GENERALIZATIONS OF THE CONVERSE OF THE CONTRACTION MAPPING PRINCIPLE

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1. Introduction. This paper is an outgrowth of studies related to the converse of the contraction mapping principle. A natural formulation of the converse statement may be stated as follows: "Let X be a complete metric space, and T be a mapping of X into itself such that for each $x \in X$, the sequence of iterates $\{T^n x\}$ converges to a unique fixed point $\omega \in X$. Then there exists a complete metric in X in which T is a contraction." This is in fact true, even in a stronger sense, as may be seen from the following result of Bessaga **(1)**.

THEOREM A. *Let X be an abstract set and T be a mapping of X into itself such that for each positive integer k, and each $x \in X$, the equation $T^k x = x$ implies $x = \omega$, the unique fixed point of T. Then for each λ, $0 \leqslant \lambda < 1$, there exists a complete metric on X such that $\rho(Tx, Ty) \leqslant \lambda\rho(x, y)$ for all $x, y \in X$.*

We are concerned here with further generalizations of Theorem A. Specifically, we ask whether there exists a metric on X in which mutually commuting mappings $T_1, T_2, \ldots, T_n$ with common unique fixed point are simultaneoulsy contractions. Note that if $T_1, T_2, \ldots, T_n$ are contractions, then every element of the commutative semi-group generated by $T_1, T_2, \ldots, T_n$ is again a contraction. In this way, we may extend the concept of a contraction to the concept of a contractive semi-group $\mathfrak{S}$. We first obtain necessary and sufficient conditions for $\mathfrak{S}$ to be contractive in terms of the existence of certain level functions on X. Sufficient conditions on $\mathfrak{S}$ are also given for $\mathfrak{S}$ to be contractive. If $\mathfrak{S}$ is generated by a finite number of mutually commuting mappings with common unique fixed point, our result reads (cf. **3**):

THEOREM B. *Let X be an abstract set with n mutually commuting mappings $T_1, T_2, \ldots, T_n$ defined on X into itself such that each iteration $T_1^{k_1} \ldots T_n^{k_n}$ (where $k_1, \ldots, k_n$ are non-negative integers not all equal to zero) possesses a unique fixed point which is common to every choice of $k_1, \ldots, k_n$. Then for each $\lambda \in (0, 1)$, there exists a complete metric ρ on X such that $\rho(T_i x, T_i y) \leqslant \lambda\rho(x, y)$ for $1 \leqslant i \leqslant n$ and for all $x, y \in X$.*

Our method of proof is based upon a "Hahn–Banach extension" type argument, whereas Bessaga's proof is purely set-theoretic.

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In §2, we introduce the basic notation and terminology needed for all later discussions. Section 3 introduces the concept of a contractive semi-group $\mathfrak{S}$, and presents a necessary and sufficient condition for $\mathfrak{S}$ to be contractive. The main theorem is proved in §4 where sufficient conditions are imposed on $\mathfrak{S}$ to ensure that it be contractive. This result is then applied in §5 to prove Theorem B. Finally, we make several remarks which lead to questions for further investigation.

2. Definitions and notation. Let X be a non-empty abstract set and $\mathfrak{S}$ be a commutative semi-group of operators on X into itself, containing the identity I. $\mathfrak{S}$ is said to be a *contractive* (*completely contractive*) *semi-group on* X if there exists a metric (complete metric) ρ on X such that for each $S \in \mathfrak{S}$, $\rho(Sx, Sy) \leqslant \lambda(S)\rho(x, y)$ for all $x, y \in X$ where $0 \leqslant \lambda(S) < 1$ for $S \neq I$ and $\lambda(I) = 1$. We say that $\mathfrak{S}$ is a *uniformly contractive* (*uniformly completely contractive*) *semi-group on* X if there exists a real number λ such that $\lambda(S) \leqslant \lambda < 1$ for all $S \in \mathfrak{S}$, $S \neq I$. In all later discussions we call $\mathfrak{S}$ contractive, completely contractive, or uniformly contractive for short. In order to avoid dealing with the trivial contractive semi-group $\mathfrak{S} = \{I\}$, we assume that $\mathfrak{S}$ contains at least one element $T \neq I$.

