Can. J. Math. Vol. **48** (2), 1996 pp. 381–396

A SUBNORMAL OPERATOR AND ITS DUAL

ROBERT F. OLIN AND LIMING YANG

ABSTRACT. It is shown that the essential spectrum of a cyclic, self-dual, subnormal operator is symmetric with respect to the real axis. The study of the structure of a cyclic, irreducible, self-dual, subnormal operator is reduced to the operator S_μ with $\mathrm{bpe}\mu = D$. Necessary and sufficient conditions for a cyclic subnormal operator S_μ with $\mathrm{bpe}\mu = D$ to be self-dual are obtained under the additional assumption that the measure on the unit circle is log-integrable. Finally, an approach to a general cyclic, self-dual, subnormal operator is discussed.

0. **Introduction.** Let $\mathcal{H}$ be a separable Hilbert space over the complex field $\mathbb{C}$ and let $\mathcal{L}(\mathcal{H})$ be the algebra of all linear bounded operators on $\mathcal{H}$. The operator S in $\mathcal{L}(\mathcal{H})$ is a subnormal operator if there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and if there is a normal operator N on $\mathcal{K}$ which leaves $\mathcal{H}$ invariant such that N restricted to $\mathcal{H}$ is S. Let S be a pure subnormal operator on $\mathcal{H}$ (that is, S is a subnormal operator with no normal direct summand). If N is the minimal normal extension of S, then the dual T of S is the restriction of N^* to the space $\mathcal{K}\ominus\mathcal{H}$. A subnormal operator S is self-dual if S is unitarily equivalent to its dual T (this notion was introduced by J. Conway in [1]). The reader can consult [2] and [3] for a thorough exposition of subnormal operators.

The relations between a subnormal operator and its dual have been examined by numerous people (for example, see the papers [1], [6] and [10]). This work continues this investigation. We give some necessary and sufficient conditions for a cyclic, irreducible, subnormal operator to be self-dual. The main tools used in this paper are Thomson's recent results (see [9]) on analytic bounded point evaluations for the space $P^2(\mu)$ and our recent results (see [8]) on the boundary behavior of functions in the algebra $P^2(\mu)\cap L^\infty(\mu)$.

In Section 1, we explain the notation used already and present the notation, terminology and well-known facts that are related to our work.

In Section 2, we obtain the following set inclusion relating the essential spectrum of the minimal normal extension to the essential spectra of a subnormal operator and its dual:

$$\sigma_e(N) \subset \sigma_e(S) \cup \sigma_e(T^*).$$

It is shown that if a function f in $P^2(\mu)$ is zero μ a.e. in neighborhood of a point on the boundary, then f has to be the zero function. We are then able to prove the essential

The first author was supported in part by NSF DMS9019865 and the second author was supported in part by NSF DMS9401234

Received by the editors April 29, 1994.

AMS subject classification: Primary: 47B20; secondary: 30H05.

https://doi.org/10.4153/CJM-1996-021-1 Published online by Cambridge University Press

spectrum of a cyclic, self-dual, subnormal operator is symmetric with respect to the real axis.

In Section 3, our analysis of the class of cyclic, irreducible, self-dual, subnormal operators is reduced to one where the focal point is on those operators in this collection whose set of bounded point evaluations is the open unit disc.

Section 4 gives some necessary and sufficient conditions for a cyclic subnormal operator, whose set of bounded point evaluations is the open unit disc, to be self-dual under additional hypotheses that the scalar spectral measure restricted to the unit circle is log-integrable.

Finally, in Section 5, we give an approach to the classification problem for an arbitrary cyclic, self-dual, subnormal operator.

1. **Preliminaries.** For an operator T in $\mathcal{L}(\mathcal{H})$, the sets $\sigma(T)$ and $\sigma_e(T)$ consist of the spectrum and essential spectrum of T, respectively. If S is a subnormal operator, then the minimal normal extension N can be written in a matrix format as follows:

$$N = \begin{bmatrix} S & X \\ 0 & T^* \end{bmatrix}$$

where T is the dual of S. For a finite compactly supported measure μ on $\mathbb{C}$, the Hilbert space $P^2(\mu)$ denotes the closure of polynomials in $L^2(\mu)$ and the operator S_μ is multiplication by z on $P^2(\mu)$. Similarly, the minimal normal extension of S_μ is N_μ, multiplication by z on $L^2(\mu)$, and the dual of S_μ will be denoted by T_μ. Let μ^* denote the measure obtained from μ as follows:

$$\mu^*(\Delta) = \mu(\Delta^*)$$

where Δ is any Borel subset of $\mathbb{C}$ and $\Delta^* = \{z : \bar{z} \in \Delta\}$. It is well-known that every cyclic subnormal operator is unitarily equivalent to S_μ for a suitable choice of μ. Recently, J. Thomson [9] proved that every pure, cyclic, subnormal operator has a nontrivial set of analytic bounded point evaluations. In fact, he obtained much more.

THOMSON'S THEOREM. *If μ is any compactly supported measure on $\mathbb{C}$, then there is a Borel partition $\{\Delta_0, \Delta_1, \ldots\}$ of the support of μ such that if $\mu_n = \mu|_{\Delta_n}$, then the following statements are true.*

(a) $P^2(\mu) = L^2(\mu_0) \oplus P^2(\mu_1) \oplus \cdots$.

