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RESEARCH ARTICLE

An application of natural matrices to the dynamic balance problem of planar parallel manipulators

Jaime Gallardo-Alvarado

Department of Mechanical Engineering, National Technological Institute of Mexico/Celaya Campus, Celaya, GTO, Mexico
Email: jaime.gallardo@itcelaya.edu.mx

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Abstract
This paper introduces a simplified matrix method for balancing forces and moments in planar parallel manipulators. The method resorts to Newton's second law and the concept of angular momentum vector, yet it is not necessary to perform the velocity and acceleration analyses, tasks that were normally unavoidable in seminal contributions. With the introduction of natural matrices, the proposed balancing method is independent of the time and the trajectory generated by the moving links of parallel manipulators. The effectiveness of the method is exemplified by balancing two planar parallel manipulators.

1. Introduction
A reactionless mechanism is one that does not exert reaction forces and moments other than gravity on its base regardless of the time and the trajectory of the mechanism. A mechanism like this is free of vibrations minimizing the noise, fatigue and wear induced by the moving links increasing its dynamic performance despite maximum speeds and accelerations [1]. The advantages of having mechanisms with these characteristics have not gone unnoticed by researchers and so the topic has been of growing interest for more than half a century, focusing in its beginnings on the balancing of planar mechanisms, see for instance [2–8]. In spite of the considerable number of contributions in the area, the topic of balancing mechanisms does not lose its relevance and is continuously enriched with new contributions, improving existing methods or proposing novel theories. Maddahi et al [9] introduced a controller for an inertial mobile vehicle using the Lyapunov's feedback control method to balance the mechanism. Meijaard and van der Wijk [10] approached the balancing of four-bar mechanisms by resorting to the theory of principal vectors. In a previous contribution, van der Wijk et al [11] applied the same theory to redundant planar parallel manipulators. Arakelian et al [12] developed a balanced Scotch yoke mechanisms endowed with a linear spring. Franco et al [13] introduced a method for partial-static balancing of four-bar linkages where torsion springs are commanded to adjust the stiffness of the model. Yao and Yang [14] proved that using non-circular gears is a viable option to reduce the fluctuations of the driving torque in planar linkages. Van der Wijk et al [15] report an approach based on the inverse dynamics for balancing a four-degree-of-freedom redundant planar parallel manipulator. Woo et al [16] introduced an algorithm to optimize the torques of the kinematic pairs of a redundant planar parallel manipulator with the virtue of being free of torque saturation. Baron et al [17] optimized the number of counter rotary links to balance kinematically redundant parallel manipulators by resorting to parallelogram-like closed chains. In short, the balancing of mechanism is a necessary task to generate robot manipulators operating at high speeds [18–22].

This work is devoted to the dynamic balance of planar parallel manipulators. The rest of the contribution is organized as follows. Section 2 reports the notation of variables associated with the closure equations of a closed kinematic chain and their inclusion in what in the contribution are called *natural*

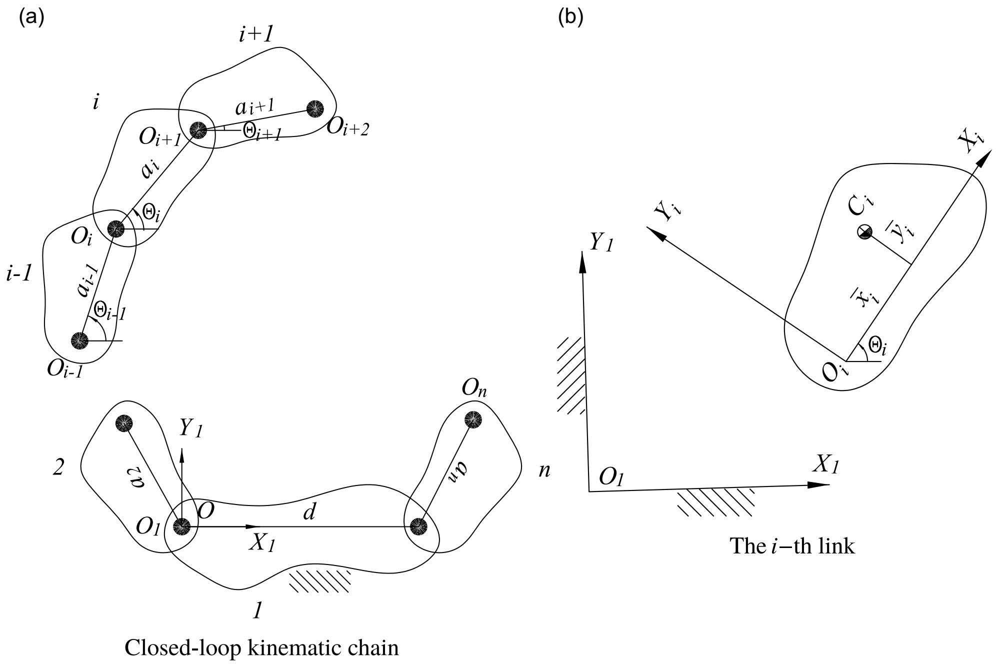

Figure 1. *Introductory notation.*

matrices. In section 3, the force balance theory of planar parallel manipulators using natural matrices is developed. The theory relies on Newton's second law, and however, it is not necessary to perform the acceleration analysis, a somewhat tedious task in most parallel manipulators. In section 4, the conditions for the moment balance of planar parallel manipulators are elucidated. The method includes the moment of inertia of the moving links as it is based on angular momentum vector, and however, similar to the force balancing, it does not require the calculation of the angular momentum vector. In this case, a natural skew-symmetric matrix is obtained which is associated with the angular velocity vector of the links. For clarity, in section 5 the method is applied to the dynamic balance of two parallel manipulators. Finally, some conclusions are given at the end of the contribution.

2. Natural matrix

Figure 1 shows a closed kinematic chain where the links are serially connected through helical pairs ${}^{i-1}\$^{i}$. Unless otherwise specified, the remainder of the contribution is considered $i = 1 \sim n$. In that regard, in planar mechanisms only zero and infinite pitch screw exist. There, 1 denotes the fixed link while the moving links are labelled $2 \sim n$ clockwise starting from the base link 1. The $i-$th link has a mass m_i and a moment of inertia J_i about the axis passing through the centre of mass C_i and normal to the $XY-$plane. The degree-of-freedom F of the linkage is the number of generalized coordinates $q_i (i = 1 \sim F)$ grouped in the vector $\boldsymbol{q} = \begin{bmatrix} q_1 & q_2 \ldots q_F \end{bmatrix}^T$.

More of the notation of the closed kinematic chain of Figure 2 is as follows. Attached to the base link 1, there is a reference frame $O_1X_1Y_1$ whose origin O is located at the nominal position of one of the kinematic pairs of body 1. Then, the orientation of the moving links is notated by angles θ_i measured ccw from the $X-$axis. Furthermore, the moving links have nominal lengths a_i while the fixed link is characterized by a length a_0. The inclusion of natural coordinates $(\bar{x}_i, \bar{y}_i)$ requires the introduction of reference frames $O_iX_iY_i$ attached to the moving links. Without loss of generality, the $i-$th X_i-axis is placed between the axes of the two helical pairs of the corresponding link. Thus, the coordinates of the centre of mass of each link are notated as $C_i = (\bar{x}_i, \bar{y}_i)$ expressed in the corresponding reference frame.

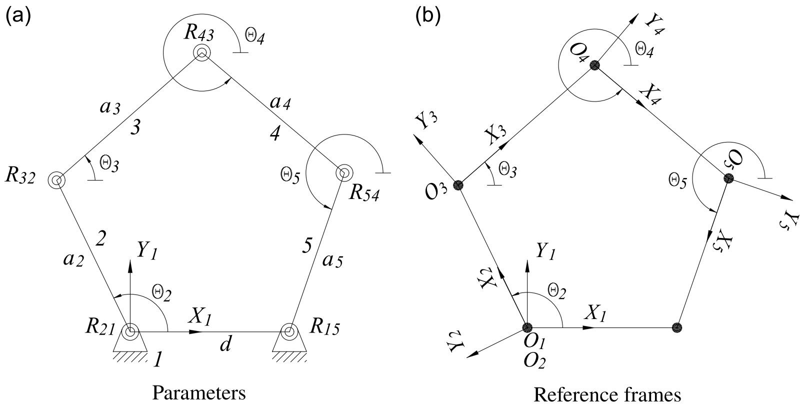

Figure 2. *The five-bar linkage.*

To generate the balancing conditions of the mechanism, it is required to express the coordinates of points C_i in the reference frame $O_1X_1Y_1$. To this end, using the rotation matrix R_i between the $i-$th link and the base link 0 we have that

$$\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} = \begin{bmatrix} \mathrm{R}_i & \boldsymbol{r}_{O_i/O} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{x}_i \\ \bar{y}_i \\ 1 \end{bmatrix} \tag{1}$$

where $\boldsymbol{r}_{O_i/O}$ is the position vector of the origin O_i with respect to the origin O.