$X_1 \subseteq X$ is called an $\mathfrak{S}$-*invariant set* if $\mathfrak{S}X_1 \subseteq X_1$. Obviously X and the empty set $\emptyset$ are $\mathfrak{S}$-invariant sets. Consider the set $[a] = \mathfrak{S}\{a\} = \{x: x = Ta$ for some $T \in \mathfrak{S}\}$. Clearly, $\mathfrak{S}[a] \subseteq [a]$, and $[a]$ is the smallest $\mathfrak{S}$-invariant set containing a. Note also that arbitrary unions and intersections preserve the $\mathfrak{S}$-invariance. A subset $\mathfrak{S}_1 \subseteq \mathfrak{S}$ is called a *left ideal* if $\mathfrak{S}\mathfrak{S}_1 \subseteq \mathfrak{S}_1$.

A function λ is called *contractive on* $\mathfrak{S}$ if $0 \leqslant \lambda(S) < 1$ for all $S \in \mathfrak{S}$, $S \neq I$, and $\lambda(I) = 1$. The function λ is called *uniformly contractive on* $\mathfrak{S}$, if there exists a λ such that $\lambda(S) \leqslant \lambda < 1$ for all $S \in \mathfrak{S}$, $S \neq I$. A function ρ is called a *level function with respect* to λ if:

(i) its domain of definition Y is an $\mathfrak{S}$-invariant set,
(ii) $0 \leqslant \rho(x) < \infty$ for all $x \in Y$,
(iii) $\rho(Tx) \leqslant \lambda(T)\rho(x)$ for all $x \in Y$,
(iv) $\rho(x_1) = \rho(x_2) = 0$ implies $x_1 = x_2$.

We call a function σ a *length function on* $\mathfrak{S}$ if it satisfies the conditions:

(i) $0 \leqslant \sigma(S) < 1$ for $S \neq I$ and $\sigma(I) = 1$,
(ii) $\sigma(ST) \leqslant \sigma(S)\sigma(T)$,
(iii) $\sigma(S_1) = \sigma(S_2) = 0$ implies $S_1 = S_2$.

A length function on $\mathfrak{S}$ is certainly contractive on $\mathfrak{S}$ and may also be regarded as a level function on $\mathfrak{S}$.

We shall use the following terminology for subsets of an arbitrary partially ordered set P. By a *transverse set* we mean a subset of P whose elements are pairwise mutually incomparable. A subset J of P is called an *end* if $x \in P$ and $x \geqslant y$ for some $y \in J$ implies $x \in J$. An end J is called *principal* if it is of the form $\{x: x \in P, x \geqslant y\}$ for some fixed element $y \in P$, and it is denoted by $\langle y \rangle$.

The element y is called the *generator* of the principal end J. For other terminology on partially ordered sets not explained here, we refer to Birkhoff **(2)**.

3. Contractive semi-groups. We first propose to prove a necessary and sufficient condition for $\mathfrak{S}$ to be contractive.

THEOREM 1. *$\mathfrak{S}$ is contractive on X if and only if there exists a level function with respect to a contractive function λ on the full set X.*

Proof. X is certainly an $\mathfrak{S}$-invariant set. Since $\mathfrak{S}$ is contractive on X, for each $T \in \mathfrak{S}$, $T \neq I$, and for each $x \in X$, we have

$$\rho(T^p x, x) \leqslant \frac{\rho(Tx, x)}{1 - \lambda(T)}$$

for all non-negative integers p. For each $T \in \mathfrak{S}$, $T \neq I$, $\{\rho(T^n x, x)\}$ is a Cauchy sequence. Denote the limit of this sequence by $\rho_T(x)$. We claim that this limit is independent of T, i.e. $\rho_T(x) = \rho_S(x)$ for each pair $S, T \in \mathfrak{S}$, $S, T \neq I$. Note that for $S, T \neq I$,

$$\begin{aligned} |\rho(T^n x, x) - \rho(S^n x, x)| &\leqslant \rho(S^n x, T^n x) \\ &\leqslant \rho(S^n x, S^n T^n x) + \rho(S^n T^n x, T^n x) \\ &\leqslant \lambda^n(S)\rho(x, T^n x) + \lambda^n(T)\rho(S^n x, x) \\ &\leqslant \lambda^n(S)\frac{\rho(x, Tx)}{1 - \lambda(T)} + \lambda^n(T)\frac{\rho(Sx, x)}{1 - \lambda(S)}. \end{aligned}$$

Since the right-hand side tends to zero as n tends to infinity, this shows that $\rho_T(x) = \rho_S(x)$ as desired. We may now denote the common limit by $\rho(x)$, i.e.