(b) *If $n \geq 1$, then S_{μ_n} is irreducible. Equivalently, $P^2(\mu_n)$ contains no nontrivial characteristic functions.*

(c) *If $n \geq 1$ and $G_n = \mathrm{abpe}(\mu_n)$, then G_n is a simply connected region with $\mathrm{spt}(\mu_n) \subset \bar{G}_n$ and $\mathrm{bpe}(\mu_n) = G_n$.*

(d) *If S_μ is an irreducible operator and G is the set of analytic bounded point evaluations for μ, then the Banach algebras $P^2(\mu) \cap L^\infty(\mu)$ and $H^\infty(G)$ are algebraically and isometrically isomorphic and weak-star homeomorphic.*

Suppose S_μ is an irreducible subnormal operator. Using Thomson's Theorem, we let $\sim$ be the isometric isomorphism from $H^\infty(G)$ onto $P^2(\mu) \cap L^\infty(\mu)$, where G is the simply connected region comprised of all bounded point evaluations for $P^2(\mu)$. For $\tilde{f}$ in

$P^2(\mu)\cap L^\infty(\mu)$, the operator $M^\mu_{\tilde f}$ consists of multiplication by $\tilde f$ on $P^2(\mu)$; that is,

$$M^\mu_{\tilde f} g = \tilde f g, \quad \text{for } g \in P^2(\mu).$$

Let φ be a Riemann map of G onto the unit disc D. From the properties of the isomorphism $\sim$, the function $\tilde\varphi$ is in $P^2(\mu)\cap L^\infty(\mu)$ and

$$(\tilde\varphi, K^\mu_\lambda) = \varphi(\lambda), \quad \text{for all } \lambda \in G$$

where K^μ_λ is the kernel function for $P^2(\mu)$. For $f \in P^2(\mu)$, let

$$\hat f(\lambda) = (f, K^\mu_\lambda).$$

The pull back of μ to the closed unit disc is denoted by ν; that is, $\nu = \mu \circ \tilde\varphi^{-1}$. Let ω be the harmonic measure for G with respect to a fixed point. Finally, we denote $\psi = \varphi^{-1}$. The following theorem is found in [8].

THEOREM 1.1. *Let μ, ν be as above. The following facts hold true:*

(1) The measure $\mu|_{\partial G}$ is absolutely continuous with respect to the harmonic measure ω.

(2) $\tilde\psi$ is one-to-one almost everywhere from a carrier of $\nu|_{\partial D}$ to a carrier of $\mu|_{\partial G}$.

(3) The operator S_μ is unitarily equivalent to $M^\nu_{\tilde\psi}$ on $P^2(\nu)$.

For $f \in H^\infty(G)$ and $\tilde f \in P^2(\mu)\cap L^\infty(\mu)$, we will need nontangential limits of $\tilde f$ on a carrier of μ restricted to ∂G. Looking at the last theorem, we may assume that $\tilde\psi$ is a one-to-one map from a Borel set $E \subset \partial D$ to a Borel set $F \subset \partial G$ with $\nu(E^c) = 0$ and $\mu(F^c) = 0$. Observing $f\circ\psi$ is in $H^\infty(D)$, we can choose a Borel set $E_1 \subset E$ with $m(E_1) = 0$ and for every point $e^{i\theta}$ in $E\setminus E_1$, we have the radial limit of $f\circ\psi$

$$\lim_{r\to 1^-} f\circ\psi(re^{i\theta}) = (f\circ\psi)^*(e^{i\theta}).$$

(Actually we may compute $(f\circ\psi)^*(e^{i\theta})$ as a nontangential limit m a.e.)

Define

$$f^*(w) = (f\circ\psi)^*(e^{i\theta}) \tag{11}$$

for each $w \in \tilde\psi(E\setminus E_1)$, where we find a unique $e^{i\theta}$ in $E\setminus E_1$ so that $w = \psi(e^{i\theta})$. This radial limit $f^*(w)$ is well-defined on a carrier of $\mu|_{\partial G}$. The following theorem was also proved in [8].

THEOREM 1.2. *If $f \in H^\infty(G)$ and $\tilde f \in P^2(\mu)\cap L^\infty(\mu)$, then $\tilde f(w) = f^*(w)$ almost everywhere with respect to $\mu|_{\partial G}$.*

2. **Essential spectra of self-dual subnormal operators.** A fundamental inclusion in this area of operator theory is that

$$\sigma(N) \subset \sigma(S),$$

a fact originally proved by Paul Halmos. We now present another central inclusion between the essential spectrum of N and the essential spectra of S and its dual T. Recalling the fact that

$$\sigma(S) = \sigma(T^*)$$

(see [1]), we can derive the Halmos result from our Proposition 2.1.

PROPOSITION 2.1. *Let S be a pure subnormal operator on $\mathcal{H}$ with the minimal normal extension N on $\mathcal{K}$ and let T be the dual of S. then*

$$\sigma_e(N) \subset \sigma_e(S) \cup \sigma_e(T^*).$$

PROOF. Suppose to the contrary that there is a point $\lambda_0 \in \sigma_e(N) \setminus \big(\sigma_e(S) \cup \sigma_e(T^*)\big)$. Choose an infinite sequence of unit vectors $\{f_n\}$ in $\mathcal{K}$ which converges to zero weakly and

$$\|(N - \lambda_0)f_n\| \longrightarrow 0.$$

For each n, let $f_n = g_n + h_n$ be the decomposition of f_n with respect to the orthogonal decomposition of $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$. It follows that

$$\|(S - \lambda_0)^* g_n\| \longrightarrow 0,$$

and that

$$\|(T - \bar{\lambda}_0)^* h_n\| \longrightarrow 0.$$

For each n let $g_n = g_n^1 + g_n^2$ be the decomposition of g_n with respect to the orthogonal decomposition

$$\mathcal{H} = \operatorname{Ker}(S - \lambda_0)^* \oplus \operatorname{Ran}(S - \lambda_0).$$

We now see g_n^2 converges to zero in norm since $\lambda_0 \in \sigma_e(S)^c$. Therefore, there is a subsequence $\{g_{n_k}\}$ converging in norm to a vector g since g_n^1 is in $\operatorname{Ker}(S - \lambda_0)^*$, a finite dimensional space. Using the same argument, we can show that there is a subsequence $\{h_{n_{kl}}\}$ converging in norm to a vector h. Hence, $f_{n_{kl}}$ converges in norm to a unit vector f. This is a contradiction to the fact that f_n goes to zero weakly.