The use of reduced sets of variables associated with the mechanism closure equations plays a central role in the contribution. For example, given the closed kinematic chain nature of the mechanism, it is possible to write a closed-loop equation as follows

$$\boldsymbol{a}_2 + \boldsymbol{a}_3 + \cdots + \boldsymbol{a}_n = \boldsymbol{d} \tag{2}$$

Thus, considering the vectorial decomposition of Eq. (2), a matrix U of variables may be defined as

$$\mathrm{U} = \begin{bmatrix} \cos\theta_2 & \cos\theta_3 & \ldots & \cos\theta_n & \sin\theta_2 & \sin\theta_3 & \ldots & \sin\theta_n \end{bmatrix}^T \tag{3}$$

However, since from Eq. (2) two linear equations are available, then we have the opportunity to cancel two variables, notated as w_1 and w_2, of U. Thereafter, two matrices V and W are defined as

$$\mathrm{V} = \begin{bmatrix} v_1 & v_2 \cdots & v_{n-2} \end{bmatrix}^T, \quad \mathrm{W} = \begin{bmatrix} w_1 & w_2 \end{bmatrix}^T \tag{4}$$

where $\mathrm{U} = \begin{bmatrix} \mathrm{V} \\ \mathrm{W} \end{bmatrix}$. Afterwards, Eq. (2) may be rewritten as

$$\mathrm{AV} + \mathrm{BW} = \mathrm{C} \tag{5}$$

where $\mathrm{C} = \begin{bmatrix} d \\ 0 \end{bmatrix}$, while A and B are matrices formed with the length links of the closed kinematic chain. Finally, matrix W is computed as

$$\mathrm{W} = \mathrm{B}^{-1}(\mathrm{C} - \mathrm{AV}) \tag{6}$$

Hereafter, w_1 and w_2 are diluted in the rest of the variables simplifying the balancing of the mechanism.

Using matrix V and Eq. (1), the coordinates $C_i = (x_i, y_i)$ of the centre of mass of the $i-$th link may be expressed as

$$x_i = g_i + \mathbb{X}_i \mathrm{V}, \quad y_i = h_i + \mathbb{Y}_i \mathrm{V} \tag{7}$$

where g_i and h_i are obtained upon the parameters of the closed kinematic chain, that is, g_i and h_i are generated with the lengths a_i and the local coordinates $\bar{x}_i$ and $\bar{y}_i$. Meanwhile, $\mathbb{X}_i$ and $\mathbb{Y}_i$ are matrices formed also with the parameters of the links and the natural coordinates of their centres of mass. Matrices $\mathbb{X}_i$ and $\mathbb{Y}_i$, named in the contribution as *natural matrices*, remain unchanged over time and therefore are considered as properties of the links.

The concepts presented in this section, especially those concerning natural matrices, even when limited to a closed kinematic chain, are so simple that they can be effortlessly extended to more complex mechanisms, such as parallel manipulators.

3. Shaking force balancing

The shaking force balancing of the mechanism is based on the method recently introduced by Gallardo-Alvarado [23]. The inclusion of this section is for the sake of completeness and for a better understanding of the rest of the contribution. The shaking force balancing of mechanisms consists of cancelling the forces transmitted by the moving links to the base link. The shaking force balancing problem is formulated as follows: Given the mass and geometric parameters of the links of the mechanism, it is necessary to determine the natural coordinates of the centres of mass of the moving links that allow the fluctuating forces to vanish regardless of the instant of time and the trajectory generated by the moving links. In this section, it is shown how this important condition may be satisfied by resorting to Newton's second law but without achieving the acceleration analysis of the mechanism.

Since g_i and h_i, see Eq. (7), are invariant with respect to the time t then the force $\boldsymbol{f}_i$ of the $i-$th link may be written according to Newton's second law as follows

$$\boldsymbol{f}_i = \frac{d^2}{dt^2}\left[m_i\mathbb{X}_i\mathrm{V}\hat{\boldsymbol{i}} + m_i\mathbb{Y}_i\mathrm{V}\hat{\boldsymbol{j}}\right] \tag{8}$$

Expression (8) allows to compute the forces $\boldsymbol{f}_i$ for each link of the closed kinematic chain. However, the purpose of the balancing is to cancel the resultant forces of the moving links on the base link. That is to say

$$\sum_{i=2}^{n}\boldsymbol{f}_i = \frac{d^2}{dt^2}\sum_{i=2}^{n}\left[m_i\mathbb{X}_i\mathrm{V}\hat{\boldsymbol{i}} + m_i\mathbb{Y}_i\mathrm{V}\hat{\boldsymbol{j}}\right] = \mathbf{0} \tag{9}$$

Hence, the shaking force balancing condition of the mechanism is established as follows

$$\sum_{i=2}^{n}\left[m_i\mathbb{X}_i\mathrm{V}\hat{\boldsymbol{i}} + m_i\mathbb{Y}_i\mathrm{V}\hat{\boldsymbol{j}}\right] = \mathbf{0} \tag{10}$$

It is therefore somewhat curious that the balancing equation (10) is obtained by resorting to Newton's second law and yet the calculation of the time derivatives of the functions describing the motion of the links is omitted, which could be considered an ambiguity. However, this apparent contradiction makes sense if one takes into account that Eq. (9) is equal to the vector zero. On the other hand, considering that in general matrix V is different to the null matrix, then to cancel the fluctuating forces of the mechanism it must satisfy that

$$\sum_{i=2}^{n} m_i\,\mathbb{X}_i = \mathbb{O}_X \tag{11}$$

and

$$\sum_{i=2}^{n} m_i\,\mathbb{Y}_i = \mathbb{O}_Y \tag{12}$$

It is evident that the force balancing is complete, perfect, if the matrices $\mathbb{O}_X$ and $\mathbb{O}_Y$ are both the zero matrix $\mathbb{O}$. Otherwise, the manipulator is partially balanced from the point of view of forces.

Expressions (11) and (12) yield the conditions to obtain a planar force balanced parallel manipulator. However, as the balancing problem was defined, the number of variables exceeds the number of available equations (11) and (12), so that, for the application of the method, it is necessary to assign values to a sufficient number of variables $\bar{x}$, $\bar{y}$. Afterwards, the natural coordinates of the centres of mass of the moving links may be calculated. It is interesting to emphasize that the force balancing method resorts to Newton's second law and yet it is not necessary to perform the acceleration analysis. Furthermore, these equations are composed only of the parameters and natural coordinates of the centres of mass of the moving links and are available regardless of the trajectory assigned to the manipulator or its degrees-of-freedom. Finally, it is interesting to note that the balancing of forces does not require the inclusion of additional links, since with a simple redistribution of the masses of the links it is theoretically possible according to Eqs. (11) and (12) to cancel the shaking forces generated by the moving links on the base link.

4. Shaking moment balancing

A mechanism that is designed without taking into account the balancing of forces may face adverse conditions, especially if it operates at high speeds. Thus, force balancing is a highly advisable task in the design process of robot manipulators [24]. In that concern, the moment balancing can certainly also contribute to the optimal design of mechanisms. The shaking moment balancing problem is stated as follows: Given the inertial properties and geometric parameters of the links of the mechanism, it is necessary to determine the natural coordinates of the centres of mass of the moving links that allow the fluctuating moments to vanish regardless of the instant of time and the trajectory generated by the moving links. This section addresses the issue of moment balancing in parallel manipulators by resorting to the concept of angular momentum vector.

The angular momentum vector $\boldsymbol{L}_i$ of the $i-$th link is defined as the combination of the moment produced by the linear momentum vector $\boldsymbol{p}_i$ about a reference pole, usually the origin of the fixed reference frame, and the effect of the corresponding moment of inertia J_i with the angular velocity $\dot{\theta}_i$. In that sense, the angular momentum vector has direction and magnitude, and both are conserved. The angular momentum vector is computed as

$$\boldsymbol{L}_i = \boldsymbol{r}_i \times m_i \boldsymbol{v}_i + J_i \dot{\theta}_i \hat{\boldsymbol{k}} \tag{13}$$

where $\boldsymbol{r}_i = x_i\hat{\boldsymbol{i}} + y_i\hat{\boldsymbol{j}}$ is the position vector of the centre of mass C_i while $\boldsymbol{v}_i$ is the velocity vector of point C_i. Furthermore, J_i is the moment of inertia of the $i-$th link about the axis passing through the corresponding centre of mass C_i. Note that the angular momentum vector $\boldsymbol{L}_i$ depends on the chosen origin. On the other hand, it is evident that the velocity vector $\boldsymbol{v}_i$ may be expressed as

$$\boldsymbol{v}_i = \dot{x}_i\hat{\boldsymbol{i}} + \dot{y}_i\hat{\boldsymbol{j}} \tag{14}$$

Assuming that $\boldsymbol{q}$ is the vector of generalized coordinates, then according to the concept of first-order influence coefficient [25, 26] it follows that

$$\dot{x}_i = \frac{\partial x_i}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}}, \quad \dot{y}_i = \frac{\partial y_i}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}}, \quad \dot{\theta}_i = \frac{\partial \theta_i}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} \tag{15}$$

where $\dot{\boldsymbol{q}}$ is the vector of generalized speeds. On the other hand, $\dot{\theta}_i$ can be written as

$$\dot{\theta}_i = \left[\cos\theta_i \frac{\partial \sin\theta_i}{\partial \boldsymbol{q}} - \sin\theta_i \frac{\partial \cos\theta_i}{\partial \boldsymbol{q}}\right]\dot{\boldsymbol{q}} \tag{16}$$