$$\rho(x) = \lim_{n \to \infty} \rho(T^n x, x)$$

where $T \in \mathfrak{S}$, $T \neq I$. Obviously $0 \leqslant \rho(x) < \infty$. Furthermore,

$$\rho(Tx) = \lim_{n \to \infty} \rho(T^n x, Tx) \leqslant \lambda(T) \lim_{n \to \infty} \rho(T^{n-1} x, x) = \lambda(T)\rho(x).$$

Finally, for each pair $x_1, x_2, \in X$, we have:

$$\begin{aligned} \rho(x_1, x_2) &\leqslant \rho(x_1, T^n x_1) + \rho(T^n x_1, T^n x_2) + \rho(T^n x_2, x_2) \\ &\leqslant \rho(x_1, T^n x_1) + \lambda^n(T)\rho(x_1, x_2) + \rho(T^n x_2, x_2). \end{aligned}$$

In particular, by choosing $T \neq I$ (hence $0 \leqslant \lambda(T) < 1$) and letting n tend to infinity, we obtain that $\rho(x_1) = \rho(x_2) = 0$ implies $x_1 = x_2$.

Conversely, let $\rho(x)$ be a level function with respect to a contractive function λ. Define a metric $\tilde{\rho}$ on X by:

$$\tilde{\rho}(x, y) = \begin{cases} \rho(x) + \rho(y) & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

from which it is easily seen that $\mathfrak{S}$ is contractive.

Repeating the arguments in Theorem 1 with λ replacing $\lambda(T)$ throughout, we obtain:

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THEOREM 2. *$\mathfrak{S}$ is uniformly contractive if and only if there exists a level function defined on the full set X with respect to a uniformly contractive function.*

COROLLARY. *If $\mathfrak{S}$ is uniformly contractive with respect to a $\lambda \in (0, 1)$, then it is also uniformly ocntractive with respect to any other $\mu \in (0, 1)$.*

The above corollary shows that the definition of uniformly contractive semi-group is actually independent of the uniformly contractive function λ. The following result characterizes the completely contractive semi-groups.

THEOREM 3. *$\mathfrak{S}$ is completely contractive if and only if $\mathfrak{S}$ is contractive and there exists an element $\omega \in X$ such that $S\omega = \omega$ for some $S \in \mathfrak{S}$, $S \neq I$.*

Proof. Let $\mathfrak{S}$ be completely contractive. Choose $S \in \mathfrak{S}$, $S \neq I$. There exists, by the contraction mapping principle, an element $\omega \in X$ such that $S\omega = \omega$. Conversely, assume that $\mathfrak{S}$ is contractive and that there exists $\omega \in X$ such that $S\omega = \omega$ for some $S \in \mathfrak{S}$, $S \neq I$. Construct a level function ρ on X and the corresponding metric $\tilde{\rho}$ as defined in Theorem 1. The value $\rho(\omega)$ of this level function at ω must be zero, since $\lambda(S) \neq 1$ and $\rho(\omega) = \rho(S\omega) \leqslant \lambda(S)\rho(\omega)$. Let $\{x_n\}$ be a Cauchy sequence in X with respect to $\tilde{\rho}$. If $\rho(x_n)$ tends to zero as n tends to infinity, then the sequence $\{x_n\}$ has the limit ω, since

$$\tilde{\rho}(x_n, \omega) \leqslant \rho(x_n) + \rho(\omega) = \rho(x_n).$$

On the other hand, if $\rho(x_n)$ does not tend to zero, then there exists a subsequence $\{y_n\} \subseteq \{x_n\}$ such that $\rho(y_n) \geqslant \delta > 0$. By assumption, there exists an $N \geqslant 0$ such that $\tilde{\rho}(y_n, y_m) < \delta$ for $n, m \geqslant N$. This implies that $y_n = y_m$ for all $n, m \geqslant N$, and the subsequence $\{y_n\}$ has a limit, namely y_N. As a Cauchy sequence, the full sequence has the same limit.

Theorem 3 shows that completely contractive semi-groups are essentially contractive semi-groups. For any non-completely contractive semi-group $\mathfrak{S}$ we may always add the point ω to X and define $T\omega = \omega$ for all $T \in \mathfrak{S}$ to make it completely contractive.