REMARK. If S has a compact self-commutator, then an easy matricial argument shows

$$\sigma_e(N) = \sigma_e(S) \cup \sigma_e(T^*).$$

When S is a self-dual subnormal operator, J. Conway [1] shows that the spectrum of S and the spectrum of N are symmetric with respect to the real axis. It is natural to ask the following question.

QUESTION. Is the essential spectrum of a self-dual subnormal operator symmetric with respect to the real axis?

We have not resolved this issue. The goal for this section is to supply an affirmative answer when S has a cyclic vector. First, we establish a result which is similar to the Riesz theorem for the classical Hardy spaces.

THEOREM 2.2. *Let S_μ be an irreducible subnormal operator on $P^2(\mu)$ with* $\mathrm{bpe}\mu = G$. *Let $\lambda_0 \in \partial G$ and let $\Delta = O(\lambda_0, \delta_0)$ be the open disc with center at λ_0 and radius δ_0. Suppose $f \in P^2(\mu)$ is zero almost everywhere with respect to $\mu|_\Delta$. Then $f \equiv 0$ almost everywhere with respect to μ.*

PROOF. Suppose that f is a nonzero function. Let L be a connected subset of $O(\lambda_0, \frac{\delta_0}{2}) \cap \partial G$ and let ϕ be the Riemann map from $\mathbb{C} \setminus L$ to the open unit disk. Letting h be a constant times ϕ, we may assume that h is in $H^\infty(G)$, $|h| < 1$ off Δ, and $|h(w)| \geq 1$ for some $w \in \Delta \cap G$ with $\langle f, K_w \rangle \neq 0$. Using Thomson's Theorem, we have $\tilde{h} \in P^2(\mu) \cap L^\infty(\mu)$ and $\tilde{h} = h$ a.e. $\mu|_{L^c}$. Now $\tilde{h}^n f$ converges to zero in $L^2(\mu)$-norm, while $|\langle \tilde{h}^n f, K_w \rangle| = |h^n(w)|\,|\langle f, K_w \rangle|$ does not converge to zero. This is a contradiction.

The following lemma is an old chestnut. For the sake of completeness we include its proof.

LEMMA 2.3. *Let S_μ be an irreducible subnormal operator on $P^2(\mu)$ with* $\mathrm{bpe}\mu = G$ *and $\lambda_0 \in G$. Then there is a small positive constant $\delta_0 > 0$ such that S_{μ_0} is similar to S_μ where $\mu_0 = \mu|_{\Delta_0^c}$ and $\Delta_0 = O(\lambda_0, \delta_0)$.*

PROOF. There is a constant $M > 0$ so that for all polynomial p

$$\begin{aligned}\int |p|^2\,d\mu \leq M \int |z-\lambda_0|^2 |p|^2\,d\mu &= M \int_{\Delta_0} |z-\lambda_0|^2|p|^2\,d\mu + M\int_{\Delta_0^c} |z-\lambda_0|^2|p|^2\,d\mu \\ &= M\delta_0^2 \int_{\Delta_0} |p|^2\,d\mu + M\|z-\lambda_0\|_\infty^2 \int_{\Delta_0^c} |p|^2\,d\mu.\end{aligned}$$

The first inequality follows since $\mathrm{bpe}\mu = a\mathrm{bpe}\mu$. Choose δ_0 to be small enough such that

$$1 - M\delta_0 > 0.$$

We then have

$$(1 - M\delta_0) \int |p|^2\,d\mu \leq M\|z-\lambda_0\|_\infty^2 \int |p|^2\,d\mu_0.$$

Obviously, we have

$$\int |p|^2\,d\mu_0 \leq \int |p|^2\,d\mu.$$

The last two inequalities yield the desired result: S_{μ_0} is similar to S_μ.

THEOREM 2.4. *Let S_μ be an irreducible, self-dual, subnormal operator on $P^2(\mu)$ and* $\mathrm{bpe}\mu = G$. *Then ∂G is symmetric with respect to real axis.*

PROOF. There is a unitary operator U from $P^2(\mu)$ to $P^2(\mu)^\perp$ so that

$$US_\mu U^* = T_\mu.$$

https://doi.org/10.4153/CJM-1996-021-1 Published online by Cambridge University Press

Suppose that there exists $\lambda_0 \in \partial G$ and $\bar{\lambda}_0 \notin \partial G$. Since $\sigma(S_\mu)$ is symmetric with respect to the real axis and

$$\sigma_e(S_\mu) = \partial G,$$

it follows that $\overline{\lambda_0} \in G$. Using Lemma 2.3, we choose $\delta_0 > 0$ small enough such that $O(\bar{\lambda}_0, \delta_0) \subset G$ and S_{μ_0} is similar to S_μ where

$$\mu_0 = \mu|_{O(\bar{\lambda}_0,\delta_0)^c}.$$

It follows then that S_{μ_0} is an irreducible subnormal operator. Using [3], p. 280, we choose a function $g \in L^2(\mu_0)$ which is orthogonal to $P^2(\mu_0)$ and $|g| > 0$ almost everywhere with respect to μ_0. Define

$$h = \begin{cases} g, & O(\bar{\lambda}_0, \delta_0)^c \\ 0, & O(\bar{\lambda}_0, \delta_0). \end{cases}$$

Clearly h is orthogonal to $P^2(\mu)$. Put $f = U1$; plainly $|f| > 0$ almost everywhere with respect to μ. Since h is orthogonal to $P^2(\mu)$ and f is a cyclic vector, we may choose a sequence of polynomials $\{p_n\}$ so that

$$p_n(\bar{z})f \longrightarrow h$$

in $L^2(\mu)$ norm. Since U is a unitary operator, we see that for any polynomial p

$$\int |p|^2\, d\mu = \int |p(\bar{z})|^2 |f|^2\, d\mu.$$

Hence $\{p_n\}$ must converge to a function t in $P^2(\mu)$. It is easy to show that

$$h(z) = f(z)t(\bar{z}) \quad \text{a.e. } \mu.$$

Therefore, $t(z) = 0$ almost everywhere with respect to $\mu|_{O(\lambda_0,\delta_0)}$. However, according to Theorem 2.2, the function t has to be zero. This is a contradiction since h is not the zero function. The proof is completed.

