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MAHLER'S MEASURE OF A POLYNOMIAL IN FUNCTION OF THE NUMBER OF ITS COEFFICIENTS

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ABSTRACT. Mahler's measure of a monic polynomial is equal to the product of modules of its roots which lie outside the unit circle. By classical theorem of Kronecker it is strictly greater than 1 for any polynomial that is not a product of cyclotomic factors. In this case a number of lower bounds of the measure, depending either on the degree of the polynomial or on the number of its non-zero coefficients, has been found. Here is given an improvement of the bound of the latter type previously found by the author, A. Schinzel and W. Lawton.

Mahler defined the measure of a polynomial g by

$$M(g) = |a_0| \prod_{i=0}^{n} \max\,(1,\ |\alpha_i|)$$

where a_0 is the leading coefficient and $\alpha_1, \ldots, \alpha_n$ are the roots of the polynomial g. D. H. Lehmer [5] asked whether for every $\epsilon > 0$ there exists a monic polynomial g such that $1 < M(g) < 1 + \epsilon$? Clearly $M(g) = 1$ if g is cylotomic. In [3], there was given a bound for g that is not a product of cyclotomic factors,

$$M(g) \geqq 1 + \frac{1}{\exp_{k+1} 2k^2},$$

depending only on the number k of non-zero coefficients of g. The aim of this paper is to sharpen this result. We shall prove

THEOREM. *If $g \in \mathbf{Z}[z]$ is a monic polynomial with $g(0) \neq 0$ that is not a product of cyclotomic polynomials then*

$$M(g) \geqq 1 + \frac{1}{a \exp\,(bk^k)},$$

where k is the number of non-zero coefficients of g, $a \leqq 13911$ and $b \leqq 2.27$.

We use the notation of [3], so that

$$(1) \qquad F(\mathbf{z}) = F(z_1, \ldots, z_N) = \sum_{\mathbf{j} \in \mathcal{J}} a(\mathbf{j})\mathbf{z}^{\mathbf{j}} = \sum_{\mathbf{j} \in \mathcal{J}} a(\mathbf{j}) z_1^{j_1} \ldots z_N^{j_N},$$

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where $\mathbf{j} = (j_1, \ldots, j_N) \in \mathbf{Z}^N$ and $\mathcal{J}$ is a finite set such that $a(\mathbf{j}) \neq 0$ for $\mathbf{j} \in \mathcal{J}$,

$$\mathcal{J} - \mathcal{J} = \{\mathbf{j}_1 - \mathbf{j}_2 : \mathbf{j}_i \in \mathcal{J}\},\ J = |\mathcal{J}|,\ D = \max_{\mathbf{j}\in\mathcal{J}} \max_{1\leqq n\leqq N} |j_n|,$$

$$d_n = \min_{\mathbf{j}\in\mathcal{J}} j_n,\ IF(\mathbf{z}) = F(\mathbf{z}) \prod_{n=1}^{N} z_n^{-d_n},\ h(F) = \max_{\mathbf{j}\in\mathcal{J}} |a(\mathbf{j})|,\ l(F) = \sum_{\mathbf{j}\in\mathcal{J}} |a(\mathbf{j})|.$$

An extended cyclotomic polynomial is a polynomial ψ of the form

$$\psi(\mathbf{z}) = I\phi_m(z_1^{v_1}, \ldots, z_N^{v_N}),$$

where $v_1, \ldots, v_N$ are coprime integers and ϕ_m denotes the mth cyclotomic polynomial. For fixed $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbf{Z}^n$ and a function of the form (1) we shall write

$$F_{\mathbf{r}}(z) = F(z^{r_1}, \ldots, z^{r_N}).$$

Further we define a derivation depending on $\mathbf{r}$, by

$$D_{\mathbf{r}}F(\mathbf{z}) = F^{(1)}(\mathbf{z}) = \sum_{i=1}^{N} r_i z_i \frac{\partial F}{\partial z_i}(\mathbf{z}).$$

