Canadian Mathematical Bulletin
doi:10.4153/CMB-2011-100-6

On Segre Forms of Positive Vector Bundles

Dincer Guler

Abstract. The goal of this note is to prove that the signed Segre forms of Griffiths' positive vector bundles are positive.

1 Introduction

Let E be a Hermitian holomorphic vector bundle over a complex manifold X. Naturally, restrictions on the curvature of E will impose some restrictions on all constructions arising from it. The goal of this note is to prove that when E has a metric with positive (Griffiths') curvature, then certain combinations of Chern forms, known as signed Segre forms, are positive. This gives evidence for a conjecture of Griffiths ([2]), which predicts that if E has a positive curvature, then a combination of Chern forms is positive if and only if it can be written as a nontrivial combination of Schur polynomials of Chern forms with nonnegative coefficients. We remark that the signed Segre forms are Schur polynomials of Chern forms.

A very similar problem was considered by Fulton and Lazarsfeld ([1]) who confirmed the aforementioned conjecture for Chern classes of an ample vector bundle. An everywhere closed positive (p, p) form on a projective manifold X^n always gives a positive (p, p) cohomology class, but for $1 < p < n - 1$, the converse is not known. Before proceeding further we state our main theorem.

Theorem 1.1 (Main Theorem) *Let X be a projective manifold and let E be a Griffiths' positive vector bundle over X. If $S_k(E)$ denote the Segre forms of E, then the form $(-1)^k S_k(E)$ is a positive (k, k)-form for any $k = 1, \dots, n$.*

2 Preliminaries

Positive forms. Let X^n be a complex manifold equipped with a Hermitian metric ω. A smooth (p, p)-form α is said to be strongly positive if in local coordinates there is a representation $\alpha = i^p \alpha_1 \wedge \overline{\alpha}_1 \wedge \alpha_2 \wedge \overline{\alpha}_2 \wedge \cdots \wedge \alpha_p \wedge \overline{\alpha}_p$, where each α_j is a smooth $(1, 0)$ form and α_j's are linearly independent. A smooth (p, p) form φ on X is said to be positive if in local coordinates we can write $\varphi \wedge \alpha = f\omega^n$, where f is a positive function on X, for any strongly positive form of bidegree $(n - p, n - p)$. This definition is independent of the choice of the metric.

In fact a (p, p)-form φ is positive if and only if φ restricts to a volume form on any p-dimensional subvariety of X or if and only if for any $x \in X$ and any linearly

Received by the editors December 2, 2008; revised July 30, 2009.
Published electronically May 20, 2011.
AMS subject classification: **53C55**, 32L05.

independent $(1,0)$ type tangent vectors $v_1, \dots, v_p$ at x it holds that

$$(-i)^{p^2}\varphi(v_1, \dots, v_p, \bar{v}_1, \dots, \bar{v}_p) > 0.$$

We finish this paragraph by remarking that if X and Y are complex manifolds and if $f\colon X \to Y$ is a holomorphic submersion, then for any strongly positive form α on Y, $f^*\alpha$ is again a strongly positive form on X.

Griffiths' positivity, Chern and Segre forms. Let (E, h) be a Hermitian holomorphic vector bundle over X. Recall that the curvature matrix of E is given by $\Theta = (\Theta^i_j)$, where each Θ^i_j is a $(1,1)$-form expressed in local coordinates by $\Theta^i_j = R^i_{j\alpha\beta} dz_\alpha \wedge d\bar{z}_\beta$. For sections u, v in E we define the $(1,1)$ form $\Theta_{u\bar{v}}$ by

$$\Theta_{u\bar{v}} = \sum_{i,j,k=1}^{r} \Theta^k_i h_{kj} u_i \bar{v}_j,$$

where $u = \sum u_i e_i$ and $v = \sum v_i e_i$ under a frame $\{e_1, \dots, e_r\}$ of E. Then $\Theta_{u\bar{v}}$ is a global $(1,1)$ form on X independent of the choice of e. The Hermitian bundle E is said to be positive in the sense of Griffiths, or Griffiths positive if $i\Theta_{v\bar{v}}$ is a positive $(1,1)$ form for any $x \in X$ and any nonzero $v \in E_x$.