In [3], p. 408, Conway uses our last theorem in the proof of Proposition 6.5. Conway does not prove the theorem; he asserts its validity follows from the fact that the spectrum is symmetric with respect to the real axis. To see that more justification is needed one should ponder why the following subnormal operator is not self-dual.

It is easy to construct a measure μ enjoying the following properties:

(1) $\sigma(S_\mu) = \bar{D}$.

(2) The support of μ is symmetric with respect to the real axis.

(3) $a\text{bpe}(\mu) = D \setminus L$, where $L = \{z : |z| < 1, \operatorname{Re} z \leq 0, \operatorname{Im} z = \frac{1}{2}\}$. It then follows from Thomson's Theorem that

$$\sigma_e(S_\mu) = \partial D \cup L.$$

COROLLARY 2.5. *Suppose S_μ is an irreducible, self-dual, subnormal operator on $P^2(\mu)$ with* $\text{bpe}(\mu) = G$. *We then have*

$$\sigma(N_\mu) = \partial G \cup \{\lambda_n\}$$

where $\{\lambda_n\} \subset G$ *is a sequence of isolated points.*

PROOF. It is well-known that for any normal operator N

$$\sigma(N) \setminus \sigma_e(N) = \{\lambda_n\}$$

where $\{\lambda_n\}$ is a sequence of isolated points. Using the remark after Proposition 2.1 and the result of Theorem 2.4, we have

$$\sigma_e(N_\mu) = \sigma_e(S_\mu) = \partial G.$$

3. **Reformulation of the problem; a reduction to the unit disc.** Suppose S_μ is an irreducible, cyclic, self-dual, subnormal operator on $P^2(\mu)$ with $\operatorname{bpe}(\mu) = G$. Theorem 2.4 implies that G is equal to G^*. Let ψ be a Riemann map from D to G where $\psi(0) = a \in G$ is a real number and $\psi'(0) > 0$. If we define the analytic function ψ_0 on D by setting $\psi_0(z) = \overline{\psi(\bar{z})}$, then ψ_0 is also a Riemann map with the properties $\psi_0(0) = a$ and $\psi_0'(0) > 0$. From the uniqueness of the Riemann map, one sees $\psi_0(z) = \psi(z)$. We define the measure ν on $\bar{D}$ as done in Theorem 1.1.

THEOREM 3.1. *We use the notation and results of preceding paragraph. Let* S_μ *be a cyclic irreducible operator on* $P^2(\mu)$ *with bounded point evaluations* $\operatorname{bpe}\mu = G$. *Then the operator* S_μ *is self-dual if and only if the following two properties hold:*

(1) The operator S_ν *is a self-dual subnormal operator on* $P^2(\nu)$.

(2) The operator S_μ *is unitarily equivalent to* $M^\nu_{\tilde{\psi}}$ *(multiplication by* $\tilde{\psi}$ *on the space* $P^2(\nu)$*) and* $\psi(z) = \overline{\psi(\bar{z})}$.

PROOF. Suppose conditions (1) and (2) are satisfied. Let U be a unitary operator from $P^2(\nu)^\perp$ to $P^2(\nu)$ such that

$$U^* S_\nu U = T_\nu.$$

If $\tilde{\psi}(N_\nu)$ denotes the operator of multiplication by $\tilde{\psi}$ on $L^2(\nu)$, then it can be written matricially as

$$\tilde{\psi}(N_\nu) = \begin{bmatrix} M^\nu_{\tilde{\psi}} & * \\ 0 & T_1^* \end{bmatrix}$$

on $L^2(\nu) = P^2(\nu) \oplus P^2(\nu)^\perp$. Using the lifting theorem [2], p. 128, there exists a unitary operator V on $L^2(\nu)$ so that

$$V^* N_\nu V = N_\nu^*.$$

Thus, for every function $f \in L^\infty(\nu)$, we have

$$V^* f(N_\nu) V = f(N_\nu^*)$$

where $f(N_\nu^*)$ is the operator obtained by multiplication by $f(\bar{z})$ on $L^2(\nu)$. Using Theorem 1.2, we know that $\tilde{\psi}|_{\partial D}$ is equal to the nontangential limit of ψ almost everywhere with respect to $\nu|_{\partial D}$. This implies

$$\tilde{\psi}(\bar{z}) = \overline{\tilde{\psi}(z)} \quad \text{a.e. } \nu$$

Hence,

$$V^*\tilde{\psi}(N_\nu)V = \tilde{\psi}(N_\nu)^*.$$

However, V can be expressed matricially as

$$V = \begin{bmatrix} 0 & U \\ U^* & 0 \end{bmatrix}$$

with respect to the decomposition $L^2(\nu) = P^2(\nu) \oplus P^2(\nu)^\perp$. A trivial matrix computation shows

$$U^*M^\nu_{\tilde{\psi}}U = T_1.$$

Since $\tilde{\psi}(N_\nu)$ is the minimal normal extension of $M^\nu_{\tilde{\psi}}$, see [5] or [7], it follows from condition (2) that S_μ is self-dual.

Now suppose that S_μ is a self-dual subnormal operator. Let $\varphi = \psi^{-1}$ be the Riemann map from G to D, then $\tilde{\varphi} \in P^2(\mu)$. Set $\nu = \mu \circ \tilde{\varphi}^{-1}$, then according to Theorem 1.1, we know that S_μ is unitarily equivalent to $M^\nu_{\tilde{\psi}}$ on $P^2(\nu)$. Also by the argument before the theorem, we can show

$$\psi(z) = \overline{\psi(\bar{z})}$$

So (2) is proved.