Thus,

$$\frac{\partial \theta_i}{\partial \boldsymbol{q}} = \cos\theta_i \frac{\partial \sin\theta_i}{\partial \boldsymbol{q}} - \sin\theta_i \frac{\partial \cos\theta_i}{\partial \boldsymbol{q}} \tag{17}$$

The angular momentum vector is directly related with the balancing of mechanical systems since the total angular momentum vector of a closed system remains constant. The shaking moment

$\boldsymbol{M}_i$ transmitted by the $i-$th moving link to the base link, see Schiehlen and Eberhard [27], is given by

$$\boldsymbol{M}_i = \frac{d}{dt}\boldsymbol{L}_i \tag{18}$$

Based on Eqs. (13)-(15) it follows that

$$\boldsymbol{M}_i = \frac{d}{dt}\left(m_i x_i \frac{\partial y_i}{\partial \boldsymbol{q}} - m_i y_i \frac{\partial x_i}{\partial \boldsymbol{q}} + J_i \frac{\partial \theta_i}{\partial \boldsymbol{q}}\right)\dot{\boldsymbol{q}} \tag{19}$$

On the other hand, the substitution of Eqs. (7) and (17) into Eq. (19) leads to

$$\boldsymbol{M}_i = \left\{m_i\left(g_i\mathbb{Y}_i - h_i\mathbb{X}_i\right)\frac{\partial \mathrm{V}}{\partial \boldsymbol{q}} + \mathrm{V}^T\left[m_i\left(\mathbb{X}_i\mathbb{Y}_i^T - \mathbb{Y}_i\mathbb{X}_i^T\right) + J_i\mathbb{Z}_i\right]\frac{\partial \mathrm{V}}{\partial \boldsymbol{q}}\right\}\dot{\boldsymbol{q}} \tag{20}$$

where

$$\mathbb{Z}_i = \begin{bmatrix} \mathrm{O} & -\mathrm{Z}_i \\ \mathrm{Z}_i & \mathrm{O} \end{bmatrix} \tag{21}$$

is a skew-symmetric matrix formed with the parameters of the links. Similar to $\mathbb{X}_i$ and $\mathbb{Y}_i$, $\mathbb{Z}_i$ is named also a natural matrix of the $i-$th link. Thereafter, the shaking moments induced by the moving links to the base link are nullified as long as the following condition is satisfied

$$\sum_{i=2}^{n}\left\{m_i\left(g_i\mathbb{Y}_i - h_i\mathbb{X}_i\right)\frac{\partial \mathrm{V}}{\partial \boldsymbol{q}} + \mathrm{V}^T\left[m_i\left(\mathbb{X}_i\mathbb{Y}_i^T - \mathbb{Y}_i\mathbb{X}_i^T\right) + J_i\mathbb{Z}_i\right]\frac{\partial \mathrm{V}}{\partial \boldsymbol{q}}\right\}\dot{\boldsymbol{q}} = \boldsymbol{0} \tag{22}$$

To obtain the shaking moment balance conditions for the closed kinematic chain note that vanishing the vector of generalized speeds $\dot{\boldsymbol{q}}$ lead to a triviality and this option is therefore immediately discarded. The only way to satisfy Eq. (22) is by reviewing matrix V and the partial derivative $\frac{\partial \mathrm{V}}{\partial \boldsymbol{q}}$. From the force balancing analysis, it was deduced that V is different from the zero matrix so it is not possible to cancel $\frac{\partial \mathrm{V}}{\partial \boldsymbol{q}}$ either. With this in mind, it follows that the shaking moment balancing conditions for the closed kinematic chain are given by

$$\sum_{i=2}^{n} m_i\left(g_i\mathbb{Y}_i - h_i\mathbb{X}_i\right) = \mathbb{O}_M \tag{23}$$

and

$$\sum_{i=2}^{n}\left[m_i\left(\mathbb{X}_i\mathbb{Y}_i^T - \mathbb{Y}_i\mathbb{X}_i^T\right) + J_i\mathbb{Z}_i\right] = \mathbb{O}_J \tag{24}$$

It is evident that the moment balancing is complete, perfect, if the matrices $\mathbb{O}_M$ and $\mathbb{O}_J$ are both the zero matrix. Otherwise, the manipulator is partially balanced from the point of view of moments. Furthermore, as with Eqs. (11) and (12), Eq. (23) dispenses with the moment of inertia J and can therefore be used to complete the equations required in the force balancing of the mechanism. On the contrary, Eq. (24) necessarily involves the moment of inertia J which presupposes a complete force and moment balancing of the mechanism. In that sense, having the balancing equations is not a guarantee that the balancing of the mechanism is carried out by completely eliminating the shaking moments, and so it is sometimes necessary to resort to alternative complementary strategies, for example by modifying the original topology of the mechanism with additional links. Finally, with Eqs. (23) and (24), together with Eqs. (11) and (12), the range of possibilities for balancing planar parallel manipulators is substantially increased improving the performance of these popular manipulators. However, the cancellation of moments generated by the inertia properties of the moving links requires the inclusion of additional links whose moments of inertia counteract these shaking moments. Although there are no restrictions on the number of additional links, it is evident that they can increase the torque of the driving links. Therefore it is necessary to find a balance between the number of additional links and the cancellation of the shaking moments. In other words, the cost/benefit of shaking moment cancellation by including additional links may be questionable in some cases even if a perfect balance is achieved. For clarity, in what follows the proposed method is applied to the dynamic balance of three parallel manipulators.

5. Applications

To exemplify the balancing method, in this section the method is applied to the dynamic balancing of two parallel manipulators with widely different kinematic characteristics: the five-bar mechanism and a non-redundant four-degree-of-freedom manipulator with configurable platform. For the sake of completeness, and as a necessary step to complete the number of equations required for the moment balancing in each case, the shaking force balancing of the complex linkages is also included.

5.1. Example 1: five-bar mechanism

Despite having only two degrees-of-freedom, the popular five-bar mechanism has been useful in a variety of applications ranging from prosthetics to haptic feedback, and other applications requiring precise control of movement. Probably, its first application in a mechanism with a high technological development is related to the ingenious Antikythera mechanism [28]. Tong [29] developed a high-stiffness five-bar linkage where the synthesis is performed by using the concept of orthogonal paths. Ouyang et al [30] designed a compliant mechanical amplifier by resorting to the five-bar topology. Ting et al [31] solved the full rotability of the five-bar mechanism by introducing gears regardless of which kinematic pairs play the role of generalized coordinates. Joubair et al [32] implemented a specialized calibration procedure for reconfigurable planar robots. Ruiz-Torres et al [33] proposed a robot named CICABOT formed with two parallel manipulators. Zi et al [34] developed a cable-driven hybrid planar parallel manipulator. Cui et al [35] designed an automatic system for ankle rehabilitation. Essomba and Nguyen [36] developed a remote motion centre with two decoupled angular degrees-of-freedom implemented in a five-bar spatial mechanism. Briot and Goldsztejn [37] introduced a topological optimization methodology for industrial robots which was successfully tested on five-bar mechanisms. Qizhi et al [38] introduced a robot whose main characteristic is to maintain a constant passive force of the mechanism, thus obtaining a simplified control system. Dealing with existing literature approaching the balancing of the five-bar mechanisms, some representative contributions are as follows. Lia and Sinatra [39] optimized the lengths of the five-bar mechanism by dynamically balancing it. Lecours and Gosselin [40] approached the balancing of the five-bar mechanism considering a range of payloads while de Jong et al [41] introduced a screw-based balancing methodology available for the five-bar mechanism.

Figure 2 shows the topology of the linkage under balancing. The mechanism is formed with four moving links labelled 2, 3, 4, and 5 and the base link labelled 1. The links are serially connected through revolute joints R within which stand out the revolutes R_{21} and R_{15} connecting the kinematic chain to the base link 1. The axes of the revolute joints are mutually parallel limiting the mobility of the mechanism, in fact, only two degree-of-freedom are available for it. An inertial reference frame $O_1X_1Y_1$ with associated unit vectors $\hat{\boldsymbol{i}}\hat{\boldsymbol{j}}\hat{\boldsymbol{k}}$ is attached to the base link with the origin O_1 coincident with R_{21}. The moving link lengths are designated as $a_i(i = 2 \sim 5)$ while their corresponding orientation is designated by angles $\theta_i(i = 2 \sim 5)$ measured ccw from the X_1-axis. To complete the notation, the base link is characterized by a length d.