THEOREM 4. *If $\mathfrak{S}$ is contractive on X, then there exists a length function defined on $\mathfrak{S}$.*

Proof. Define

$$\sigma(S) = \sup_{x \neq y} \frac{\rho(Sx, Sy)}{\rho(x, y)}.$$

If $S \neq I$, then $0 \leqslant \sigma(S) \leqslant \lambda(S) < 1$. If $S = I$, then $\sigma(S) = 1$. For any $S, T \in \mathfrak{S}$ and $x \neq y$, we have $\rho(STx, STy) \leqslant \sigma(S)\rho(Tx, Ty)$ for $Tx \neq Ty$. But this inequality obviously holds even if $Tx = Ty$. Moreover,

$$\rho(STx, STy) \leqslant \sigma(S)\sigma(T)\rho(x, y).$$

Hence by dividing through by $\rho(x, y)$, we easily conclude that $\sigma(ST) \leqslant \sigma(S)\sigma(T)$. Finally, assume that

$$\sigma(S_1) = \sigma(S_2) = 0;$$

then

$$\rho(S_1 x, S_1 y) = \rho(S_2 x, S_2 y) = 0 \qquad \text{for all } x, y \in X, x \neq y.$$

This implies the existence of ω_1, ω_2 such that $S_1 x = \omega_1$ and $S_2 x = \omega_2$ for all $x \in X$. Now for any $x \in X$, we have

$$\rho(S_1 x, S_2 x) \leqslant \rho(S_1 x, S_1 S_2 x) + \rho(S_1 S_2 x, S_2 x) = \rho(\omega_1, \omega_1) + \rho(\omega_2, \omega_2) = 0.$$

Thus $\sigma(S_1) = \sigma(S_2) = 0$ implies that $S_1 = S_2$.

We observe that if $\mathfrak{S}$ is contractive (uniformly contractive) on X, then $\mathfrak{S}$ is also contractive (uniformly contractive) on itself. The onverse is obviously not true.

4. The main theorem. In this section, we prove our main result, which provides a set of sufficient conditions on $\mathfrak{S}$ for it to be contractive.

THEOREM 5. *Suppose $\mathfrak{S}$ satisfies the conditions:*

(a) *for each $T \in \mathfrak{S}$, $T \neq I$, $Tx_1 = x_1$ and $Tx_2 = x_2$ imply that $x_1 = x_2$,*

(b) *there exists a length function σ defined on $\mathfrak{S}$,*

(c) *for any given left ideal $\mathfrak{J} \subseteq \mathfrak{S}$, there exists a finite set $\mathfrak{B} \subseteq \mathfrak{J}$ such that to each $T \in \mathfrak{J}$ there corresponds a $U \in \mathfrak{B}$ satisfying $T = US$ for some $S \in \mathfrak{S}$ and $\sigma(T) = \sigma(U)\sigma(S)$.*

Then $\mathfrak{S}$ is contractive.

Note that conditions (a) and (b) have been proved to be necessary and are easily seen to be independent of each other. Before proving Theorem 5, we need the following two lemmas.

LEMMA 1. *Suppose that ρ_1, ρ_2 are two level functions defined with respect to the same contractive function λ and let X_1, X_2 be their respective domains of definition. If there exist positive constants c_1, c_2 such that $c_1 \rho_1(x) \leqslant c_2 \rho_2(x)$ for all $x \in X_1 \cap X_2$, then ρ_1 can be extended to $X_1 \cup X_2$.*

Proof. Define the function ρ on $X_1 \cup X_2$ by

$$\rho(x) = \begin{cases} \rho_1(x) & \text{if } x \in X_1, \\ \dfrac{c_2}{c_1}\rho_2(x) & \text{if } x \in X_2, x \notin X_1. \end{cases}$$

Conditions (i), (ii) of the definition of a level function are obviously satisfied. Condition (iii) is also obvious if $x \in X_1$ or $Tx \notin X_1$. Suppose now that $x \in X_2$, $x \notin X_1$, and $Tx \in X_1$; then

$$\lambda(T)\rho(x) = \lambda(T)\frac{c_2}{c_1}\rho_2(x) \geqslant \frac{c_2}{c_1}\rho_2(Tx) \geqslant \rho_1(Tx) = \rho(Tx).$$

Finally, to prove that $\rho(x_1) = \rho(x_2) = 0$ implies $x_1 = x_2$, we need to prove only that $\rho_1(x_1) = \rho_2(x_2) = 0$ implies $x_1 = x_2$. Choose $T \in \mathfrak{S}$, $T \neq I$; then

$\rho_1(Tx_1) \leqslant \lambda(T)\rho_1(x_1) = 0$ and hence $Tx_1 = x_1$. Similarly $Tx_2 = x_2$. By assumption (a), we conclude that $x_1 = x_2$.