Looking at Theorem 1.1 again, we know that S_ν is unitarily equivalent to $M^\mu_{\tilde{\varphi}}$ on $P^2(\mu)$. It also follows from Theorem 1.2 that every function in the algebra $P^2(\mu) \cap L^\infty(\mu)$ has nontangential limit almost everywhere with respect to $\mu|_{\partial G}$ which guarantees

$$\overline{\tilde{\varphi}(z)} = \tilde{\varphi}(\bar{z}) \quad \text{a.e. } \mu.$$

Using the same argument as above, we can show that $M^\mu_{\tilde{\varphi}}$ is self-dual. That is, S_ν is self-dual. The assertion in (1) is verified.

Theorem 3.1 says that the study of a cyclic, self-dual, subnormal operator can be done under the additional assumption that bpe$\mu = D$.

4. **Self-dual, cyclic, subnormal operators having the unit disc as their set of bounded point evaluations.** In this section, we study the class of self-dual operators mentioned at the end of the last section. That is, a cyclic, self-dual, subnormal operator S_μ with bpe$\mu = D$. We always assume that $\frac{d\mu|_{\partial D}}{dm}$ is log-integrable. That is,

$$\int \log \frac{d\mu|_{\partial D}}{dm}\, dm > -\infty.$$

where m is the normalized Lebesgue measure ($dm = \frac{1}{2\pi}d\theta$). The latter assumption implies (in fact it's equivalent to) that the operator, multiplication by z on $P^2(\mu|_{\partial D})$, is pure. By Szego's Theorem (see [4], p. 136), there is an outer function $r \in H^2$ such that

$$\mu|_{\partial D} = |r|^2 m.$$

From the results in [1] and Corollary 2.5, we can assume

$$(*) \qquad \mu = |r|^2 m + \sum_{i=1}^{\infty} \beta_i \delta_{a_i} + \sum_{j=1}^{\infty} (\gamma_j \delta_{b_j} + \gamma_j' \delta_{\bar{b}_j})$$

where the notation δ_a denotes point mass measure at a; the constants $\beta_i, \gamma_j, \gamma_j'$ are strictly positive; the constants a_i are real; and the constants b_j have nonzero imaginary parts. For $a \in D$, we define

$$\varphi_a(z) = \frac{z-a}{1-\bar{a}z}.$$

LEMMA 4.1. *Let φ be an infinite Blaschke product whose zeros are exactly $a_1, a_2, \dots$ with each having multiplicity one. Let h be a function in H^2. If*

$$\sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|\frac{\varphi}{\varphi_{a_n}}(a_n)|} |h(a_i)|\,|h(\bar{a}_i)| < \infty,$$

then

$$\int p(z)h(z)\overline{h(\bar{z})} z\bar{\varphi}\, dm = \sum_{n=1}^{\infty} \frac{1-|a_n|^2}{\frac{\varphi}{\varphi_{a_n}}(a_n)} p(a_n)h(a_n)\overline{h(\bar{a}_n)}$$

for every polynomial p.

PROOF. If $\varphi_n = \varphi_{a_1}\varphi_{a_2}\cdots\varphi_{a_n}$, then the sequence $\{\varphi_n\}$ converges to φ in the weak-star topology. Hence, using Cauchy integral formula, we have

$$\begin{aligned}\int p(z)h(z)\overline{h(\bar{z})} z\bar{\varphi}\, dm &= \lim \int p(z)h(z)\overline{h(\bar{z})} z\bar{\varphi}_n\, dm \\ &= \lim \frac{1}{2\pi i} \int \frac{p(z)h(z)\overline{h(\bar{z})}}{\varphi_n(z)}\, dz \\ &= \lim \sum_{i=1}^{n} \frac{1-|a_i|^2}{\frac{\varphi_n}{\varphi_{a_i}}(a_i)} p(a_i)h(a_i)\overline{h(\bar{a}_i)}.\end{aligned}$$

Now note that

$$\begin{aligned}\sum_{i=1}^{n} \frac{1-|a_i|^2}{|\frac{\varphi_n}{\varphi_{a_i}}(a_i)|} |p(a_i)|\,|h(a_i)|\,|h(\bar{a}_i)| &\le \sum_{i=1}^{n} \frac{1-|a_i|^2}{|\frac{\varphi}{\varphi_{a_i}}(a_i)|} |p(a_i)|\,|h(a_i)|\,|h(\bar{a}_i)| \\ &\le \|p\|_\infty \sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|\frac{\varphi}{\varphi_{a_n}}(a_n)|} |h(a_n)|\,|h(\bar{a}_n)| < \infty.\end{aligned}$$

An easy application of the Lebesgue dominated theorem yields the desired result.

THEOREM 4.2. *Let μ be as in (*). The operator S_μ is self-dual if and only if*

(a) The set $\{a_i, b_j, \bar{b}_j\}$ is the zero set of a nonzero function in H^∞. In the case, our notation for the Blaschke factor of this function is

$$\varphi = \prod \varphi_{a_i} \prod \varphi_{b_j} \varphi_{\bar{b}_j};$$

(b) The zeros of φ and the weights of μ are related as follows:

$$\beta_i = |r(a_i)|^2 \frac{1-a_i^2}{|\frac{\varphi}{\varphi_{a_i}}(a_i)|},$$

$$\sqrt{\gamma_j\gamma_j'} = |r(b_j)|\,|r(\bar{b}_j)| \frac{1-|b_j|^2}{|\frac{\varphi}{\varphi_{b_j}}(b_j)|},$$