As an initial step, from the closed kinematic chain condition, it is possible to write a vector loop expression as follows

$$\boldsymbol{a}_2 + \boldsymbol{a}_3 + \boldsymbol{a}_4 + \boldsymbol{a}_5 = \boldsymbol{d} \tag{25}$$

Therefore, the vector decomposition of Eq.(25) leads to two closure equations given by

$$a_2 \cos\theta_2 + a_3 \cos\theta_3 + a_4 \cos\theta_4 + a_5 \cos\theta_5 = d \tag{26a}$$

$$a_2 \sin\theta_2 + a_3 \sin\theta_3 + a_4 \sin\theta_4 + a_5 \sin\theta_5 = 0 \tag{26b}$$

Considering that the trigonometric functions $\cos\theta_i$ and $\sin\theta_i$ are variables no matter if they refer in their case to the same angle θ_i, then the matrix of variables U is defined as follows

$$\mathrm{U} = \begin{bmatrix}\cos\theta_2 & \cos\theta_3 & \cos\theta_4 & \cos\theta_5 & \sin\theta_2 & \sin\theta_3 & \sin\theta_4 & \sin\theta_5\end{bmatrix}^T \tag{27}$$

Moreover, from Eq. (26) two linear equations are available. Choosing $\cos\theta_4$ and $\sin\theta_4$ as the variables w_1 and w_2, then the matrix of reduced variables V and the matrix of suppressed variables W are established as

$$\mathrm{V} = \begin{bmatrix}\cos\theta_2 & \cos\theta_3 & \cos\theta_5 & \sin\theta_2 & \sin\theta_3 & \sin\theta_5\end{bmatrix}^T \tag{28}$$

and

$$\mathrm{W} = \begin{bmatrix}\cos\theta_4 & \sin\theta_4\end{bmatrix}^T \tag{29}$$

To continue with the balancing of the mechanism, it is natural to consider a symmetric five-bar mechanism. For example, it can be assumed that $a_2 = a_5 = a$ and $a_3 = a_4 = b$. With this in mind, matrices A, B and C, see Eq. (5), are obtained as

$$\begin{cases} \mathrm{A} = \begin{bmatrix} a & b & a & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b & a \end{bmatrix} \\ \mathrm{B} = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} \\ \mathrm{C} = \begin{bmatrix} d \\ 0 \end{bmatrix} \end{cases}$$

Thus, by applying Eq. (6) the matrix of suppressed variables W results in

$$\mathrm{W} = \frac{1}{b}\begin{bmatrix} d - av_1 - bv_2 - av_3 \\ -av_4 - bv_5 - av_6 \end{bmatrix} \tag{30}$$

Once the variables $\cos\theta_4$ and $\sin\theta_4$ have been suppressed, the parameters g_i and h_i as well as the $\mathbb{X}_i$, $\mathbb{Y}_i$, and $\mathbb{Z}_i$ natural matrices for each moving link are calculated using the reference frames of the mechanism, see Figure 2a. For brevity, only the parameters and natural matrices of link 4 are provided here

$$\begin{cases} g_4 = \bar{x}_4 d/b, \quad h_4 = \bar{y}_4 d/b \\ \mathbb{X}_4 = \begin{bmatrix} a(1-\bar{x}_4/b) & b - \bar{x}_4 & -a\bar{x}_4/b & a\bar{y}_4/b & \bar{y}_4 & a\bar{y}_4/b \end{bmatrix} \\ \mathbb{Y}_4 = \begin{bmatrix} -a\bar{y}_4/b & -\bar{y}_4 & -a\bar{y}_4/b & a(1-\bar{x}_4/b) & b-\bar{x}_4 & -a\bar{x}_4/b \end{bmatrix} \\ \mathbb{Z}_4 = \frac{1}{b^2}\begin{bmatrix} 0 & 0 & 0 & a^2 & ab & a^2 \\ 0 & 0 & 0 & ab & b^2 & ab \\ 0 & 0 & 0 & a^2 & ab & a^2 \\ -a^2 & -ab & -a^2 & 0 & 0 & 0 \\ -ab & -b^2 & -ab & 0 & 0 & 0 \\ -a^2 & -ab & -a^2 & 0 & 0 & 0 \end{bmatrix} \end{cases} \tag{31}$$

Note that $\mathbb{Z}_4$ is effectively a skew-symmetric matrix, as it occurs with the infinitesimal rotation matrix. Once the natural matrices have been determined, the balancing of forces and moments of the five-bar mechanism can be carried out.

5.1.1. Shaking force balancing of the five-bar mechanism

Based on Eq. (11), six force balancing conditions for the five-bar mechanism are obtained as follows

$$m_2 b\bar{x}_2 + m_3 ab - m_4 a\bar{x}_4 + m_4 ab = 0 \tag{32a}$$

$$m_3 b\bar{x}_3 - m_4 b\bar{x}_4 + m_4 b^2 = 0 \tag{32b}$$

$$-m_4 a\bar{x}_4 + m_5 b\bar{x}_5 - m_5 ab = 0 \tag{32c}$$

$$-m_2 b\bar{y}_2 + m_4 a\bar{y}_4 = 0 \tag{32d}$$

$$-m_3 b\bar{y}_3 + m_4 b\bar{y}_4 = 0 \tag{32e}$$

$$m_4 a\bar{y}_4 - m_5 b\bar{y}_5 = 0 \tag{32f}$$

It is interesting to note that the rank of the system of equations (32) is less than six considering the natural coordinates as variables. This is explained by the decoupled nature of the involved equations. In fact, it is easy to verify that the system of six equations is in reality composed of two independent subsystems:

i) Eqs. (32a)-(32c) form a system of three equations in the unknowns $\bar{x}_i(i = 2 \sim 5)$
ii) Eqs. (32d)-(32f) form a system of three equations in the unknowns $\bar{y}_i(i = 2 \sim 5)$

Therefore, if the balancing of the mechanism is limited to the balancing of forces then it is necessary to assign, perhaps intuitively, values to a pair of variables $\bar{x}$ and $\bar{y}$ and then determine the remaining ones in terms of the chosen pair of variables. Otherwise, to complete the number of equations it is necessary to resort to the moment balancing of the mechanism.

Finally, it is worth mentioning that the same balancing conditions are obtained when expression (12) is applied instead of expression (11).

5.1.2. Shaking moment balancing of the five-bar mechanism

The first part of the moment balancing problem focuses on the linear momentum effect. The application of Eq. (23) yields four conditions for the shaking moment balancing of the mechanism as follows

$$m_4\bar{y}_4 = 0 \tag{33a}$$

$$m_5\bar{y}_5 = 0 \tag{33b}$$

$$\bar{x}_4^2 - b\bar{x}_4 + \bar{y}_4^2 = 0 \tag{33c}$$

$$m_4 a\left(\bar{x}_4^2 + \bar{y}_4^2\right) + m_5 b^2(a - \bar{x}_5) = 0 \tag{33d}$$

With the formulation of Eqs. (33), the dilemma of assigning intuitive values to two variables of the force balance is automatically eliminated. Solving Eqs. (32) and (33) one obtains that the centres of mass of the moving links must be given by

$$C_2 = (-am_3/m_2, 0), \quad C_3 = (0, 0), \quad C_4 = (b, 0), \quad C_5 = (a(m_4 + m_5)/m_5, 0) \tag{34}$$

Condition (34) produces a balanced mechanism meeting expressions (32) and (33). However, the cancellation of the angular momentum vector of the linkage requires that expression (24) must be satisfied. In that concern, the application of Eq. (24) yields four particular conditions as follows

$$m_3 a^2 b^2(m_2 + m_3) + m_2(J_2 b^2 + J_4 a^2) = 0 \tag{35a}$$

$$m_4 a^2 b^2(m_4 + m_5) + m_5(J_4 a^2 + J_5 b^2) = 0 \tag{35b}$$

$$J_3 + J_4 = 0 \tag{35c}$$

$$J_4 = 0 \tag{35d}$$

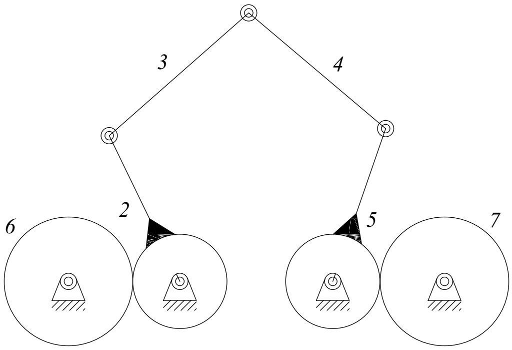

Figure 3. *The five-bar mechanism with additional links and attached spur gears to links 2 and 5.*

The first two conditions require the consideration of negative inertial properties of some links, which must obviously be disregarded. The remaining conditions are valid only if the moments of inertia J of links 3 and 4 are considered insignificant, which undoubtedly leads to an incomplete momentum balance. Thus, to obtain a full shaking moment balance of the mechanism, it is necessary to modify the original topology of the planar linkage. For example, the addition of spur gear pairs, as shown in Figure 3, is a recommended practice [4]. This option does not affect the results obtained for the shaking force balance; however, it does increase the torque required to control the motion of the mechanism.