LEMMA 2. *Suppose that $\mathfrak{S}$ satisfies the hypotheses of Theorem 5. Then there exists, for each $a \in X$, a level function on $[a]$ with respect to the length function σ.*

Proof. Define for $x \in [a]$,

$$\rho(x) = \inf_{T\in\mathfrak{A}_x} \sigma(T),$$

where $\mathfrak{A}_x = \{T\colon Ta = x\}$. Since $x \in [a]$ implies that $\mathfrak{A}_x$ is non-empty, we have $0 \leqslant \rho(x) < \infty$. Next we note that

$$\rho(Sx) = \inf_{U\in\mathfrak{A}_{Sx}} \sigma(U) \leqslant \inf_{T\in\mathfrak{A}_x} \sigma(ST) \leqslant \sigma(S) \inf_{T\in\mathfrak{A}_x} \sigma(T) = \sigma(S)\rho(x).$$

Finally, let $x \in [a]$, $\rho(x) = 0$. Consider the invariant set $\mathfrak{J}_x = \mathfrak{S}\mathfrak{A}_x$ and denote by $\mathfrak{B}_x$ the finite set corresponding to $\mathfrak{J}_x$ according to assumption (c).

Suppose $\mathfrak{A}_x \neq \mathfrak{B}_x$; then there exists a $T_1 \in \mathfrak{A}_x$ such that $T_1 = U_1 S_1$ where $U_1 \in \mathfrak{B}_x$, $S_1 \in \mathfrak{S}$, and $S_1 \neq I$. Since $U_1 \in \mathfrak{B}_x \subseteq \mathfrak{J}_x$, there exists $T_2 \in \mathfrak{A}_x$ such that $U_1 = T_2 S_2$ where $S_2 \in \mathfrak{S}$. Since $\sigma(S_1 S_2) \leqslant \sigma(S_1)\sigma(S_2) \leqslant \sigma(S_1) < 1$, then $S_1 S_2 \neq I$. Now $x = T_1 a = U_1 S_1 a = T_2 S_1 S_2 a = S_1 S_2 x$. For any $T \in \mathfrak{S}$, note that $TS_1 S_2 x = Tx = S_1 S_2(Tx)$. Hence by assumption (a), $Tx = x$ for all $T \in \mathfrak{S}$.

The same conclusion can be reached in the case when $\mathfrak{A}_x = \mathfrak{B}_x$. Indeed, since $\mathfrak{B}_x$ is finite, there exists an $S \in \mathfrak{A}_x$ such that $\sigma(S) = 0$. For any $Y \in \mathfrak{S}$, $\sigma(ST) \leqslant \sigma(S)\sigma(T) = 0$. Thus $ST = S$. Therefore $x = Sa = STa = Tx$. If now $\rho(x_1) = \rho(x_2) = 0$, then $Tx_1 = x_1$ and $Tx_2 = x_2$ for all $T \in \mathfrak{S}$. In particular, by choosing $T \neq I$, we conclude from assumption (a) that $x_1 = x_2$.

Proof of Theorem 5. Let X_1 be an invariant set in X and ρ_1 be a level function on X_1 defined with respect to the length function σ. Suppose $a \in X_1$. We claim that ρ_1 can be extended to $X_1 \cup [a]$. Denote by ρ_2 the level function defined on $[a]$ according to Lemma 2. Consider the set

$$\mathfrak{J} = \{T\colon T \in \mathfrak{S},\ Ta \in X_1 \cap [a]\}.$$

Clearly, $\mathfrak{J}$ is a left ideal. Let $\mathfrak{B}$ be the finite set corresponding to $\mathfrak{J}$ according to assumption (c). In addition, let

$$\mathfrak{B}' = \{U\colon U \in \mathfrak{B},\ \rho_1(Ua) \neq 0\} \quad\text{and}\quad \mathfrak{B}'' = \{U\colon U \in \mathfrak{B},\ \rho_2(Ua) \neq 0\}.$$

Define

$$c_1 = \min_{U\in\mathfrak{B}''} \rho_2(Ua) \quad\text{and}\quad c_2 = \max_{U\in\mathfrak{B}'} \rho_1(Ua).$$

