13.7] we find that $k < \frac{1}{2}n$ and G_Δ^Δ is primitive on Δ .

Moreover, if $g \in G$ is chosen so that $\Gamma^g \neq \Gamma$ and $|\Gamma^g \cup \Gamma|$ is as small as possible, then $B = \Gamma^g \cup \Gamma - \Gamma$ is a block of imprimitivity for $G(\Delta \cap \Delta^g)$. Because G is not doubly primitive we have $|B| \geq 2$. Now X^g has a subgroup which acts transitively on B , so G_Δ^Δ satisfies the same hypotheses as G .

Proof of Theorem A. Let α be an element of $\Delta \cap \Delta^g$. Since $X \subseteq G_\alpha$

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and $|\Gamma| > \frac{1}{2}n$ it follows that $\Delta - \{\alpha\}$ is a union of blocks of imprimitivity for G_α. By induction G_Δ^Δ is $(|\Delta \cap \Delta^g|+1)$-fold transitive, so if $\Delta \cap \Delta^g \neq \{\alpha\}$, then $G_{\Delta,\alpha}^\Delta$ is primitive on $\Delta - \{\alpha\}$; hence this latter set constitutes a single block of imprimitivity for G_α. Thus for all $\beta \in \Delta - \{\alpha\}$ we have $G_{\alpha\beta} \subseteq G_\Delta$. But G_Δ^Δ is doubly transitive on Δ, so $G_{\alpha\beta}$ is transitive on the blocks of $\mathcal{D}$ containing α and β, contradicting $\Delta \cap \Delta^g \neq \{\alpha\}$. It follows that $\Delta \cap \Delta^g = \{\alpha\}$ and hence $\lambda = 1$.

Since X_{Δ^g} is transitive on $\Gamma \cap \Delta^g$, it follows that X_{Δ^g} is the unique subgroup of X of order $k - 1$. Now choose $\gamma \in \Gamma \cap \Delta^g$, $\beta \in \Delta - \{\alpha\}$ such that $\{\beta, \gamma\} \subseteq \Delta$. Then $X_{\Delta^g} = X_{\Delta^h}$, yet $X_{\Delta^g} \cap X_{\Delta^h} \subseteq X_\gamma = 1$. This contradiction completes the proof of Theorem A.

Proof of Theorem B. Proceeding as in Theorem A we find that either $\mathcal{D}$ is a 2-design with $\lambda = 1$ or else G_Δ^Δ is not doubly primitive.

Suppose that G_Δ^Δ is not doubly primitive. By induction $k = 7$, $G_\Delta^\Delta \simeq \mathrm{PSL}(3, 2)$, and the sets B^x, $x \in G_\Delta^\Delta$, form a 2-design which is the 7-point plane. From the Veblen and Young axioms [4], it is easily verified that the 2-design with blocks B^x, $x \in G$, is a projective geometry over $\mathrm{GF}(2)$. However, none of the groups $\mathrm{PGL}(d, 2)$, $d \geq 4$, satisfy the hypotheses of the theorem so this case cannot arise.

It follows that $\lambda = 1$ and as before, if Δ_1 is a block of $\mathcal{D}$ meeting Δ in α, then $|X_{\Delta_1}| = k - 1$. If Δ_2 is a block meeting Δ in $\beta \neq \alpha$ and meeting Δ_1 in γ, then $X_{\Delta_1} \cap X_{\Delta_2} \subseteq X_\gamma = 1$ since X acts regularly on Γ. As X contains a cyclic subgroup of index 2, we must have $k = 3$. Let t be the number of involutions in X and suppose that X has order $2m$. If $x \in X$ is an involution and γ and δ are points of Ω interchanged by x, then the third point of the block through γ and δ is fixed by x and hence belongs to Δ. Each point of Δ is in m blocks other than Δ and each such block is fixed by a unique involution of X. Thus $3m = mt$ and hence $t = 3$. Therefore, if x and y are involutions of X and $D = \langle x, y \rangle$, then D is either a Klein four-group or a dihedral group of order 6. If Γ' is an orbit of D in Γ, then $\Delta \cup \Gamma'$ is a subspace of $\mathcal{D}$ and the design is either a projective geometry over GF(2) or an affine geometry over GF(3) (see Hall [1]). Thus G is a subgroup of $\mathrm{PGL}(d, 2)$ or $\mathrm{AGL}(d, 3)$ for some d. The only examples which satisfy the hypotheses of the theorem are PGL(2, 2), PGL(3, 2), and AGL(2, 3), and this completes the proof.

References

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