Considering that the angular velocity ratio between two spur gears is constant, then it is possible to write restrictions credited to the additional links as follows

$$r_2 \frac{\partial \theta_2}{\partial \boldsymbol{q}} + r_6 \frac{\partial \theta_6}{\partial \boldsymbol{q}} = 0, \quad r_5 \frac{\partial \theta_5}{\partial \boldsymbol{q}} + r_7 \frac{\partial \theta_7}{\partial \boldsymbol{q}} = 0 \tag{36}$$

where $r_i (i = 2 \sim 7)$ are the radii of the spur gears. Hence, one obtains the following natural matrices for the additional links 6 and 7 as follows

$$\mathbb{Z}_6 = -\frac{r_2}{r_6}\mathbb{Z}_2, \quad \text{and} \quad \mathbb{Z}_7 = -\frac{r_5}{r_7}\mathbb{Z}_5 \tag{37}$$

With the additional links 6 and 7 by applying Eqs. (24), most of the unrealistic conditions are eliminated. Furthermore, one obtains that the moments of inertia J_6 and J_7 of the additional spur gears must be given by

$$J_6 = r_6 \left[m_3 a^2 b^2 \left(m_2 + m_3 \right) + m_2 (J_2 b^2 + J_4 a^2) \right] / \left(m_2 r_2 b^2 \right) \tag{38}$$

and

$$J_7 = r_7 \left[m_4 a^2 b^2 \left(m_4 + m_5 \right) + m_5 (J_4 a^2 + J_5 b^2) \right] / \left(m_5 r_5 b^2 \right) \tag{39}$$

Despite the progress achieved in balancing the five-bar mechanism, the condition $J_3 = J_4 = 0$ is not eliminated so that links 3 and 4 of the linkage are partially balanced. The links 3 and 4 can be fully balanced by adding gears with rotation motions opposite to the corresponding links. However, such a solution leads to a more complex mechanism and will also require a considerable increase in the torque of the servomotors of the mechanism due to the increase of masses and inertias. On the other hand, according to the calculated moments of inertia J_6 and J_7, the gears 6 and 7 can be chosen in such a way that the performance of the mechanism is not compromised since, according to Eqs. (38)-(39), it is suggested relationships like $r_6/r_2 < 1$ and $r_7/r_5 < 1$. For clarity, Figure 4 conceptually summarizes the shaking force and shaking moment balancing of the five-bar linkage using additional links.

Finally, as expected, the symmetry of the moving links imposed to the mechanism is reflected in the symmetry of the balancing results obtained.

Table I. *Parameters of the five-bar mechanism.*

$a = 300$ [mm]	$b = 480.044$ [mm]	$d = 300$ [mm]
$m_2 = m_5 = 2.065$ kg	$J_2 = J_5 = 337.438$ [kg . mm^2]	
$m_3 = m_4 = 3.188$ kg	$J_3 = J_4 = 524.708$ [kg . mm^2]	
$r_2 = r_5 = 400$ [mm]	$r_6 = r_7 = 10$ [mm]	

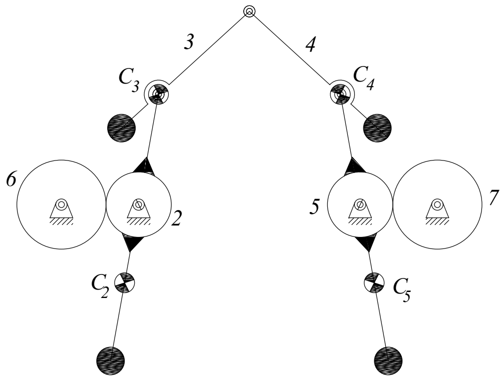

Figure 4. *Balanced five-bar linkage schematic.*

5.1.3. Numerical application for the five-bar mechanism

Let us consider that the five-bar mechanism is composed of links whose parameters are listed in Table I.

According to Eqs. (34) one obtains the following natural coordinates

$$\begin{cases} \bar{x}_2 = -0.463 \text{ [m]}, \quad \bar{x}_3 = 0 \text{ [m]}, \quad \bar{x}_4 = 0.480 \text{ [m]}, \quad \bar{x}_5 = 0.763 \text{ [m]} \\ \bar{y}_2 = \bar{y}_3 = \bar{y}_4 = \bar{y}_5 = 0 \text{ [m]} \end{cases}$$

Afterwards, dealing with the shaking force balancing of the linkage, one obtains that matrices $\mathbb{O}_X$ and $\mathbb{O}_Y$ are both precisely the zero matrix, which ensures that the balance of forces of the five-bar mechanism is complete or fully satisfied, an expected result. On the other hand, to achieve the moment balancing of the mechanism, based on expression (23) the first condition of the moment balancing yields that the matrix $\mathbb{O}_M$ is also the zero matrix. Meanwhile, to compute matrix $\mathbb{O}_J$, the application of Eqs. (38) and (39) yield $J_6 = J_7 = 0.018$ [kg . m^2]. Then, after a lengthy procedure, the second condition of the moment balancing, see expression (24), leads us to the matrix

$$\mathbb{O}_J = \begin{bmatrix} 0. & 0. & 0. & 0. & 0.0003 & 0.0002 \\ 0. & 0. & 0. & 0.0003 & 0.001 & 0.0003 \\ 0. & 0. & 0. & 0.0002 & 0.0003 & 0. \\ 0. & -0.0003 & -0.0002 & 0. & 0. & 0. \\ -0.0003 & -0.0001 & -0.0003 & 0. & 0. & 0. \\ -0.0002 & -0.0003 & 0. & 0. & 0. & 0. \end{bmatrix} \quad (40)$$

Finally, since not all the elements of matrix $\mathbb{O}_J$ vanish, then the five-bar mechanism is partially balanced from the point of view of moment balancing. However, matrix $\mathbb{O}_J$ is close to the zero matrix so the results obtained are, in reality, reasonably acceptable.

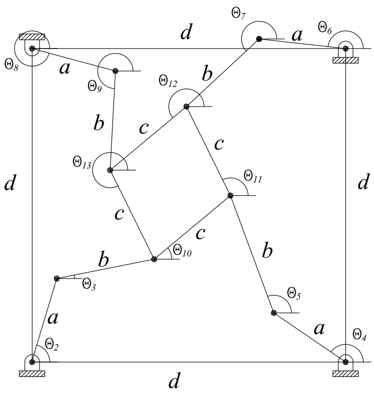

Figure 5. *Configurable parallel manipulator.*

5.2. Example 2: configurable parallel manipulator

Parallel manipulators with configurable platforms are mechanisms where the typical rigid moving platform is replaced with a moving platform able to modify its contour improving the dexterity of the parallel manipulator [42–44]. Usually, the configurable platform is a closed kinematic chain with at least one internal degree-of-freedom. In this section, the balancing method is applicable to the configurable parallel manipulator introduced by Gallardo-Alvarado et al [45].

The 4-RRR redundant planar parallel manipulator consists of a rigid moving platform shaping a quadrilateral connected to the fixed platform by means of four RRR-type limbs [46–49]. If the rigid quadrilateral is replaced by a closed articulated kinematic chain, see Figure 5, then the redundant actuation of the manipulator and the internal forces generated by it are ameliorated, yet the four degrees-of-freedom of the mechanism are preserved thanks to the newest internal degree-of-freedom of the configurable platform. The complexity of the mechanism is certainly interesting to test the effectiveness of the balancing method. According to four closed loops of the mechanism, eight closure equations may be written as follows

$$a\cos\theta_2 + b\cos\theta_3 + c\cos\theta_{10} - a\cos\theta_4 - b\cos\theta_5 = d \tag{41a}$$

$$a\sin\theta_2 + b\sin\theta_3 + c\sin\theta_{10} - a\sin\theta_4 - b\sin\theta_5 = 0 \tag{41b}$$

$$a\cos\theta_4 + b\cos\theta_5 + c\cos\theta_{11} - a\cos\theta_6 - b\cos\theta_7 = 0 \tag{41c}$$

$$a\sin\theta_4 + b\sin\theta_5 + c\sin\theta_{11} - a\sin\theta_6 - b\sin\theta_7 = d \tag{41d}$$

$$a\cos\theta_6 + b\cos\theta_7 + c\cos\theta_{12} - a\cos\theta_8 - b\cos\theta_9 = -d \tag{41e}$$

$$a\sin\theta_6 + b\sin\theta_7 + c\sin\theta_{12} - a\sin\theta_8 - b\sin\theta_9 = 0 \tag{41f}$$

$$a\cos\theta_8 + b\cos\theta_9 + c\cos\theta_{13} - a\cos\theta_2 - b\cos\theta_3 = 0 \tag{41g}$$

$$a\sin\theta_8 + b\sin\theta_9 + c\sin\theta_{13} - a\sin\theta_2 - b\sin\theta_3 = -d \tag{41h}$$

The configurable platform consists of four articulated links, and eight equations associated with the position analysis of the mechanism are available, see Eqs. (41). It is therefore logical to choose the variables associated with the links of the configurable platform as the variables to be suppressed. Under this consideration, the matrices V and W are defined as

$$\mathrm{V} = [\cos\theta_2 \quad \cos\theta_3 \quad \ldots \quad \cos\theta_8 \quad \cos\theta_9 \quad \sin\theta_2 \quad \sin\theta_3 \quad \ldots \quad \sin\theta_8 \quad \sin\theta_9]^T \tag{42}$$

and

$$\mathrm{W} = \begin{bmatrix} \cos\theta_{10} & \cos\theta_{11} & \cos\theta_{12} & \cos\theta_{13} & \sin\theta_{10} & \sin\theta_{11} & \sin\theta_{12} & \sin\theta_{13} \end{bmatrix}^T \tag{43}$$