Choose $x \in X_1 \cap [a]$ and consider the sets $\mathfrak{A}_x$, $\mathfrak{B}_x$, and $\mathfrak{J}_x$ as introduced in Lemma 2. We first note that if $\mathfrak{B}' = \emptyset$, then for each $T \in \mathfrak{A}_x$, $T = US$ for some $U \in \mathfrak{B}$, we have $\rho_1(x) = \rho_1(Ta) = \rho_1(USa) \leqslant \sigma(S)\rho_1(Ua) = 0$. Now $\rho_1(x) = 0$ implies that $Sx = x$ for all $S \in \mathfrak{S}$, since $\rho_1(Sx) \leqslant \lambda(S)\rho_1(x)$ for all

$S \neq I$. Choose any $S \neq I$; we conlude from $\rho_2(x) = \rho_2(Sx) \leqslant \lambda(S)\rho_2(x)$ that $\rho_2(x) = 0$. Hence in this case the inequality $c_1 \rho_1(x) \leqslant c_2 \rho_2(x)$ holds in a trivial way. A similar conclusion holds if $\mathfrak{B}'' = \emptyset$. We may now assume that $\mathfrak{B}' \neq \emptyset$ and $\mathfrak{B}'' \neq \emptyset$. It is readily seen from the above that $\rho_1(x) = 0$ if and only if $\rho_2(x) = 0$. So, we may also assume that $\rho_2(x) \neq 0$. Suppose $\mathfrak{A}_x \neq \mathfrak{B}_x$; then by repeating the same argument as in Lemma 2, we conclude that $Tx = x$ for all $T \in \mathfrak{S}$; in particular if $T \neq I$, then $\rho_2(x) = \rho_2(Tx) \leqslant \sigma(T)\rho_2(x)$ implies that $\rho_2(x) = 0$. Again, the desired inequality holds trivially. Suppose now that $\mathfrak{A}_x = \mathfrak{B}_x$; then we may choose $T \in \mathfrak{A}_x$ such that $\rho_2(x) = \sigma(T)$. Note that $\rho_2(x) \neq 0$ implies that $\rho_2(Ua) \neq 0$, and from the definition of ρ_2, $\rho_2(a) = 1$. For otherwise there exists $V \neq I$ such that $Va = a$, and thus $Sa = a$ for all $S \in \mathfrak{S}$. In particular, $a = Ta = x \in X_1$, contradicting $a \notin X_1$. Thus

$$\begin{aligned} c_1 \rho_1(x) &= c_1 \rho_1(USa) \leqslant c_1 \sigma(S)\rho_1(Ua) \leqslant \rho_2(Ua)\sigma(S)\rho_1(Ua) \\ &\leqslant \sigma(S)\sigma(U)\rho_2(a)\rho_1(Ua) \leqslant c_2 \sigma(T)\rho_2(a) = c_2 \sigma(T) = c_2 \rho_2(x). \end{aligned}$$

We then apply Lemma 1 to extend ρ_1 over $X_1 \cup [a]$.

Let Φ be the family of all level functions defined with respect to the length function σ. Φ is non-empty for it contains the level function on the empty set $\emptyset$. Let X_ρ be the domain of definition corresponding to $\rho \in \Phi$. We say that $\rho_1 \leqslant \rho_2$ if (i) $X_{\rho_1} \subseteq X_{\rho_2}$, (ii) $\rho_1 = \rho_2$ on X_{ρ_1}. Clearly this defines a partial ordering on Φ. Suppose now that Ψ is a totally ordered subset of Φ. Define a level function $\tilde{\rho}$ on $\bigcup_{\rho \in \Psi} X_\rho$ by $\tilde{\rho}(x) = \rho(x)$ if $x \in X_\rho$ for some $\rho \in \Psi$. Since Ψ is totally ordered, this definition of $\tilde{\rho}$ is unambiguous. $\tilde{\rho}$ is clearly an upper bound for Ψ and thus Φ satisfies the hypothesis of Zorn's lemma. Therefore, there must exist a maximal element $\rho_M \in \Phi$. We claim that $X = X_{\rho_M}$. For otherwise there exists $a \in X$, $a \notin X_{\rho_M}$ and we may extend ρ_M to $X_{\rho_M} \cup [a]$, contradicting the maximality of ρ_M. Knowing the existence of a level function on the full set X, we conclude by Theorem 1 that $\mathfrak{S}$ is contractive.