and

$$\sum \beta_i + \sum(\gamma_j + \gamma_j') < \infty.$$

PROOF. Suppose both (a) and (b) hold. Let

$$f(z) = \begin{cases} \bar{z}\varphi(z)\frac{\overline{r(\bar{z})}}{r(z)} & \text{on } \partial D, \\ -|\frac{\varphi}{\varphi_{a_i}}(a_i)|/\left(\overline{\frac{\varphi}{\varphi_{a_i}}(a_i)}\right) & z = a_i, \\ -\sqrt{\frac{\gamma_j'}{\gamma_j}}\left(|\frac{\varphi}{\varphi_{b_j}}(b_j)|/\left(\overline{\frac{\varphi}{\varphi_{b_j}}(b_j)}\right)\right)\frac{\overline{r(b_j)}}{|r(b_j)|}\frac{r(\bar{b}_j)}{|r(\bar{b}_j)|} & z = b_j, \\ -\sqrt{\frac{\gamma_j}{\gamma_j'}}\left(|\frac{\varphi}{\varphi_{\bar{b}_j}}(\bar{b}_j)|/\left(\overline{\frac{\varphi}{\varphi_{\bar{b}_j}}(\bar{b}_j)}\right)\right)\frac{r(b_j)}{|r(b_j)|}\frac{\overline{r(\bar{b}_j)}}{|r(\bar{b}_j)|} & z = \bar{b}_j. \end{cases}$$

CLAIM 1. $f \perp P^2(\mu)$.
The validity of the claim is a simple computation:

$$\begin{aligned} \langle p, f\rangle &= \int pr\overline{r(\bar{z})}z\bar{\varphi}\, dm + \sum \beta_i p(a_i)\overline{f(a_i)} + \sum\left(\gamma_j p(b_j)\overline{f(b_j)} + \gamma_j' p(\bar{b}_j)\overline{f(\bar{b}_j)}\right) \\ &= \int pr\overline{r(\bar{z})}z\bar{\varphi}\, dm - \sum(1-a_i^2)p(a_i)r(a_1)\overline{r(\bar{a}_i)}\Big/\left(\frac{\varphi}{\varphi_{a_i}}(a_i)\right) \\ &\quad - \sum(1-|b_j|^2)p(b_j)r(b_j)\overline{r(\bar{b}_j)}\Big/\left(\frac{\varphi}{\varphi_{b_j}}(b_j)\right) \\ &\quad - \sum(1-|\bar{b}_j|^2)p(\bar{b}_j)r(\bar{b}_j)\overline{r(b_j)}\Big/\left(\frac{\varphi}{\varphi_{\bar{b}_j}}(\bar{b}_j)\right) \\ &= 0. \end{aligned}$$

Note that the last equality follows from Lemma 4.1.

CLAIM 2. $\{\overline{p(z)}f : p \text{ is a polynomial}\}$ is dense in $P^2(\mu)^\perp$.

It is sufficient to show that if $g \in L^2(\mu)$ is orthogonal to both $P^2(\mu)$ and $\text{span}\{\overline{p(z)}f\}$, then g is the zero function. If g is the function with these properties, then for every polynomial p

$$\langle \bar{\varphi}\bar{p}f, g\rangle = 0$$

because

$$\overline{P^\infty(\mu)}f \subset L^2(\mu) \text{ closure of } \{\overline{p(z)}f\}.$$

It follows that for every polynomial p

$$\int pr\overline{r(\bar{z})}zg\, dm = 0.$$

Since $\overline{r(\bar{z})}$ is an outer function, we see that

$$\int przg\,dm = 0$$

for every polynomial p. Thus, there is a function $g_0 \in H^2(\partial D)$ such that

$$\text{(a)} \qquad rg|_{\partial D} = g_0.$$

On the other hand, we see that for every polynomial p

$$\left\langle \bar{p}\overline{\left(\frac{\varphi}{\varphi_{a_i}}\right)}f, g\right\rangle = 0.$$

This implies that for any polynomial p

$$\int pr\overline{r(\bar{z})}z\bar{\varphi}_{a_i}g\,dm = |r(a_i)|^2(1-a_i^2)p(a_i)g(a_i) = \int p\overline{r(\bar{z})}g_0(1-a_iz)\bar{k}_{a_i}\,dm$$

where $k_\lambda(z) = \frac{1}{1-\bar{\lambda}z}$. Therefore,

$$\overline{r(a_i)}(1-a_i^2)p(a_i)\tilde{g}_0(a_i) = |r(a_i)|^2(1-a_i^2)p(a_i)g(a_i)$$

where $\tilde{g}_0$ is the analytic extension to the disc. It follows that for all i

$$\text{(b)} \qquad r(a_i)g(a_i) = \tilde{g}_0(a_i).$$

Using the same method, we can show that for all j

$$(c) \qquad r(b_j)g(b_j) = \tilde{g}_0(b_j), \quad r(\bar{b}_j)g(\bar{b}_j) = \tilde{g}_0(\bar{b}_j).$$

Now let K_λ be the reproducing kernel for $P^2(\mu)$ and let

$$\varphi_n = \prod_{i=1}^n \varphi_{a_i} \prod_{j=1}^n \varphi_{b_j}\varphi_{\bar{b}_j}.$$

For each $\lambda \in D$ and for each polynomial p we have

$$\begin{aligned}
p(\lambda)\frac{\varphi(\lambda)}{\varphi_n(\lambda)}
&= \int \frac{\varphi(z)}{\varphi_n(z)}p(z)\bar{K}_\lambda\,d\mu \\
&= \int \frac{\varphi(z)}{\varphi_n(z)}p(z)|r|^2\bar{K}_\lambda\,dm + \sum_{i=1}^n \beta_i\frac{\varphi}{\varphi_n}(a_i)p(a_i)\overline{K_\lambda(a_i)} \\
&\quad + \sum_{i=1}^n\left(\gamma_i\frac{\varphi}{\varphi_n}(b_i)p(b_i)\overline{K_\lambda(b_i)} + \gamma_i'\frac{\varphi}{\varphi_n}(\bar{b}_i)p(\bar{b}_i)\overline{K_\lambda(\bar{b}_i)}\right) \\
&= \int \frac{\varphi(z)}{\varphi_n(z)}p(z)r(z)\overline{\left(rK_\lambda + \sum_{i=1}^n\left(\beta_i\frac{K_\lambda(a_i)}{\overline{r(a_i)}}k_{a_i} + \gamma_i\frac{K_\lambda(b_i)}{\overline{r(b_j)}}k_{b_i} + \gamma_i'\frac{K_\lambda(\bar{b}_i)}{\overline{r(\bar{b}_i)}}k_{\bar{b}_i}\right)\right)}\,dm.
\end{aligned}$$

https://doi.org/10.4153/CJM-1996-021-1 Published online by Cambridge University Press