We write $F^{(0)} = F, F^{(n)} = D_{\mathbf{r}}^n F$ as usual, and put $f(z) = F_{\mathbf{r}}(z)$. Thus we have

$$F_{\mathbf{r}}^{(t)}(z) = \sum_{j=1}^{t} c_j z^j f^{(j)}(z) \tag{2}$$

where the constants $c_1, \ldots, c_t$ are natural numbers different from zero. If $F = F_1F_2$ where F_1 and F_2 are of the form (1) then

$$F^{(t)} = \sum_{j=0}^{t} F_1^{(j)} F_2^{(t-j)}. \tag{3}$$

The following lemma is a generalization of Lemma 3 in [4] (c.f. also Lemma 2 in [3]).

LEMMA 1. *Let $F(\mathbf{z})$ be a polynomial of the form (1) and $\mathbf{r} \in \mathbf{Z}^N$. Suppose that $F = F_1F_2$ where F_1 and F_2 are polynomials; F_2 is not divisible by any extended cyclotomic polynomial but $\phi_m(z) \mid IF_2(z^{r_1}, \ldots, z^{r_N})$. In addition suppose that F and f have the same number of terms. Then there are linearly independent vectors $\mathbf{v}^{(i)} \in \mathcal{J} - \mathcal{J}, i = 1, 2$ for which*

$$m \mid (\mathbf{v}^{(1)}\mathbf{r}, \mathbf{v}^{(2)}\mathbf{r})P,$$

where

$$P = \prod_{p\leqq J} p.$$

PROOF. Let γ be the multiplicity of $\phi_m(z)$ in $IF(z) = IF_{\mathbf{r}}(z)$. Then by (2)

$$\phi_m(z) \mid IF_{\mathbf{r}}^{(t)}(z)$$

for

$$0 \leqq t \leqq \gamma - 1.$$

Further

$$F^{(t)}(\mathbf{z}) = \sum_{\mathbf{j} \in \mathcal{J}_t} (\mathbf{j}^{\mathbf{r}})^t a(\mathbf{j}) \mathbf{z}^{\mathbf{j}}$$

where $\mathcal{J}_0 = \mathcal{J}$ and $\mathcal{J}_t = \mathcal{J} \setminus \{\mathbf{0}\}$ for $t \geqq 1$. Now we apply the reasoning of the proof of the Lemma 3 in [4] up to formula (9) on page 200, to polynomials $F^{(0)}, \ldots, F^{(\gamma-1)}$. For each t we have a partition of $\mathcal{J}_t$ into subsets $\mathcal{J}_{t,i}$, $1 \leqq i \leqq I_t$. We choose $\mathbf{h}_t^{(i)} \in \mathcal{J}_{t,i}$ so that $\mathbf{h}_t^{(i)}\mathbf{r}$ is minimal (instead of minimizing

$$\sum_{n=1}^{N} h_{t,n}^{(i)}$$

as in the original proof). We get sets

$$W_t = \bigcup_{i=0}^{I_t} \{\mathbf{v} = \mathbf{j} - \mathbf{h}_t^{(i)} : \mathbf{j} \in \mathcal{J}_{t,i}\}$$

and we put

$$W = \bigcup_{t=0}^{\gamma - t} W_t \subset \mathcal{J} - \mathcal{J}$$

Following the original proof, if W contains two independent vectors $\mathbf{v}^{(i)}$, $i = 1, 2$, then $m \mid (\mathbf{v}^{(i)}\mathbf{r}, \mathbf{v}^{(2)}\mathbf{r})$ and our Lemma is proved. We shall show that the other case is not possible. Suppose to the contrary that W lies on a line L through the origin. Our choice of $\mathbf{h}_t^{(i)}$ assures that for a suitable generator $\mathbf{v}$ of L, $\mathbf{vr} \geqq 0$ and the exponents $b(\mathbf{j})$ corresponding to the formula (9) on page 200 in [4], are positive. The fact that F and f have the same number of terms implies that $\mathbf{vr} \neq 0$, so $\mathbf{vr} > 0$. Hence for $0 \leqq t \leqq \gamma - 1$ and $l = m/(m, \mathbf{vr})$