We remark that if the length function σ on $\mathfrak{S}$ is in addition uniformly contractive on $\mathfrak{S}$, then Theorem 5 together with Theorem 2 implies that $\mathfrak{S}$ is uniformly contractive on X.

5. Semi-groups generated by a finite number of elements. Let X be a non-empty abstract set and $T_1, T_2, \ldots, T_n$ be mutually commuting mappings defined on X into itself. Denote by $\mathfrak{S}$ the commutative semi-group containing the identity which is generated by $T_1, T_2, \ldots, T_n$. Obviously, we may restrict ourselves to the case where all the T_i's are different from the identity. In this case, there exists a set of necessary and sufficient conditions for $\mathfrak{S}$ to be uniformly contractive. In particular, assumptions (a) and (b) of Theorem 5 are both necessary and sufficient. In fact, we can prove that $\mathfrak{S}$ is uniformly contractive under assumption (a) and only part of assumption (b).

Let Q be the product N^n of the additive semi-group N of natural numbers with its usual cardinal product partial ordering. It is not difficult to see that every transverse subset of Q is finite. Obviously, Q forms a semi-group with respect to vector addition, and the mapping

$$k = (k_1, \ldots, k_n) \to T^k = T_1^{k_1} \ldots T_n^{k_n}$$

defines a homomorphism of Q into $\mathfrak{S}$. Denote

$$\varphi(k) = \sum_{i=1}^{n} k_i \qquad \text{for all } k \in Q.$$

With these preliminary remarks, we proceed to prove Theorem B in the following form:

THEOREM 6. *Suppose $\mathfrak{S}$ satisfies the conditions:*
(a) *for each $T \in \mathfrak{S}$, $T \neq I$, $Tx_1 = x_1$ and $Tx_2 = x_2$ imply $x_1 = x_2$,*
(b) *for each pair $S, T \in \mathfrak{S}$, $ST = I$ implies $S = T = I$.*
Then $\mathfrak{S}$ is uniformly contractive.

Proof. Assumption (a) of Theorem 5 is satisfied by hypothesis. For each $S \in \mathfrak{S}$, let $K(S) = \{p: p \in Q, T^p = S\}$. Choose any $\lambda \in (0, 1)$, and define

$$\sigma(S) = \inf_{k \in K(S)} \lambda^{\varphi(k)}.$$

Clearly $0 < \sigma(S) \leqslant \lambda < 1$ for all $S \neq I$, and $\sigma(I) = 1$. Moreover

$$\sigma(ST) = \inf_{k \in K(ST)} \lambda^{\varphi(k)} \leqslant \inf_{p \in K(S)} \inf_{q \in K(T)} \lambda^{\varphi(p+q)} = \inf_{p \in K(S)} \lambda^{\varphi(p)} \inf_{q \in K(T)} \lambda^{\varphi(q)} = \sigma(S)\sigma(T).$$

For any $S \in \mathfrak{S}$, if $K(S)$ as a subset of Q is transverse (hence finite), then

$$\sigma(S) = \min_{k \in K(S)} \lambda^{\varphi(k)} > 0.$$

On the other hand, if $K(S)$ is not transverse, then there exist $p, q \in K(S)$, $p > q$. Now $T^{p-q}(Sx) = Sx$ for all $x \in X$, and $T^{p-q} \neq I$. Again by assumption (a), we have $Sx = \theta$ for all $x \in X$. Hence $\sigma(S) = 0$ implies that $Sx = \theta$ for all $x \in X$. Now suppose that $\sigma(S_1) = \sigma(S_2) = 0$; then there exist θ_1, θ_2 such that $S_1 x = \theta_1$ and $S_2 x = \theta_2$ for all $x \in X$. Since $\theta_1 = S_1 S_2 \theta_1 = S_2 S_1 \theta_1 = \theta_2$, we have $S_1 = S_2$. Thus $\sigma(S)$ satisfies (b) of Theorem 5.

We next show that assumption (c) of Theorem 5 is also satisfied. Let $\mathfrak{J}$ be a left ideal in $\mathfrak{S}$. Consider the set $J = \{\rho: p \in Q, T^p \in \mathfrak{J}\}$, which is clearly an end in Q. It is easy to see that $J = \bigcup_{p \in B} \langle p \rangle$, where

$$B = \{p: p \in J, \quad \text{and if} \quad q \in J \text{ and } q \leqslant p, \text{ then } q = p\}$$

and B is a transverse set.