Since there are polynomials p_n such that $p_n r$ converges to g_0 in H^2, we now have that for all n

$$\begin{aligned}\frac{g_0(\lambda)}{r(\lambda)}\frac{\varphi(\lambda)}{\varphi_n(\lambda)}\\ &= \int \frac{\varphi(z)}{\varphi_n(z)} g_0(z) \overline{\left(rK_\lambda + \sum_{i=1}^{n} \left(\beta_i \frac{K_\lambda(a_i)}{\overline{r(a_i)}} k_{a_i} + \gamma_i \frac{K_\lambda(b_i)}{\overline{r(b_j)}} k_{b_i} + \gamma_i' \frac{K_\lambda(\bar{b}_i)}{\overline{r(\bar{b}_i)}} k_{\bar{b}_i} \right) \right)}\, dm \\ &= \int \frac{\varphi(z)}{\varphi_n(z)} g(z) \bar{K}_\lambda \, d\mu.\end{aligned}$$

The last equality follows from (a), (b), and (c). The sequence of functions $\frac{\varphi}{\varphi_n}$ converges to 1 in both $P^\infty(\mu)$ and $H^\infty(\partial D)$ in the weak-star topology. Hence, for all $\lambda \in D$ we have

$$\int g\bar{K}_\lambda \, d\mu = \frac{g_0(\lambda)}{r(\lambda)} = 0.$$

Therefore, $g_0 = 0$. It now follows that g is the zero function. This establishes Claim 2.

Let U be the operator form $P^2(\mu)$ to $P^2(\mu)^\perp$ defined by

$$(Up)(z) = p(\bar{z})f(z)$$

for every polynomial p. By the assumption, it is easy to check $\mu = \left(|f(z)|^2\mu\right)^*$. Thus,

$$\|Up\|^2 = \int |p(\bar{z})|^2|f(z)|^2 \, d\mu = \int |p(z)|^2\left(|f(z)|^2 \, d\mu\right)^* = \int |p(z)|^2 \, d\mu.$$

This means U is a unitary operator. Also we have

$$US_\mu p = \bar{z}p(\bar{z})f(z) = \bar{z}Up = T_\mu Up$$

for every polynomial p. Therefore, S_μ is a self-dual subnormal operator.

Now suppose S_μ is a self-dual subnormal operator, that is, there is a unitary operator U such that

$$US_\mu U^* = T_\mu.$$

If

$$h = \begin{cases} \frac{\bar{z}}{r(z)}, & \partial D \\ 0, & D, \end{cases}$$

then h is orthogonal to $P^2(\mu)$. Let $s = U^*h$. The function UK_λ is the kernel function for $P^2(\mu)^\perp$ and

$$\langle s, K_\lambda \rangle = \langle h, UK_\lambda \rangle.$$

From the definition of h and the fact that the defining values of h agree with the analytic extension of h to D almost everywhere μ, we see

$$\langle h, UK_{a_i} \rangle = \langle h, UK_{b_j} \rangle = \langle h, UK_{\bar{b}_j} \rangle = 0.$$

Hence,

$$\langle s, K_{a_i} \rangle = \langle s, K_{b_j} \rangle = \langle s, K_{\bar{b}_j} \rangle = 0.$$

It now follows that $\{a_i, b_j, \bar{b}_j\}$ is a zero set of a nonzero bounded analytic function. As before, we write $\varphi = \prod \varphi_{a_i} \prod \varphi_{b_j} \varphi_{\bar{b}_j}$ and let $U1 = f$. It follows then that for all polynomials p that $(Up)(z) = p(\bar{z})f(z)$. Since U is a unitary operator we now see that for all polynomials p_1 and p_2

$$\begin{aligned}\int p_1(z)\bar{p}_2(z)\,d\mu(z) &= \int p_1(\bar{z})\bar{p}_2(\bar{z})|f(z)|^2\,d\mu(z) \\ &= \int p_1(z)\bar{p}_2(z)|f(\bar{z})|^2\,d\mu^*(z).\end{aligned}$$

It now follows from Stone-Weierstrass theorem that

$$\mu = |f(\bar{z})|^2\mu^*.$$

This means

$$|f(z)| = \frac{|r(\bar{z})|}{|r(z)|} \quad \text{a.e. } m \text{ on } \partial D,$$

$$|f(a_i)| = 1,$$

and

$$|f(b_j)| = \sqrt{\frac{\gamma_j'}{\gamma_j}}, \quad |f(\bar{b}_j)| = \sqrt{\frac{\gamma_j}{\gamma_j'}}.$$

For every polynomial p,

$$\int p\varphi\bar{f}\,d\mu = 0.$$

Thus,

$$\int p\varphi\bar{f}\bar{r}\,dm = 0.$$

since r is an outer function. Hence, on ∂D the function $\varphi\bar{f}\bar{r} \in H_0^2$. Therefore, there are an inner function ϕ and an outer function k so that

$$\varphi\bar{f}\bar{r} = z\phi k.$$

Thus, on ∂D, we have

$$|k(z)| = |\overline{r(\bar{z})}|.$$

So $k(z) = a\overline{r(\bar{z})}$ where a is a constant of modulus one. This means on ∂D, we may assume that

$$f = \frac{r(\bar{z})}{r(z)}\bar{z}\varphi\bar{\phi}.$$

Let

$$q = \begin{cases} \bar{\phi}\varphi, & \text{on } \partial D \\ 0, & \text{on } D, \end{cases}$$

we notice that

$$\int p(\bar{z})f\bar{q}\,d\mu = \int p(\bar{z})\bar{z}r(\bar{z})\overline{r(z)}\,dm = 0.$$

