ON THE LAWS OF SOME VARIETIES OF GROUPS

M. R. VAUGHAN-LEE

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1. Introduction

D. E. Cohen has shown in [1] that a variety consisting of metabelian groups has a finite basis for its laws. In this paper I make use of Cohen's result to prove the following theorem.

THEOREM. *A variety of groups which satisfy an identity*

$$[[x_1, x_2, \cdots, x_m], [x_{m+1}, x_{m+2}]] = 1$$

has a finite basis for its laws.

($[x_1, x_2, \cdots, x_m]$ denotes the commutator $[[\cdots[[x_1, x_2], x_3], \cdots], x_m]$.) The theorem is an extension of Cohen's result, which covers the cases $m = 1, 2$.

2. Notation

The notation is that of [2]. The laws of a variety, $\boldsymbol{U}$, of groups are in one-one correspondence with the elements, or words, of a fully invariant subgroup, U, of the free group, X, on the variables $x_1, x_2, \cdots$. The laws of $\boldsymbol{U}$ are finitely based if U is finitely generated as a fully invariant subgroup of X. If $\boldsymbol{U}$ is determined by its n variable laws then $\boldsymbol{U}$ is determined by a fully invariant subgroup, W, of the free group of rank n, X_n. Then the laws of $\boldsymbol{U}$ are finitely based if W is finitely generated as a fully invariant subgroup of X_n.

Two sets of words are equivalent if they generate the same fully invariant subgroup of X. If $w_1, w_2, \cdots, w_r$ are words of a group F, then $V(w_1, w_2, \cdots, w_r)$ denotes the fully invariant subgroup of F generated by $w_1, w_2, \cdots, w_r$.

X'' denotes the second derived group of X. X'' is generated by

$$\{[[a, b], [c, d]] | a, b, c, d \in X\}.$$

I use the following notation for the lower central series of a group F. Put

$$\gamma_1(F) = F, \ \gamma_{i+1}(F) = [\gamma_i(F), F].$$

I shall use the construction of commutators given in Chapter 3 of [2], applying it to the free group X. The commutators of weight one are the elements x_i^n; $i = 1, 2, \cdots$; n positive or negative, but not zero. If c_1, c_2 are commutators of weight k, l respectively then, provided $[c_1, c_2] \neq 1$, $[c_1, c_2]$ is a commutator of weight $k+l$.

Let δ_i denote the endomorphism of X mapping x_i to 1, and mapping x_j to x_j for $j \neq i$. Then the commutator c involves the variable x_i if and only if $c\delta_i = 1$. If a commutator involves n distinct variables then its weight is at least n, and it is contained in $\gamma_n(X)$.

3. Preliminary lemmas

LEMMA 1. *If* $a = a_1 a_2 \cdots a_m$, $b = b_1 b_2 \cdots b_n$ *then* $[a, b]$ *is a product of elements of the form* $[c_1, c_2, \cdots, c_k]$, *where* $k \geqq 2$ *and* $c_i \in \{a_1, a_2, \cdots, a_m, b_1, \cdots, b_n\}$ *for* $i = 1, 2, \cdots, k$.

PROOF. The proof is by induction on $m+n$, the result being trivial if $m+n = 2$. If $m+n > 2$ then either $m > 1$ or $n > 1$.

Suppose that $m > 1$. Then using the identity $[xy, z] = [x, z]^y[y, z]$, we obtain:

$$[a, b] = [a_1 a_2 \cdots a_m, b_1 b_2 \cdots b_n]$$
$$= [a_1 \cdots a_{m-1}, b_1 \cdots b_n]^{a_m}[a_m, b_1 \cdots b_n].$$

By induction $[a_1 \cdots a_{m-1}, b_1 \cdots b_n] = d_1 d_2 \cdots d_k$, $[a_m, b_1 \cdots b_n] = e_1 e_2 \cdots e_l$, where $d_1, d_2, \cdots, d_k, e_1, e_2, \cdots, e_l$ are of the required form.

Hence

$$[a, b] = (d_1 d_2 \cdots d_k)^{a_m} e_1 e_2 \cdots e_l$$
$$= d_1^{a_m} d_2^{a_m} \cdots d_k^{a_m} e_1 e_2 \cdots e_l$$
$$= d_1[d_1, a_m]d_2[d_2, a_m] \cdots d_k[d_k, a_m]e_1 e_2 \cdots e_l.$$

Since $[d_1, a_m]$, $[d_2, a_m]$, $\cdots$, $[d_k, a_m]$ are clearly of the required form this shows that $[a, b]$ is a product of elements of the required form.