Let $\mathbb{P}(E)$ denote the projectivized bundle of lines of E^*. Then $\mathbb{P}(E)$ is a projective manifold which carries the so-called tautological line bundle $\mathcal{O}_E(-1)$ defined by the short exact sequence

$$0 \longrightarrow \mathcal{O}_E(-1) \longrightarrow \pi^* E^* \longrightarrow \mathcal{O}_E(-1) \otimes T_{\mathbb{P}(E)/X} \longrightarrow 0.$$

The Hermitian metric h on E naturally induces a Hermitian metric on $\mathcal{O}_E(-1)$ and it is well known that for any $p \in X$ and any $v \neq 0$, $v \in E_p$ the curvature of $L = \mathcal{O}_E(-1)$ at the point $(p, [v]) \in \mathbb{P}(E)$ is given by

$$\Theta(L)|_{(p,[v])} = \frac{1}{|v|^2}\Theta(E^*)_{v\bar{v}} - \omega_{FS},$$

where ω_{FS} is the Fubini–Study metric on the fibers. It follows that if E is a Griffiths positive vector bundle over X, then $\mathcal{O}_E(1)$ is an ample line bundle over $\mathbb{P}(E)$.

Let X be a complex manifold and let (E, h) be a Hermitian vector bundle over X. The Chern forms $C_k(E)$ are defined as $f_k(\frac{i}{2\pi}\Theta)$, where

$$\det(P + tI) = f_n(P) + f_{n-1}(P)t + \cdots + f_1(P)t^{n-1} + t^n$$

and the Segre forms are defined inductively by the relation

$$S_k(E) + C_1(E)S_{k-1}(E) + \cdots + C_k(E) = 0.$$

In particular, $S_1(E) = -C_1(E)$, $S_2(E) = C_1^2(E) - C_2(E)$ and so on.

If E is a positive line bundle, then all the forms $C_1^k(E)$ are positive for k no greater than the dimension of X. For the rest of this paper define $\Phi = C_1(\mathcal{O}_{\mathbb{P}(E)}(1))$. Then the forms Φ^k are positive for $k \leq \dim(X) + \operatorname{rank}(E) - 1$.

Push forward of forms. Let M and N be oriented differentiable manifolds of respective dimensions m and n and let $f\colon M \to N$ be a proper submersion. That is, f is surjective and has surjective differential everywhere and the fibers are compact and connected. Write $r = m - n$.

For any smooth $(p+r)$-form η on M there exists a unique smooth p-form ξ on N such that the equality

$$\int_M \eta \wedge f^*\varphi = \int_N \xi \wedge \varphi$$

holds for any smooth $(n-p)$-form φ on N with compact support. We call this form ξ the push-forward of η and denote it by $f_*\eta$.

Lemma 2.1 *Let X and Y be compact Kähler manifolds of respective dimensions m and n and let $f\colon X \to Y$ be a holomorphic fibration without singular fibers. If η is a positive $(p+r, p+r)$ form on X, then $f_*\eta$ is a positive (p,p) form on Y, where $r = m - n$.*

Proof First let η be a top degree positive form. Denote the volume form of Y by dV_Y and let ω be the Kähler form on X. Since f is of maximum rank everywhere, the form $\omega^r \wedge f^*(dV_Y)$ is a positive (m,m)-form on X. So we can write $\eta = g(x)\omega^r \wedge f^*(dV_Y)$ for some positive function g on X. But then

$$f_*\eta = \Big(\int_F g(x)\omega^r\Big)\, dV_Y$$

is a positive form, since $(\int_F g(x)\omega^r)$ is a positive function on Y, where F denotes a fiber of f.

Now let η be a positive form of degree $(p+r, p+r)$ on X and let τ be a strongly positive form on Y with complementary degree to $f_*\eta$. Then $f^*\tau$ is strongly positive on X, so $\eta \wedge f^*\tau$ is a positive top degree form on X. By the argument above,

$$f_*(\eta \wedge f^*\tau) = (f_*\eta) \wedge \tau$$

is positive. Thus the push forward $f_*\eta$ is positive. ■

3 Proof of the Main Theorem

Our main theorem follows from the next proposition and the previous lemma.

Proposition 3.1 *Let X be a projective manifold and (E,h) a Hermitian vector bundle over X of rank r. Let $\pi\colon \mathbb{P}(E) \to X$ be the projectivization of E. Then the push forward form $\pi_*(\Phi^{k+r-1})$ is exactly equal to the signed Segre form $(-1)^k S_k(E)$ on X for any $1 \le k \le n$.*

Proof Recall that $\Phi = C_1(\mathcal{O}_{\mathbb{P}(E)}(1)) = \frac{i}{2\pi}\Theta(\mathcal{O}_{\mathbb{P}(E)}(1))$ is a global $(1,1)$-form on $\mathbb{P}(E)$ and at $p = (x,[v]) \in \mathbb{P}(E)$, where $v \in E^*$, we have

$$\Phi = \frac{i}{2\pi}\Big(-\frac{1}{|v|^2}\Theta_{v\bar{v}} + \omega_{FS}\Big),$$

https://doi.org/10.4153/CMB-2011-100-6 Published online by Cambridge University Press

where Θ is the curvature form of E^* and ω_{FS} is the Fubini–Study metric on the fiber $\pi^{-1}(x) = \mathbb{P}(E_x^*) \cong \mathbb{P}^{r-1}$ induced from the metric on E_x^* (see [3] or [4]).