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Therefore, $q \in P^2(\mu) \cap L^\infty(\mu)$. It follows now from Theorem 1.2 that $\bar{\phi}\varphi$ is an inner function, I, that has zeros at all atoms of μ. However, since

$$\phi I = \varphi$$

and φ has single zeros at precisely those atoms, it follows that ϕ is a constant. Without loss of generality, we may assume on ∂D, the function f is equal to $\bar{z}\varphi\frac{r(\bar{z})}{r(z)}$. For every polynomial p,

$$\int \frac{\varphi}{\varphi_{a_i}} p\bar{f}\, d\mu = 0.$$

Hence,

$$\int \frac{\varphi}{\varphi_{a_i}} p\bar{f}|r|^2\, dm + \frac{\varphi}{\varphi_{a_i}}(a_i)p(a_i)\beta_i\overline{f(a_i)} = 0.$$

Therefore,

$$\int p\overline{r(\bar{z})}r(z)z\bar{\varphi}_{a_i}\, dm + \frac{\varphi}{\varphi_{a_i}}(a_i)p(a_i)\beta_i\overline{f(a_i)} = 0.$$

A simple computation shows that

$$\left|\frac{\varphi}{\varphi_{a_i}}(a_i)p(a_i)\beta_i f(a_i)\right| = |r(a_i)|^2(1-a_i^2)|p(a_i)|.$$

This implies

$$\beta_i = |r(a_i)|^2(1-a_i^2)\Big/\left|\frac{\varphi}{\varphi_{a_i}}(a_i)\right|.$$

Using the same method, we can prove that for all j

$$\sqrt{\gamma_j\gamma_j'} = |r(b_j)|\,|r(\bar{b}_j)|(1-|b_j|^2)\Big/\left|\frac{\varphi}{\varphi_{b_j}}(b_j)\right| = |r(b_j)|\,|r(\bar{b}_j)|(1-|\bar{b}_j|^2)\Big/\left|\frac{\varphi}{\varphi_{\bar{b}_j}}(\bar{b}_j)\right|.$$

Also we have

$$\sum \beta_i + \sum(\gamma_j + \gamma_j') < \infty$$

since μ is a finite measure. The proof of the Theorem is now completed.

5. **An approach to the general case.** Theorem 4.2 shows that the atoms of the scalar spectral measure play an important role in the study of self-dual cyclic subnormal operators. The next theorem shows if the set of atoms is not too large, then the structure of this cyclic operator is understood.

THEOREM 5.1. *Let S_μ be a cyclic irreducible subnormal operator on $P^2(\mu)$ with* bpe$\mu = G$. *Suppose the set of atoms of the scalar spectral measure μ is a zero set of a nonzero function in $H^\infty(G)$. Then S_μ is a self-dual subnormal operator if and only if the following two properties hold:*

(1) G is symmetric with respect to the real axis. In this case, we let ψ be the Riemann map from D to G so that $\operatorname{Im}\psi(0) = 0$, $\psi'(0) > 0$, *and* $\psi(z) = \overline{\psi(\bar{z})}$. *Moreover, the analytic Toeplitz operator T_ψ is cyclic on $H^2(\partial D)$.*

(2) There is μ_0 satisfying the conditions of Theorem 4.2 such that S_μ is unitarily equivalent to the operator of multiplication by $\tilde{\psi}$ ($= M^{\mu_0}_{\tilde{\psi}}$) on $P^2(\mu_0)$.

PROOF. The sufficiency is obvious. We assume that S_μ is a self-dual subnormal operator. Using Theorem 2.4, we know that G is symmetric with respect to the real axis; so we choose a Riemann map as in (1). Let $\nu = \mu \circ \tilde{\varphi}^{-1}$, where $\varphi = \psi^{-1}$. According to Theorem 3.1, the operator S_μ is unitarily equivalent to multiplication by $\tilde{\psi}$ on $P^2(\nu)$ and S_ν is a self-dual subnormal operator with bpe$\nu = D$. Using the hypotheses, we conclude that there is a nonzero function in $H^\infty(D)$ whose zero set is the set of all atoms of ν. Let $f \in P^2(\nu)^\perp$ be a cyclic vector and let ϕ be an inner function in H^∞ whose zero set is precisely the set of all atoms of ν. We have then that

$$\int p\phi\bar{f}\,d\nu = 0.$$

Thus, $P^2(\nu|_{\partial D})$ is pure. Therefore,

$$\int \log\Big(\frac{d\nu|_{\partial D}}{dm}\Big)\,dm > -\infty.$$

According to Theorem 4.2, there is a measure μ_0 satisfying the conditions of the theorem so that S_ν is unitarily equivalent to S_{μ_0}. Hence, S_μ is unitarily equivalent to the multiplication by $\tilde{\psi}$ on $P^2(\mu_0)$.

Now we need only show T_ψ is cyclic. In fact, using Clary's Theorem (see [3], p. 370), we know that S_{μ_0} is quasisimilar to S_m; therefore, $M^{\mu_0}_{\tilde{\psi}}$ is quasisimilar to T_ψ. This means T_ψ has a cyclic vector. The theorem is proved.

REMARK. We believe that the conditions (1) and (2) are the necessary and sufficient conditions for an irreducible cyclic subnormal operator to be self-dual. We believe that our hypothesis on the atoms is a by product of the hypothesis of self-duality.

ACKNOWLEDGMENT. The authors express their deep gratitude to the referee for pointing out the simple proof of Theorem 2.2 and other comments.

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Department of Mathematics
Virginia Tech
Blacksburg, Virgina 24061
U.S.A.

Current address of second author:
Department of Mathematics
University of Hawaii
Honolulu, HI 96822
U.S.A.