Canad. Math. Bull. Vol. 15 (2), 1972

σ-REFLEXIVE SEMIGROUPS AND RINGS

BY

M. CHACRON(1) AND G. THIERRIN(2)

We shall call a semigroup S a *σ-reflexive* semigroup if any subsemigroup H in S is reflexive (i.e. for all $a, b \in S$, $ab \in H$ implies $ba \in H$ ([2], [5]). It can be verified that any group G is a σ-reflexive semigroup if and only if any subgroup of G is normal. In this paper, we characterize subdirectly irreducible σ-reflexive semigroups. We derive the following commutativity result: any generalized commutative ring R ([1]) in which the integers $n=n(x, y)$ in the equation $(xy)^n=(yx)^m$ can be taken equal to 1 for all $x, y \in R$ must be a commutative ring.

CONVENTIONS. If S (R) is a semigroup (ring), then the multiplicative subsemigroup that is generated by a given element x is written $[x]$. A polynomial $f(t) \in Z[t]$ (the ring of integral polynomials) having the form

$$f=f(t) = t^k+r_{k+1}t^{k+1}+\cdots+r_{k+m}t^{k+m} \quad (k \geq 1)$$

is termed *lower monic* polynomial of *co-degree k*. Henceforth, all polynomials $f(t) \in Z[t]$ are assumed to be without constant term.

1. In this part S is a multiplicative semigroup. Our aim is to characterize subdirectly irreducible σ-reflexive semigroups S. The following proposition is evident.

PROPOSITION 1. *Any semigroup S is σ-reflexive if and only if it satisfies the following condition*:

$$\forall a, b \in S, \qquad \exists m = m(a, b) \geq 1; \qquad ab = (ba)^m.$$

From Proposition 1 follows

PROPOSITION 2. *Let a, b be any two noncommuting elements of a σ-reflexive semigroup S. Then for some $m>1$, $(ab)^m=ab$.*

Proof. There exists $r \geq 1$ such that $ba=(ab)^r$. As $ab \neq ba$, $r>1$. As $ba \in [(ab)^r]$, we have $ab \in [(ab)^r]$.

Therefore for some $s \geq 1$, $(ab)^{rs}=ab$ with $rs>1$.

Proposition 2 is elementary and is an important tool for the present considerations. We can now prove our first theorem.

Received by the editors July 3, 1970 and, in revised form, January 11, 1971.

(1) This research has been supported by National Research Council Grant A-4807, while the author was at the University of Windsor.

(2) This research has been supported by National Research Council Grant A-4096, while the author was at the Université de Montréal.

THEOREM 1. *Any group G is σ-reflexive if and only if every subgroup of G is normal.*

Proof. The "only if" is evident. To prove the "if" suppose that G is not commutative. Then G is a Hamiltonian group and all one has to show is that if a, b are any two elements of G which do not commute, then $[ab]$ coincides with the cyclic subgroup that is generated by ab, that is to say, that ab is of finite order. Since ab does not commute with a^{-1}, ab is not central. Now it is well known (see [3]) that any noncentral element of a Hamiltonian group is of finite order. Hence ab is of finite order and the theorem is established.

THEOREM 2. (1) *Any σ-reflexive semigroup S is a central idempotent semigroup.*

(2) *Any σ-reflexive semigroup S without central idempotents is commutative.*

Proof (1). Let e be an idempotent in S. Let $x \in S$. There are r, $s \geq 1$ such that $ex=(xe)^r$, $xe=(ex)^s$ (Proposition 1). Then $exe=(xe)^r e=(xe)^r=ex$ and $exe=e(ex)^s =(ex)^s=xe$.

(2) By (1), S does not have idempotents. By Proposition 2, no elements a, $b \in S$ do not commute pairwise.

Let us recall now some properties of subdirectly irreducible semigroups [6].

A semigroup S is said to be subdirectly irreducible if the intersection of all the congruences of S different from the equality is a congruence different from the equality. Every semigroup is a subdirect product of subdirectly irreducible semigroups.

Let S be a semigroup and let I be an ideal of S. Define $a \rho b$ to mean either $a=b$ or else both a and b belong to I. We call ρ the Rees congruence modulo I. The equivalence classes of S mod ρ are I itself and every one-element set not in I. If S is subdirectly irreducible, it is immediate that the intersection of all ideals of S containing more than one element is an ideal with more than one element.

If S is a subdirectly irreducible semigroup, then every central idempotent e of S is either the zero of S or the identity of S. Indeed let us suppose that e is not the zero element of S. Then $I=eS$ is an ideal of S containing more than one element and the Rees congruence ρ mod I is not the equality. Define $a \sigma b$ to mean that $ea=eb$. It is immediate that σ is a congruence and that the intersection of ρ and σ is the equality. Since S is subdirectly irreducible, then σ must be the equality. From $e x \sigma x$ follows that e is the identity of S.

The following proposition is evident.

PROPOSITION 3. *Any σ-reflexive semigroup is a subdirect product of subdirectly irreducible σ-reflexive semigroups.*

We are now in a position to show our main result.

THEOREM 3. *Let S be a noncommutative σ-reflexive semigroup which is subdirectly irreducible. Then S satisfies the following conditions:*

(1) *S has an identity and $G=\{x \mid x \in S, y \in S, xy=1\}$ is a σ-reflexive group which is noncommutative (Hamiltonian group).*

(2) *If $D=S-G$ is nonempty, then S is a semigroup with zero $0 \in D$, D is the maximum ideal of S and D is contained in the center of S.*

Proof. In view of Theorem 2, S must contain at least one central idempotent. Since S is subdirectly irreducible, an idempotent element of S is the zero of S or the identity element 1.