The same result follows similarly if $n > 1$, using the identity $[x, yz] = [x, z][x, y]^z$. This proves Lemma 1.

COROLLARY. *An element of* X'' *can be written as a product of commutators of the form* $[c', d']$, *where* c', d' *are both commutators of weight at least two.*

PROOF. It is sufficient to show that an element $[[a, b], [c, d]]$ can be written as a product of commutators of the form $[c', d']$ where c', d' are commutators of weight at least two.

Now a, b, c, d can all be written as products of commutators of weight one.

Hence, by Lemma 1, $[a, b]$ and $[c, d]$ can both be written as products of commutators of weight at least two. By Lemma 1, again, this implies that $[[a, b], [c, d]]$ can be written as a product of commutators of the form $[c_1, c_2, \cdots, c_k]$; $k \geqq 2$; $c_1, c_2, \cdots, c_k$ commutators of weight at least two. This proves the corollary since $[c_1, c_2, \cdots, c_k] = [[c_1, c_2, \cdots, c_{k-1}], c_k]$.

LEMMA 2. A word in X'' is equivalent to a set of $2(m-1)$ variable words and a set of words in $[\gamma_m(X), \gamma_2(X)]$.

PROOF. Theorem 33.45 of [2] states that a word w in n variables $x_1, x_2, \cdots, x_n$ is equivalent to a set of words each of which is a product of commutators involving precisely the variables x, $i \in M$, for some subset M of $\{1, 2, \cdots, n\}$.

Now, if $i_1, i_2, \cdots, i_k$ are all distinct, a word $u(x_{i_1}, x_{i_2}, \cdots, x_{i_k})$ is equivalent to $u(x_1, x_2, \cdots, x_k)$, and so this theorem implies that a word w in n variables is equivalent to a set of words each of which is, for some $k \leqq n$, a product of commutators involving precisely the variables $x_1, x_2, \cdots, x_k$. But a word u which is a product of commutators each involving the variables $x_1, x_2, \cdots, x_k$ satisfies $u\delta_i = 1$ for $i = 1, 2, \cdots, k$, and so a word w in X'' is equivalent to a set of $2(m-1)$ variable words and a set of words $\{u_\alpha\}$ in X'' satisfying $u_\alpha \delta_i = 1$, $i = 1, 2, \cdots, 2m-1$. I shall show that a word in X'' which is in the kernel of δ_i for $i = 1, 2, \cdots, 2m-1$ is contained in $[\gamma_m(X), \gamma_2(X)]$, and this will complete the proof of Lemma 2.

I shall call a commutator $[c, d]$ a commutator of type 2 if c and d are commutators of weight at least two. Let $u \in X''$, and suppose that $u\delta_i = 1$ for $i = 1, 2, \cdots, 2m-1$. By the corollary to Lemma 1 u can be written as a product $d_1 d_2 \cdots d_r$, where $d_1, d_2, \cdots, d_r$ are commutators of type 2. Suppose that d_i involves x_1, but that d_{i+1} does not involve x_1. Then

$$u = d_1 \cdots d_{i-1} d_i d_{i+1} d_{i+2} \cdots d_r$$
$$= d_1 \cdots d_{i-1} d_{i+1} d_i [d_i, d_{i+1}] d_{i+2} \cdots d_r .$$

Clearly $[d_i, d_{i+1}]$ is of type 2 and involves x_1, and so in this way we can shift the commutators not involving x_1 to the left of those that do. Hence we may suppose that $u = d_1 d_2 \cdots d_r$, where each d_i is of type 2, and where $d_1, \cdots, d_s$ do not involve x_1, $d_{s+1}, \cdots, d_r$ do involve x_1, for some $s \leqq r$.