Since $\omega_{FS}^k = 0$ for $k \geq r$ we have by the binomial formula that for any $1 \leq k \leq n$,

$$\Phi^{k+r-1} = \Big(\frac{i}{2\pi}\Big)^{k+r-1} \sum_{j=k}^{k+r-1} (-1)^j \binom{k+r-1}{j} \Big(\frac{1}{|v|^2}\Theta_{v\bar{v}}\Big)^j \wedge \omega_{FS}^{k+r-1-j}.$$

When we push forward this form, we are integrating over the fibers of $\pi\colon \mathbb{P}(E) \to X$, so only the first term in the right-hand side survives:

$$\pi_*\Phi^{k+r-1} = \Big(\frac{i}{2\pi}\Big)^{k+r-1} (-1)^k \binom{k+r-1}{k} \int_{[v]\in\mathbb{P}(E_x^*)} \Big(\frac{1}{|v|^2}\Theta_{v\bar{v}}\Big)^k \wedge \omega_{FS}^{r-1}.$$

Fix a point $x \in X$ and let $\{e_1, \dots, e_r\}$ be a local unitary frame of E^* near x. For $v = \sum_{i=1}^r v_i e_i$ write $U = \{[v] \in \mathbb{P}(E_x^*) : v_r \neq 0\}$ and $t_i = v_i/v_r$, $1 \leq i \leq r$. Then $U \cong \mathbb{C}^{r-1}$ is an open subset of the fiber $\mathbb{P}(E_x^*)$ and $(t_1, \dots, t_{r-1})$ are its coordinates.

On this fiber we have

$$\Big(\frac{1}{|v|^2}\Theta_{v\bar{v}}\Big)^k \wedge \omega_{FS}^{r-1} = \Big(\sum_{i,j=1}^r \Theta^i_j t_i \bar{t}_j\Big)^k \frac{dt \wedge d\bar{t}}{(1+|t|^2)^{k+r}},$$

where $|t|^2 = |t_1|^2 + \cdots + |t_{r-1}|^2$, $dt = dt_1 \wedge \cdots \wedge dt_{r-1}$ and we wrote for convenience $t_r = 1$. Plug in this expression for $\pi_*\Phi^{k+r-1}$ and we get that at $x \in X$,

$$\pi_*\Phi^{k+r-1} = (-1)^k \binom{k+r-1}{k} \sum_{i_1,\dots,i_k,j_1,\dots,j_k}^{r} B_{IJ} \Theta^{i_1}_{j_1} \dots \Theta^{i_k}_{j_k},$$

where

$$B_{IJ} = \Big(\frac{i}{2\pi}\Big)^{k+r-1} \int_{t\in\mathbb{C}^{r-1}} \frac{t_{i_1}\cdots t_{i_k}\bar{t}_{j_1}\cdots\bar{t}_{j_k}}{(1+|t|^2)^{k+r}} dt \wedge d\bar{t},$$

and $I = (i_1, \dots, i_k)$, $J = (j_1, \dots, j_k)$ are multi-indices, namely, they both belong to the set $\{1, \dots, r\}^k$.

Let us denote for the moment $I = J$ if I and J are equal as sets with multiplicities. If $I \neq J$, then in the expression of B_{IJ} there will be terms of the form $e^{\pm i\theta}$ and since $\int_0^{2\pi} e^{\pm i\theta}\, d\theta = 0$, we observe that $B_{IJ} = 0$ for $I \neq J$. If $I = J$, then using the well-known formula

$$\Big(\frac{i}{2\pi}\Big)^{r-1} \int_{\mathbb{P}^{r-1}} \frac{|t_1|^{2m_1}\cdots|t_{r-1}|^{2m_{r-1}}}{(1+|t|^2)^{k+r}} dt \wedge d\bar{t} = \frac{(r-1)!\prod m_i!}{(r-1+k)!},$$