$$I\phi_l(\mathbf{z}^{\mathbf{v}}) \mid F^{(t)}(\mathbf{z})$$

and in view of (3) and by the definition of F_2 we get

$$I\phi_l(\mathbf{z}^{\mathbf{v}}) \mid F_1(\mathbf{z}) \tag{4}$$

Clearly $m \mid l\mathbf{vr}$ so that $\phi_m(z) \mid \phi_l(z^{\mathbf{vr}})$ and (2) and (4) imply that

$$\phi_m^{\gamma}(z) \mid IF_{1,\mathbf{r}}(z).$$

However $\phi_m(z)$ divides also $IF_{2,\mathbf{r}}(z)$ by the assumption of the Lemma. Hence the multiplicity of $\phi_m(z)$ in $IF_{\mathbf{r}}(z) = IF_{1,\mathbf{r}}(z) IF_{2,\mathbf{r}}(z)$ is at least $\gamma+1$ which contradicts the definition of γ. ∎

The Lemma 3 of [3], in our situation takes the following shape:

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LEMMA 2. *Let $F(\mathbf{z})$ be a polynomial of the form (1),* $\mathbf{r} \in \mathbf{Z}^N$, $\Delta = \deg IF_{\mathbf{r}}(z)$ *and* $\Delta_2 = \deg IF_{2,\mathbf{r}}(z)$. *If F_2 is not divisible by any extended cyclotomic polynomial but the sum of degrees of all cyclotomic factors of $IF_{2,\mathbf{r}}(z)$ counted with multiplicities exceeds 1/2 the degree of $IF_{2,\mathbf{r}}(z)$ then* $\mathbf{vr} = 0$ *for some vector* $\mathbf{v} \in \mathbf{Z}^n$ *such that* $0 < h(\mathbf{v}) < 2PJ^5D\Delta/\Delta_2$.

PROOF. This is the same as proof of Lemma 3 in [3], where we replace Δ by Δ_2. Consequently we get $PJ^5\Delta/\Delta_2$ as the bound for $|u|$ and $|v|$ instead of PJ^5. Also in the argument we refer to Lemma 1 in place of Lemma 2 of [3]. ■

We shall modify also Lemma 4 of [3]. For this we need some definitions. For a matrix $A = [a_{ij}]_{i\leqq m, j\leqq n}$ we define a function

$$j_A(i) = \min\{j : a_{ij} \neq 0\}, \quad i \leqq m \tag{5}$$

We denote by $\mathcal{E}$ the class of all matrices A verifying $j_A(i+1) \geqq j_A(i)+1$ for every $i \leqq m$. Remark that $A, B \in \mathcal{E}$ implies $A, B \in \mathcal{E}$, provided that the product is defined.

LEMMA 3. *Let* $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbf{Z}^k$ *and $\psi(x, y)$ be an arbitrary function non-decreasing with respect to $y \geqq 1$ for each fixed x. Define the constants $h_1, \ldots, h_k$ by the recurrence relation*

$$h_{k-t} = (k-t)(k-t+1)h_{k-t+1}\psi(k-t+1,\ h_{k-t+1}) \text{ for } t \leqq k-1 \tag{6}$$

and by putting $h_k = 1$.

There exists a vector $\mathbf{r} \in \mathbf{Z}^N$ *and an integral matrix M such that* $N \leqq k$ *and*

$$\mathbf{n} = \mathbf{r}M \tag{7}$$

$$\mathbf{M} \in \mathcal{E} \tag{8}$$

$$h(\mathbf{M}) \leqq h_n \tag{9}$$

$$\mathbf{vr} = 0,\ \mathbf{v} \in \mathbf{Z}^n \textit{ implies } \mathbf{v} = 0 \textit{ or } h(\mathbf{v}) > \psi(N,\ h(\mathbf{M})). \tag{10}$$