Let $\mathfrak{B} = \{T^p\text{: for some } p \in B\}$. Obviously $\mathfrak{B} \subseteq \mathfrak{J}$ and $\mathfrak{B}$ is finite. For $T \in \mathfrak{J}$, $\sigma(T) = 0$, choose any $U \in \mathfrak{B}$, and observe that $UT = T$ and $\sigma(T) = \sigma(T)\sigma(U)$. On the other hand, if $\sigma(T) \neq 0$, we may choose $p \in K(T)$ such that

$$\sigma(T) = \lambda^{\varphi(p)} = \lambda^{\varphi(r)+\varphi(p-r)} \geqslant \sigma(U)\sigma(S),$$

where $S = T^{p-r} \in \mathfrak{S}$. Since the reverse inequality always holds, we conclude that $\sigma(T) = \sigma(U)\sigma(S)$. Now the set $\mathfrak{B} \subseteq \mathfrak{J}$ satisfies the condition required by assumption (c) of Theorem 5. Applying Theorems 5 and 2, we conclude that $\mathfrak{S}$ is uniformly contractive. The corollary follows immediately from Theorem 3.

COROLLARY. *Let X and $\mathfrak{S}$ be given as in Theorem 6. Suppose there exists an element $\omega \in X$, such that $S\omega = \omega$ for some $S \in \mathfrak{S}$, $S \neq I$. Then $\mathfrak{S}$ is uniformly completely contractive.*

6. Remarks. We first remark that Theorem 6 cannot be extended to the corresponding case where $\mathfrak{S}$ is generated by a countably infinite number of mappings. To see this, we consider the following example. Let $X = [0, \infty)$ and $T_i x = x + 1/i$, $i = 1, 2, 3, \ldots$. Clearly X and the commutative semi-group $\mathfrak{S}$ generated by all the T_i's satisfy the hypothesis of Theorem 6. But $\mathfrak{S}$ is not uniformly contractive. Assume the contrary; then by Theorem 3 there exists a level function ρ on X such that $\rho(Tx) \leqslant \lambda\rho(x)$ for all $x \in X$ and all $T \in \mathfrak{S}$, $T \neq I$, where $0 \leqslant \lambda < 1$. Since $\infty \notin X$, we have therefore $\rho(x) \neq 0$ for all $x \in X$. For any m, we may write $\rho(T_2 x) = \rho(T_{2m}{}^m x) \leqslant \lambda^m \rho(x)$. Letting m tend to infinity, we obtain the desired contradiction. Nevertheless, in this case $\mathfrak{S}$ is contractive on X. Indeed, $\rho(x) = \lambda^x$, for any $\lambda \in (0, 1)$, is a level function on X. (Note that in this case the contractive function is clearly not uniform.)

Let X be a metrizable space and T be a mapping of X into itself such that for each positive integer k and each $x \in X$, the equation $T^k x = x$ implies $x = \omega$, the unique fixed point of T. We now ask: Does there exist a metric in which T is a contraction and which at the same time reproduces the original topology? The answer is negative even in case X is compact. Let X be any compact metrizable space, and T be a mapping of X into itself which possesses a unique inverse. In this case, we claim that there does not exist a metric on X which satisfies the above-mentioned requirements unless X is only a singleton set. Assume the contrary, i.e. there exists a metric ρ on X such that $\rho(Tx, Ty) \leqslant \lambda\rho(x, y)$ for all $x, y \in X$ and ρ induces the given topology on X. Since X is compact in the original tooplogy, it is also compact in the metric topology induced by ρ. Denote by D the diameter of X with respect to ρ, i.e.

$$D = \sup_{x,y \in X} \rho(x, y).$$

Choose $x, y \in X$, such that $x = y$. Note that $\rho(x, y) \leqslant \lambda^n \rho(T^{-n}x, T^{-n}y) \leqslant \lambda^n D$. Letting n tend to infinity, we arrive at the desired contradiction.

We finally list two open questions.

(i) What is a set of necessary and sufficient conditions for $\mathfrak{S}$ to be contractive. (This is not known even in the case when $\mathfrak{S}$ is generated by a countably infinite number of mappings.)

(ii) Let X be any compact metrizable space and T a mapping of X into itself satisfying the condition imposed in the previous paragraph. What additional conditions are sufficient to ensure the existence of a metric that will induce the original topology and at the same time make the mapping T a contraction. (This is not known even when $X = [0, 1]$.)

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