where $\sum m_i = k$ we obtain that $B_{IJ} = (i/2\pi)^k\beta$ for some $\beta \in \mathbb{Q}$. Therefore,

$$\pi_*\Phi^{k+r-1} = P\Big(\frac{i}{2\pi}\Theta\Big)$$

becomes a homogeneous polynomial of degree k in the entries of the curvature $\frac{i}{2\pi}\Theta$ of E^* with coefficients in $\mathbb{Q}$. On the other hand, the push forward $\pi_*\Phi^{k+r-1}$ is a global (k,k)-form on X independent of the choice of local frames of E^*. That is, the polynomial P is invariant under the change $A \mapsto A\Theta A^{-1}$ for any $A \in \mathrm{GL}(r,\mathbb{C})$. Therefore, it must be a polynomial of Chern forms $\pi_*\Phi^{k+r-1} = f_1(C_1,\dots,C_r)$, where f_1 is a weighted homogeneous polynomial of the Chern forms of E with rational coefficients. Of course, $(-1)^k S_k(E) = f_2(C_1,\dots,C_n)$ is also a weighted homogeneous polynomial of the Chern classes of E.

Let $[A]$ denote the cohomology class of a given form A. From the theory of Chern classes we know that $\pi_*[\Phi^{k+r-1}] = [(-1)^k S_k(E)] = (-1)^k s_k(E)$, where $s_k(E)$ is the k-th Segre class of E. Moreover the push forward commutes with the d-operator, hence $[\pi_*\Phi^{k+r-1}] = \pi_*[\Phi^{k+r-1}]$. It follows that the difference $f = f_1 - f_2$ is a closed global (k,k)-form on X which represents the trivial cohomology class.

Note that $f(C_1,\dots,C_r)$ is the same weighted homogeneous polynomial of Chern forms and $[f] = 0$ regardless of what vector bundle we begin with. More precisely, if $\mathcal{J}_k$ denote the set of all r-tuples of positive integers $(j_1,\dots,j_r)$ such that

$$j_1 + 2j_2 + \cdots + rj_r = k$$

and if for $J \in \mathcal{J}_k$ we define $C_J = C_1^{j_1} \wedge \cdots \wedge C_r^{j_r}$, then we have

$$f(C_1,\dots,C_r) = \sum_J a_J C_J,$$

where the coefficients a_J are independent of E.

In particular if we choose $E = H^{x_1} \oplus \cdots \oplus H^{x_r}$, where H is an ample line bundle on X and $x_1,\dots,x_r$ are positive integers, we obtain that f is a polynomial in x_i's with coefficients a_J. On the other hand,

$$f(C_1(E),\dots,C_r(E)) = h(x_1,\dots,x_r)C_1(H)^k$$

and

$$[f] = h(x_1,\dots,x_r)c_1(H)^k = 0 \text{ in } H^{2k}(X)$$

for some homogeneous polynomial h of degree k with rational coefficients. It follows that $h(x_1,\dots,x_r) = 0$ for any positive integers $x_1,\dots,x_r$ and by the homogenity of h we get that $h \equiv 0$. This implies that all the coefficients $a_J \equiv 0$, so that $f_1 = f_2$. This establishes the fact that $\pi_*\Phi^{k+r-1} = (-1)^k S_k(E)$ for any $1 \le k \le n$. ■

Combining the above proposition with Lemma 2.1, we obtain that the signed Segre forms $(-1)^k S_k(E)$ are positive for all $1 \le k \le n$.

Acknowledgements The author would like to thank Fangyang Zheng for invaluable discussions during the preparation of this paper. He is also very grateful to the referee for his/her kind comments and suggestions which definitely made this paper more readable.

References

[1] W. Fulton and R. Lazarsfeld, *Positive polynomials for ample vector bundles.* Ann. of Math. **118**(1983), no. 1, 35–60. doi:10.2307/2006953

[2] P. Griffiths, *Hermitian differential geometry, Chern classes, and positive vector bundles*, Global Analysis, University of Tokyo Press, Tokyo, 1969, pp. 85–251.

[3] S. T. Yau, and F. Zheng, *On a borderline class of non-positively curved compact Kähler manifolds.* Math. Z. **212**(1993), no. 4, 587–599. doi:10.1007/BF02571678

[4] F. Zheng, *Complex Differential Geometry.* AMS/IP Studies in Advanced Mathematics 18, American Mathematical Society, Providence, RI 2000.

Department of Mathematics, Park University, Parkville, MO, USA
e-mail: dincer.guler@park.edu