Then $1 = u\delta_1 = (d_1\delta_1)(d_2\delta_1) \cdots (d_r\delta_1) = d_1 d_2 \cdots d_s$. Hence $u = d_{s+1} \cdots d_r$, i.e. u is a product of commutators of type 2, each involving x_1. By induction u is a product of commutators of type 2 each involving all of $x_1, x_2, \cdots, x_{2m-1}$. Let d be a commutator of type 2 involving each of $x_1, x_2, \cdots, x_{2m-1}$. Then the weight of $d (wt\, d)$ is at least $2m-1$. Since d is of type 2, $d = [c_1, c_2]$ where $wt\, c_1$, $wt\, c_2 \geqq 2$ and $wt\, c_1 + wt\, c_2 = wt\, d \geqq 2m-1$. It follows that at least one of $wt\, c_1$, $wt\, c_2 \geqq m$, i.e. that at least one of c_1, c_2 is contained in $\gamma_m(X)$. Since both c_1 and c_2 are contained in $\gamma_2(X)$ this implies that $d = [c_1, c_2] \in [\gamma_m(X), \gamma_2(X)]$. Therefore $u \in [\gamma_m(X), \gamma_2(X)]$ and this proves Lemma 2.

4. Proof of theorem

Let $\mathbf{U}$ be variety of groups determined by the fully invariant subgroup U of X. Suppose that $[[x_1, x_2, \cdots, x_m], [x_{m+1}, x_{m+2}]] \in U$ for some $m \geqq 2$.

Let π be the natural projection of X onto X/X''. Then $X\pi$ is isomorphic to the free metabelian group of countable rank, and so $U\pi$ is a fully invariant subgroup of $X\pi$, since any endomorphism of $X\pi$ is induced by an endomorphism of X([2], 13.24).

It follows ([1]) that $U\pi$ is finitely generated as a fully invariant subgroup of $X\pi$, since $U\pi$ determines a variety of metabelian groups. So $U\pi = V(u_1\pi, u_2\pi, \cdots, u_r\pi)$ for some $u_1, u_2, \cdots, u_r \in U$. Let $u \in U$. Then there is an element $v \in V(u_1\pi, u_2\pi, \cdots, u_r\pi)$ such that $u\pi = v$. Since any endomorphism of $X\pi$ is induced by an endomorphism of X there is an element $w \in V(u_1, u_2, \cdots, u_r)$ such that $w\pi = v = u\pi$. Then $(uw^{-1})\pi = 1$, and so $uw^{-1} \in X''$. Hence U, as a fully invariant subgroup of X, is generated by $u_1, u_2, \cdots, u_r$ and by a set of words in X''.

Hence, by Lemma 2, U, as a fully invariant subgroup of X, is generated by $u_1, u_2, \cdots, u_r$ a set of $2(m-1)$ variable words and a set of words in $[\gamma_m(X), \gamma_2(X)]$. Since $[[x_1, x_2, \cdots, x_m], [x_{m+1}, x_{m+2}]] \in U$ this implies that U, as a fully invariant subgroup of X, is generated by $u_1, u_2, \cdots, u_r$, a set of $2(m-1)$ variable words and the word $[[x_1, x_2, \cdots, x_m], [x_{m+1}, x_{m+2}]]$. Suppose that u_i is an n_i variable word for $i = 1, 2, \cdots, r$ and let $n = \max\{n_1, n_2, \cdots, n_r, 2(m-1), m+2\}$. Then $\mathbf{U}$ is determined by a set of n variable words, and so $\mathbf{U}$ is determined by a fully invariant subgroup of X_n, the free group of rank n. Since $[[x_1, x_2, \cdots, x_m], [x_{m+1}, x_{m+2}]]$ is a law in $\mathbf{U}$, $\mathbf{U}$ is determined by a fully invariant subgroup of X_n containing $[\gamma_m(X_n), \gamma_2(X_n)]$. But fully invariant subgroups of X_n containing $[\gamma_m(X_n), \gamma_2(X_n)]$ are in one-one correspondence with fully invariant subgroups of $X_n/[\gamma_m(X_n), \gamma_2(X_n)]$, which is a finitely generated abelian by nilpotent group. Now finitely generated abelian by nilpotent groups satisfy the maximal condition on normal subgroups ([3]), and a fortiori satisfy the maximal condition on fully invariant subgroups. Hence fully invariant subgroups of X_n containing $[\gamma_m(X_n), \gamma_2(X_n)]$ satisfy the maximal condition, and so are finitely generated as fully invariant subgroups. This proves the Theorem.

References

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Vanderbilt University
Nashville
Tennessee

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