Thereafter, to meet Eq. (5) it follows that

$$\mathrm{A} = \begin{bmatrix} a & a & b & -a & -b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & b & -a & -b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b & -a & -b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & b & -a & -b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & b & -a & -b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & b & -a & -b \\ -a & -a & -b & 0 & 0 & 0 & 0 & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a & -b & 0 & 0 & 0 & 0 & a & b \end{bmatrix} \tag{44}$$

while

$$\mathrm{B} = \begin{bmatrix} c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c \end{bmatrix} \tag{45}$$

and

$$\mathrm{C} = \begin{bmatrix} d & 0 & 0 & d & -d & 0 & 0 & -d \end{bmatrix}^T \tag{46}$$

After a few computations, matrix W is obtained in terms of the variables $v_i (i = 1 \sim 16)$ as follows

$$\mathrm{W} = \frac{1}{c} \begin{bmatrix} -av_1 - bv_2 + d + av_3 + bv_4 \\ -av_3 - bv_4 + av_5 + bv_6 \\ -d - av_5 - bv_6 + av_7 + bv_8 \\ -av_7 - bv_8 + av_1 + bv_2 \\ -av_9 - bv_{10} + av_{11} + bv_{12} \\ -av_{11} - bv_{12} + d + av_{13} + bv_{14} \\ -av_{13} - bv_{14} + av_{15} + bv_{16} \\ -d - av_{15} - bv_{16} + av_9 + bv_{10} \end{bmatrix} \tag{47}$$

and thus the variables $w_i (i = 1 \sim 8)$ can be suppressed. Once the precise number of variables has been established, the parameters g_i and h_i as well as the natural matrices of each of the moving links are determined using the reference frames of the manipulator, see Figure 6.

For brevity in what follows only the results for link 11 are provided. The parameters result to be as $g_{11} = d(c - \bar{y}_{11})/c$ and $h_{11} = \bar{x}_{11} d/c$. Meanwhile, the natural matrices $\mathbb{X}_{11}$ and $\mathbb{Y}_{11}$ were determined as follows

$$\mathbb{X}_{11} = \begin{bmatrix} 0 & 0 & a(1-\bar{x}_{11}/c) & b(1-\bar{x}_{11}/c) & a\bar{x}_{11}/c & b\bar{x}_{11}/c & 0 & 0 \\ & & 0 & 0 & a\bar{y}_{11}/c & b\bar{y}_{11}/c & -a\bar{y}_{11}/c & -b\bar{y}_{11}/c & 0 & 0 \end{bmatrix} \tag{48}$$

and

$$\mathbb{Y}_{11} = \begin{bmatrix} 0 & 0 & -a\bar{y}_{11}/c & -b\bar{y}_{11}/c & a\bar{y}_{11}/c & b\bar{y}_{11}/c & 0 & 0 \\ 0 & 0 & a(1-\bar{x}_{11}/c) & b(1-\bar{x}_{11}/c) & a\bar{x}_{11}/c & b\bar{x}_{11}/c & 0 & 0 \end{bmatrix} \tag{49}$$

To complete the natural matrices of link 11, the skew-symmetric matrix $\mathbb{Z}_{11}$ results to be a 16×16 matrix composed of the parameters a, b, and c as

$$\mathbb{Z}_{11} = \frac{1}{c^2} \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&a^2&ab&-a^2&-ab&0&0 \\ 0&0&0&0&0&0&0&0&0&0&ab&b^2&-ab&-b^2&0&0 \\ 0&0&0&0&0&0&0&0&0&0&-a^2&-ab&a^2&ab&0&0 \\ 0&0&0&0&0&0&0&0&0&0&-ab&-b^2&ab&b^2&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&-a^2&-ab&a^2&ab&0&0&0&0&0&0&0&0&0&0 \\ 0&0&-ab&-b^2&ab&b^2&0&0&0&0&0&0&0&0&0&0 \\ 0&0&a^2&ab&-a^2&-ab&0&0&0&0&0&0&0&0&0&0 \\ 0&0&ab&b^2&-ab&-b^2&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix} \tag{50}$$

Once the natural matrices have been determined, the configurable parallel manipulator may be balanced.

5.2.1. Shaking force balancing of the configurable parallel manipulator

As can be expected, the complexity of the kinematic analysis of the configurable parallel manipulator translates into a dynamic balancing that requires a systematic procedure given the large number of variables involved in this process. By applying Eq. (11) or Eq. (12), one obtains sixteen force balancing equations as follows

$$cm_2\bar{y}_2 - m_{10}a\bar{y}_{10} + m_{13}a\bar{y}_{13} = 0 \tag{51a}$$

$$cm_3\bar{y}_3 - m_{10}b\bar{y}_{10} + m_{13}b\bar{y}_{13} = 0 \tag{51b}$$

$$cm_4\bar{y}_4 + m_{10}a\bar{y}_{10} - m_{11}a\bar{y}_{11} = 0 \tag{51c}$$

$$cm_5\bar{y}_5 + m_{10}b\bar{y}_{10} - m_{11}b\bar{y}_{11} = 0 \tag{51d}$$

$$cm_6\bar{y}_6 + m_{11}a\bar{y}_{11} - m_{12}a\bar{y}_{12} = 0 \tag{51e}$$

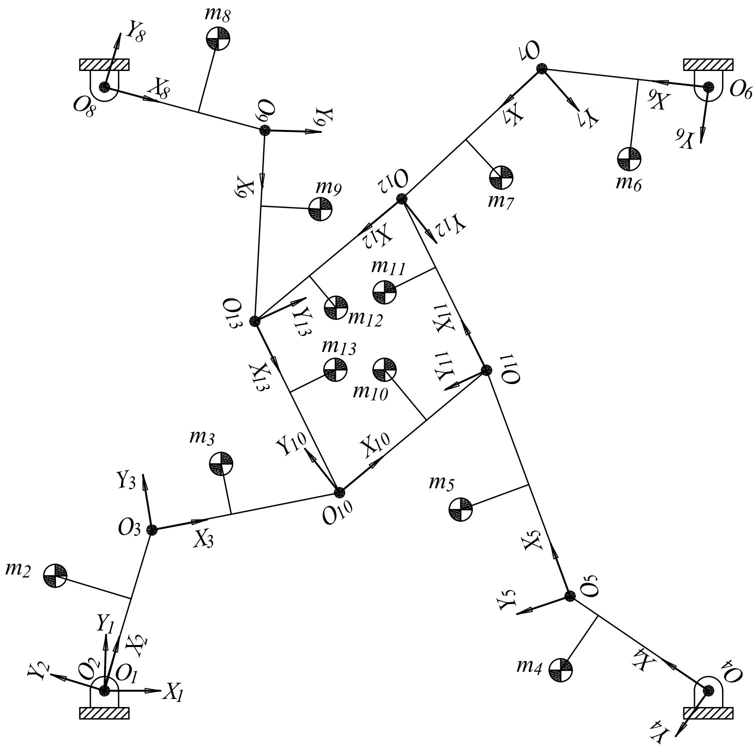

Figure 6. *Configurable parallel manipulator. Reference frames.*

$$cm_7\bar{y}_7 + m_{11}b\bar{y}_{11} - m_{12}b\bar{y}_{12} = 0 \tag{51f}$$

$$cm_8\bar{y}_8 + m_{12}a\bar{y}_{12} - m_{13}a\bar{y}_{13} = 0 \tag{51g}$$

$$cm_9\bar{y}_9 + m_{12}b\bar{y}_{12} - m_{13}b\bar{y}_{13} = 0 \tag{51h}$$

$$cm_2\bar{x}_2 + cm_3a - m_{10}a\bar{x}_{10} + cm_{10}a + m_{13}a\bar{x}_{13} = 0 \tag{51i}$$

$$cm_3\bar{x}_3 - m_{10}b\bar{x}_{10} + cm_{10}b + m_{13}b\bar{x}_{13} = 0 \tag{51j}$$

$$cm_4\bar{x}_4 + cm_5a + m_{10}a\bar{x}_{10} - m_{11}a\bar{x}_{11} + cm_{11}a = 0 \tag{51k}$$

$$cm_5\bar{x}_5 + m_{10}b\bar{x}_{10} - m_{11}b\bar{x}_{11} + cm_{11}b = 0 \tag{51l}$$

$$cm_6\bar{x}_6 + cm_7b + m_{11}a\bar{x}_{11} - m_{12}a\bar{x}_{12} + cm_{12}a = 0 \tag{51m}$$

$$cm_7\bar{x}_7 + m_{11}b\bar{x}_{11} - m_{12}b\bar{x}_{12} + cm_{12}b = 0 \tag{51n}$$

$$cm_8\bar{x}_8 + cm_9a + m_{12}a\bar{x}_{12} - m_{13}a\bar{x}_{13} + cm_{13}a = 0 \tag{51o}$$

$$cm_9\bar{x}_9 + m_{12}b\bar{x}_{12} - m_{13}b\bar{x}_{13} + cm_{13}b = 0 \tag{51p}$$

As with the five-bar mechanism, the force balancing expressions (51) can be separated into two subgroups:

i) Eqs. (51a)-(51h) include $\bar{y}$ coordinates and exclude $\bar{x}$ coordinates

ii) Eqs. (51i)-(51p) include $\bar{x}$ coordinates and exclude $\bar{y}$ coordinates

With these resources at hand, the force balancing problem of the complex linkage may be formulated as follows. Given the mass of the links as well as the parameters of the configurable parallel manipulator, it is necessary to determine the variables $\bar{x}_i, \bar{y}_i (i = 2 \sim 13)$, or natural coordinates, that allow that Eqs. (51) must be fulfilled. It is therefore clear that in order to obtain an accurate and less intuitive force balancing of the configurable parallel manipulator, it is necessary to resort to the moment balancing of the linkage, one of the motives of the next subsection. In other words, at this point in the balancing process there are more variables than available equations.