Let us suppose that S has no identity element 1. Then S must have a zero element 0. For some $a, b \in S$, we have $ab \neq ba$. Hence, by Proposition 2, $(ab)^m = ab$ for some $m > 1$ and $(ab)^{m-1}$ is an idempotent. Therefore $(ab)^{m-1}=0$, $ab=0$, and $ba=ab$, which is a contradiction and R has an identity follows. If $x \in G$ and $xy=1$, then, since 1 is a subsemigroup of S, $yx=1$. This shows that G is the group of invertible elements of S and that G is σ-reflexive.

Assuming (2), it is evident that G is noncommutative.

It remains to show (2). It is immediate that D is the maximum ideal of S. Let $x \in S$, $a \in D$. Suppose $ax \neq xa$. Then for some $m > 1$ we have $(ax)^m = ax$ (Proposition 2). But $ax \neq 0$ and $(ax)^{m-1}$ is an idempotent $\neq 0$. Hence $(ax)^{m-1}=1$ and $a \notin D$, a contradiction.

To see that S is a semigroup with zero, we proceed as follows. Let H be the intersection of all ideals of S containing more than one element. If D is reduced to one element z, then z is the zero of S. In the opposite case $H \subseteq D$ and H is in the center of S. As S is subdirectly irreducible, H contains more than one element. If for each $x \in H$ we have $Sx = xS = H$, then H is a group, hence contains a nonzero idempotent so H must be S, a contradiction. Therefore there exists at least one element $z \in H$ such that $Sz=\{z'\}$. As S has an identity element $z=z'$ follows and $0=z$ is the zero of S.

2. In this part, R is a ring. In view of Proposition 2, one can give the following generalization of σ-reflexive semigroups. A ring R is *Σ-reflexive* if for any two elements $a, b \in R$ either $ab=ba$ or $ab=f(ba)$ for some lower monic integral polynomial $f(t)$ depending on a and b of co-degree $m \geq 2$.

Clearly if the multiplicative semigroup of R is σ-reflexive, then R is Σ-reflexive. Our aim is to show that any Σ-reflexive ring is commutative. The analog of Proposition 2 reads as follows:

Proposition 4. *Let a, b be any two noncommuting elements of a Σ-reflexive ring. Then for some lower monic polynomial f of co-degree* 1 *we have $f(ab)=0$.*

Proof. There are $g(t)$ and $h(t)$ of degrees ≥ 2 such that $ab=g(ba)$, $ba=h(ab)$. Hence $ab=gh(ab)$ and $f(t)=t-gh(t)$ is the required polynomial.

Proposition 5. *Any Σ-reflexive ring R is a central idempotent ring.*

https://doi.org/10.4153/CMB-1972-033-3 Published online by Cambridge University Press

Proof. Let e be an idempotent in R. Let $x \in R$. We can find two polynomials $f, g \in Z(t)$ of degree $m \geq 1$ such that $ex = f(xe)$, $xe = g(ex)$. Then

$$exe = f(xe)e = f(xe) = ex, \qquad exe = eg(ex) = g(ex) = xe.$$

THEOREM 4. *Any Σ-reflexive ring R is commutative.*

Proof. Our proof will go by reduction to the case where R is subdirectly irreducible. As a result of Herstein [4, Theorem 17], all we will have to show is that for any $a \in R$ there is some lower monic polynomial f of co-degree 1 such that $f(a) \in C$, the center of R. Assume by contradiction that some a fails to satisfy the latter condition. Then $a \notin C$ and there must be some b such that $ab \neq ba$. By Proposition 4, there is some lower monic polynomial $s(t)$ of co-degree 1 such that $s(ab) = 0$. Since the co-degree of $s(t)$ is 1, we have for some r, $ab = (ab)^2 r$ and $(ab)r = r(ab)$. Then $e = (ab)r$ is an idempotent. If $e = 0$ then $ab = 0$ and $ba = 0 = ab$, contrary to the hypothesis. Therefore e is nonzero idempotent. Since R is subdirectly irreducible and since, by Proposition 5, e is central then e must be the identity of R. Therefore $(ab)r = r(ab) = 1$.

Repeating for ba, we see that b is invertible. Consider $b^{-1}a$ and b. If $(b^{-1}a)b = b(b^{-1}a)$ then $b^{-1}ab = a$ and $ab = ba$, contrary to the hypothesis. Therefore $b^{-1}a$ and b do not commute. By Proposition 4 again, there is some lower monic polynomial $f(t)$ of co-degree 1 such that $f(b^{-1}ab) = 0$. As $f(b^{-1}ab) = b^{-1}f(a)b$ we have $b^{-1}f(a)b = 0$. Hence $f(a) = 0$ and $f(a) \in C$, a contradiction. This establishes the theorem.

COROLLARY 1. *Any σ-reflexive semigroup which is the multiplicative semigroup of a ring is commutative.*

COROLLARY 2. *Any generalized commutative ring R in which the integers $n = n(x, y)$ in the equation $(xy)^n = (yx)^m$ can be taken equal to 1 for all $x, y \in R$ is a commutative ring.*

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CARLETON UNIVERSITY,
OTTAWA, ONTARIO
UNIVERSITY OF WESTERN ONTARIO,
LONDON, ONTARIO