5.2.2. Shaking moment balancing of the configurable parallel manipulator

Following a logical order, the linear moment balancing is the first to be approached. By applying Eq. (23) one obtains fourteen moment balancing equations as follows

$$m_{10}\bar{y}_{10}c - m_{13}\bar{y}_{13}^2 - m_{13}\bar{x}_{13}^2 + m_{13}\bar{x}_{13}c = 0 \tag{52a}$$

$$m_4\bar{y}_4c^2 + m_{11}a\bar{y}_{11}^2 - m_{11}a\bar{y}_{11}c + m_{11}a\bar{x}_{11}^2 - m_{11}a\bar{x}_{11}c = 0 \tag{52b}$$

$$m_5\bar{y}_5c^2 + m_{11}b\bar{y}_{11}^2 - m_{11}b\bar{y}_{11}c + m_{11}b\bar{x}_{11}^2 - m_{11}b\bar{x}_{11}c = 0 \tag{52c}$$

$$m_6\bar{y}_6c^2 - m_6\bar{x}_6c^2 - m_7ac^2 - m_{11}a\bar{y}_{11}^2 + m_{11}a\bar{y}_{11}c - m_{11}a\bar{x}_{11}^2 + m_{12}ac\bar{x}_{12} - m_{12}ac^2 = 0 \tag{52d}$$

$$m_7\bar{y}_7c^2 - m_7\bar{x}_7c^2 - m_{11}b\bar{y}_{11}^2 + m_{11}b\bar{y}_{11}c - m_{11}b\bar{x}_{11}^2 + m_{12}bc\bar{x}_{12} - m_{12}bc^2 = 0 \tag{52e}$$

$$-m_8\bar{x}_8c^2 - m_9ac^2 + m_{12}a\bar{y}_{12}c - m_{12}ac\bar{x}_{12} - m_{13}\bar{y}_{13}^2a - m_{13}a\bar{x}_{13}^2 + 2m_{13}a\bar{x}_{13}c - m_{13}ac^2 = 0 \tag{52f}$$

$$-m_9\bar{x}_9c^2 + m_{12}b\bar{y}_{12}c - m_{12}bc\bar{x}_{12} - m_{13}\bar{y}_{13}^2b - m_{13}b\bar{x}_{13}^2 + 2m_{13}b\bar{x}_{13}c - m_{13}bc^2 = 0 \tag{52g}$$

$$-m_{10}\bar{x}_{10}^2 + m_{10}\bar{x}_{10}c - m_{10}\bar{y}_{10}^2 + m_{13}\bar{y}_{13}c = 0 \tag{52h}$$

$$m_4\bar{x}_4c^2 + m_5ac^2 + m_{10}\bar{x}_{10}^2a + m_{10}\bar{y}_{10}^2a - m_{11}a\bar{y}_{11}c - m_{11}a\bar{x}_{11}c + m_{11}ac^2 = 0 \tag{52i}$$

$$m_5\bar{x}_5c^2 + m_{10}\bar{x}_{10}^2b + m_{10}\bar{y}_{10}^2b - m_{11}b\bar{y}_{11}c - m_{11}b\bar{x}_{11}c + m_{11}bc^2 = 0 \tag{52j}$$

$$m_6\bar{x}_6c^2 + m_6\bar{y}_6c^2 + m_7bc^2 + m_{11}a\bar{x}_{11}c + m_{12}a\bar{x}_{12}^2 - 2m_{12}ac\bar{x}_{12} + m_{12}ac^2 + m_{12}a\bar{y}_{12}^2 - m_{12}a\bar{y}_{12}c = 0 \tag{52k}$$

$$m_7\bar{x}_7c^2 + m_7\bar{y}_7c^2 + m_{11}b\bar{x}_{11}c + m_{12}b\bar{x}_{12}^2 - 2m_{12}bc\bar{x}_{12} + m_{12}bc^2 + m_{12}b\bar{y}_{12}^2 - m_{12}b\bar{y}_{12}c = 0 \tag{52l}$$

$$m_8\bar{y}_8c^2 - m_{12}a\bar{x}_{12}^2 + m_{12}ac\bar{x}_{12} - m_{12}a\bar{y}_{12}^2 + m_{12}a\bar{y}_{12}c = 0 \tag{52m}$$

$$m_9\bar{y}_9c^2 - m_{12}b\bar{x}_{12}^2 + m_{12}bc\bar{x}_{12} - m_{12}b\bar{y}_{12}^2 + m_{12}b\bar{y}_{12}c = 0 \tag{52n}$$

Owing to the excessive number of variables involved in Eqs. (23), with the experience acquired from balancing the five-bar mechanism, a practical choice is to assume that the configurable parallel manipulator belongs to the inline mechanism class, that is $\bar{y}_i = 0 (i = 2 \sim 13)$. Therefore, with this condition

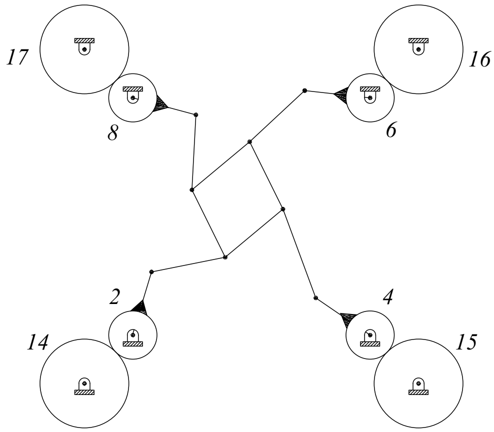

Figure 7. *Configurable parallel manipulator with spur gears as additional links.*

and solving Eqs. (51) and (52) it follows that $b = a$. Furthermore, the natural coordinates of the centres of mass of the moving links result in:

$$\begin{cases} C_2 = (-a(m_3 + m_{13})/m_2, 0), \quad C_3 = (-am_{13}/m_3, 0), \quad C_4 = (-a(m_5 + m_{10})/m_4, 0) \\ C_5 = (-am_{10}/m_5, 0), \quad C_6 = (-a(m_7 + m_{11})/m_6, 0), \quad C_7 = (-am_{11}/m_7, 0) \\ C_8 = (-a(m_9 + m_{12})/m_8, 0), \quad C_9 = (-m_{12}a/m_9, 0), \quad C_{10} = (c, 0) \\ C_{11} = (c, 0), \quad C_{12} = (c, 0), \quad C_{13} = (c, 0) \end{cases} \tag{53}$$

If the inertial properties of the links such as the moment of inertia J are neglected then the balancing conditions so far obtained for the configurable parallel manipulator are sufficient to cancel out the fluctuating forces, and fluctuating moments credited to linear momentums, on the fixed platform. However, the moment balancing cannot be considered complete unless Eq. (24) is satisfied. Certainly, the inertia moments must be incorporated to cancel the fluctuating moments attributed to angular momentum effects. In that concern, like the five-bar mechanism, to satisfy Eq. (24) it is necessary the inclusion of additional links to the configurable parallel manipulator. Consequently, spur gears are added to the original mechanism as shown in Figure 7. At this point, it is worth mentioning that the abuse in the inclusion of additional links that naturally do not contribute to improve the performance of the mechanism from the point of view of concepts such as manipulability or mechanical interference problems, among others, with the objective of obtaining a perfect dynamic balance must be taken into account in order not to compromise the topology and assembly of the links.

With the addition of four spur gears, we dispose of restrictions attributed to the additional links as follows

$$r_2 \frac{\partial \theta_2}{\partial \boldsymbol{q}} + r_{14} \frac{\partial \theta_{14}}{\partial \boldsymbol{q}} = 0, \quad r_4 \frac{\partial \theta_4}{\partial \boldsymbol{q}} + r_{15} \frac{\partial \theta_{15}}{\partial \boldsymbol{q}} = 0 \tag{54}$$

and

$$r_6 \frac{\partial \theta_6}{\partial \boldsymbol{q}} + r_{16} \frac{\partial \theta_{16}}{\partial \boldsymbol{q}} = 0, \quad r_8 \frac{\partial \theta_8}{\partial \boldsymbol{q}} + r_{17} \frac{\partial \theta_{17}}{\partial \boldsymbol{q}} = 0 \tag{55}$$

Hence, the natural matrices $\mathbb{Z}_i (i = 14 \sim 17)$ associated with the additional links maintain relationships as follows

$$\mathbb{Z}_{14} = -\frac{r_2}{r_{14}} \mathbb{Z}_2, \quad \mathbb{Z}_{15} = -\frac{r_4}{r_{15}} \mathbb{Z}_4, \quad \mathbb{Z}_{16} = -\frac{r_6}{r_{16}} \mathbb{Z}_6, \quad \mathbb{Z}_{17} = -\frac{r_8}{r_{17}} \mathbb{Z}_8 \tag{56}$$

Furthermore, since the extra spur gears have stationary centres of mass, then the computation of their natural matrices $\mathbb{X}_i$, $\mathbb{Y}_i (i = 14 \sim 17)$ is unnecessary. On the other hand, one of the benefits of the method

Table II. *Parameters of the configurable parallel manipulator.*

$a = 80$ [mm], $b = c = 120$ [mm], $d = 283.019$ [mm]
$m_2 = m_4 = m_6 = m_8 = 0.297$ kg
$J_2 = J_4 = J_6 = J_8 = 19.434$ [kg . mm^2]
$m_3 = m_5 = m_7 = m_9 = m_{10} = m_{11} = m_{12} = m_{13} = 0.422$ kg
$J_3 = J_5 = J_7 = J_9 = J_{10} = J_{11} = J_{12} = J_{13} = 27.755$ [kg . mm^2]
$r_2 = r_4 = r_6 = r_8 = r_{14} = r_{15} = r_{16} = r_{17} = 40$ [mm]

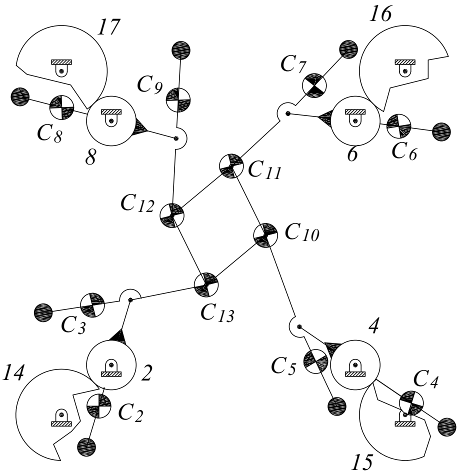

Figure 8. *Balanced configurable parallel manipulator schematic.*

is that the required moment of inertia of the extra spur gears can be determined to minimize the effect of the fluctuating moments attributed to the angular momentum of the moving links. The application of Eqs. (24) yields, among other relevant results, that the moments of inertia $J_i (i = 14 \sim 17)$ of the additional spur gears must be given by

$$J_{14} = r_{14}[a^2c^2(m_3^2 + m_{13}^2 + m_2m_3 + m_2m_{13} + 2m_3m_{13}) + m_2a^2(J_{10} + J_{13}) + m_2J_2c^2]/\left(m_2r_2c^2\right) \quad (57a)$$

$$J_{15} = r_{15}\left[a^2c^2\left(m_5^2 + m_{10}^2 + m_4m_{10} + m_4m_{10} + 2m_5m_{10}\right) + m_4a^2(J_{10} + J_{11}) + m_4J_4c^2\right]/\left(m_4r_4c^2\right) \quad (57b)$$

$$J_{16} = r_{16}\left[a^2c^2\left(m_7^2 + m_{11}^2 + m_6m_7 + m_6m_{11} + 2m_7m_{11}\right) + m_6a^2(J_{11} + J_{12}) + m_6J_6c^2\right]/\left(m_6r_6c^2\right) \quad (57c)$$

$$J_{17} = r_{17}\left[a^2c^2\left(m_9^2 + m_{12}^2 + m_8m_9 + m_8m_{12} + 2m_9m_{12}\right) + m_8a^2(J_{12} + J_{13}) + m_8J_8c^2\right]/\left(m_8r_8c^2\right) \quad (57d)$$

Figure 8 conceptually summarizes the shaking force and shaking moment balancing of the configurable parallel manipulator with additional links.

5.2.3. Numerical application for the configurable parallel manipulator

Let us consider that the configurable parallel manipulator is composed of links whose parameters are listed in Table II. The topology of the robot is such that, where possible, links with the same parameters were selected. Symmetry is always an appreciated feature for robot manipulators.

After a few computations, the matrices $\mathbb{O}_X$ and $\mathbb{O}_Y$ associated with the shaking force balancing result to be the zero matrix, which ensures that the balance of forces of the configurable parallel manipulator is complete or fully satisfied. On the other hand, to perform the moment balancing of the mechanism, based on expression (23), the first condition of the moment balancing yields that the matrix $\mathbb{O}_M$ is also the zero matrix. Meanwhile, to compute matrix $\mathbb{O}_J$, the application of Eqs. (57) yield $J_{14} = J_{15} = J_{16} = J_{17} = 0.0207$ [kg . m^2]. Then, after a lengthy procedure, the second condition of the moment balancing, see expression (24), leads us to a 16 × 16 matrix $\mathbb{O}_J$ as follows

$$\mathbb{O}_J = \begin{bmatrix} 0 & -\mathbb{M} \\ \mathbb{M} & 0 \end{bmatrix} \tag{58}$$

where $\mathbb{M}$ is a symmetric matrix given by

$$\mathbb{M} = \begin{bmatrix} 0.100e-10 & -0.138e-2 & 0.123e-4 & 0.185e-4 & -0.337e-12 & -0.506e-12 & 0.123e-4 & 0.185e-4 \\ -0.138e-2 & -0.886e-2 & 0.185e-4 & 0.277e-4 & 0. & 0. & 0.185e-4 & 0.277e-4 \\ 0.123e-4 & 0.185e-4 & 0. & -0.138e-2 & 0.123e-4 & 0.185e-4 & 0. & 0. \\ 0.185e-4 & 0.277e-4 & -0.138e-2 & -0.886e-2 & 0.185e-4 & 0.277e-4 & 0. & 0. \\ -0.337e-12 & 0. & 0.123e-4 & 0.185e-4 & -0.135e-2 & -0.370e-4 & 0.123e-4 & 0.185e-4 \\ -0.506e-12 & 0. & 0.185e-4 & 0.277e-4 & -0.138e-2 & -0.886e-2 & 0.185e-4 & 0.277e-4 \\ 0.123e-4 & 0.185e-4 & 0. & 0. & 0.123e-4 & 0.185e-4 & 0.900e-11 & -0.138e-2 \\ 0.185e-4 & 0.277e-4 & 0. & 0. & 0.185e-4 & 0.277e-4 & -0.138e-2 & -0.886e-2 \end{bmatrix} \tag{59}$$

Finally, since not all the elements of matrix $\mathbb{O}_J$ vanish, then the configurable parallel manipulator is partially balanced from the point of view of moment balancing. However, matrix $\mathbb{O}_J$ is close to the zero matrix so the results obtained are, in reality, reasonably acceptable.

6. Conclusions

In this work, the concept of natural matrix is introduced with the purpose of obtaining the general symbolic equations for the balancing of forces and moments in planar parallel manipulators. As shown in the contribution, a natural matrix is a property of the rigid body since its constitutive elements are invariant with respect to time and the trajectory generated by the rigid body undergoing an instantaneous change of pose. A natural matrix makes it possible to model physical phenomena in which time, although present, does not intervene in the behaviour of the rigid body, as in the case of the balance of mechanisms. A mechanism is balanced when the forces and moments transmitted by the moving links to the base link vanish, which leads to an improvement in the performance of the mechanism since, among other problems, vibrations, noise and fatigue due to fluctuations in the forces are ameliorated. A balanced mechanism is said to be a reactionless mechanism. In particular, the balancing of parallel manipulators is currently of great interest given the high speeds and stability at which they must operate. For example, the Adept robot is capable of performing 240 pick-and-place operations per minute.

In this contribution, a method for the shaking force balancing and shaking moment balancing devoted to planar parallel manipulators is proposed. The balancing of forces is achieved by resorting to Newton's second law and yet it is not necessary to perform the acceleration analysis, a task that is usually rather tedious in most parallel manipulators. The force balance equations are centred on two natural matrices which arise from the closure equations associated with the position analysis of the linkage. In that sense, the reduction variable plays a central role as well as the coordinate transformation. Moment balancing, on the other hand, is based on the concept of linear and angular momentum vector but with the advantage that it is not necessary to perform the velocity analysis. The moment balancing theory is based on skew-symmetric natural matrices associated with the moving links of the parallel manipulator, which follows the fashion of the force balancing. The method is easy to follow so that non-experts in the matter can find in this contribution a parallel manipulator balancing theory easy to follow that can be implemented in algorithms without major effort, despite the apparent laboriousness of the obtained expressions. The effectiveness of the method is exemplified by balancing two multi-degree-of-freedom planar parallel manipulators. The results show that inline mechanisms are the most suitable option for the balancing of parallel manipulators. Furthermore, the inclusion of additional links, for example spur gears, is an advisable option in the case of the moment balancing.

Finally, a requirement of the method is that the variables of the position analysis must be expressed in sine and cosine functions, which is only possible in mechanisms articulated by revolute joints. Therefore, the method is not applicable to mechanisms with prismatic pairs.

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