PROOF. The proof of Lemma 4 in [3] requires a slight modification to satisfy the condition (8). However we repeat it here in full detail for the convenience of the reader. So, we remark that the identity matrix $\mathbf{M}$ and $\mathbf{r} = \mathbf{n}$ satisfy the conditions (7), (8) and (9). Now take a matrix $\mathbf{M}_0$ and a vector $\mathbf{r}_0$ that satisfy these conditions and correspond to the least possible N. Following [3] we shall show that they also satisfy (10). Suppose to the contrary that there exists $\mathbf{v} \in \mathbf{Z}^N$ such that

$$\mathbf{vr} = 0 \quad \text{and} \quad 0 < h(\mathbf{v}) \leqq \psi(N,\ h(\mathbf{M}_0)). \tag{11}$$

Consider the lattice L of all integral vectors $\mathbf{r}$ satisfying (11). Let m be the largest number such that $v_m \neq 0$. The lattice L contains $N-1$ linearly independent vectors

$$\mathbf{r}_1 = (v_m,\ 0\ldots 0,\ -v_1,\ 0\ldots 0),\ \mathbf{r}_2 = (0,\ v_m,\ 0\ldots 0,\ -v_2,\ 0\ldots 0), \ldots,$$
$$\mathbf{r}_{m-1} = (0\ldots 0,\ v_m,\ -v_{m-1},\ 0\ldots 0),\ \mathbf{r}_m = (0,\ldots 0,\ 1,\ 0\ldots 0),\ \mathbf{r}_{N-1} = (0\ldots 0,\ 1)$$

(where $-v_1, \ldots, -v_{m-1}, 0, \ldots, 0$ are the mth components of these vectors).

Then there exists a basis $\mathbf{b}_1, \ldots, \mathbf{b}_{N-1}$ of L of the form

$$\mathbf{b}_j = \sum_{i=j}^{N-1} \lambda_{ij}\mathbf{r}_i,\ 1 \leqq j \leqq N-1,$$

with $\lambda_{ij} \neq 0$ for $1 \leqq j \leqq N-1$ and $|\lambda_{ij}| \leqq 1$ for $1 \leqq j \leqq i \leqq N-1$. The existence of such a basis easily follows from Corollary 2 to the Theorem 1 in [1], with $\mathbf{a}_i = \mathbf{r}_i$ in the notation of the Corollary. This means that the matrix

$$\mathbf{B} = \begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_{N-1} \end{bmatrix}$$

belongs to $\mathcal{E}$. Further

$$h(\mathbf{B}) \leqq (N-1)h(\mathbf{v}), \qquad \mathbf{r}_0 = \sum_{i=1}^{N-1} s_i\mathbf{b}_i,$$

$$\mathbf{s} = (s_1, \ldots, s_{N-1}) \in \mathbf{Z}^{N-1}$$

and

$$\mathbf{n} = \sum_{i=1}^{N-1} s_i\mathbf{b}_i\mathbf{M}_0 = \mathbf{sBM}_0$$

Hence

$$\mathbf{M} = \mathbf{BM}_0 \in \mathcal{E}$$

and

$$h(\mathbf{M}) = h(\mathbf{BM}_0) \leqq Nh(\mathbf{B})h(\mathbf{M}_0) \leqq N(N-1)h(\mathbf{v})h_N$$
$$\leqq N(N-1)h_N\psi(N,\ h_N) \leqq h_N.$$

which contradicts the choice of N. ■

Finally we quote Lemma 9 from [3], here as

LEMMA 4. *Let $\mathbf{a} \in \mathbf{Z}^J$ be a vector and $B > 1$ a real number. Then there exist vectors $\mathbf{c} \in \mathbf{Z}^J$ and $\mathbf{r}' \in \mathbf{Q}^J$ and a rational number q such that*

$$\mathbf{a} = \mathbf{r}' + q\mathbf{c}, \tag{12}$$

$$0 < l(\mathbf{c}) \leqq (JB)^J + B^{-1}, \tag{13}$$

$$q \geqq Bl(\mathbf{r}'). \tag{14}$$

PROOF OF THE THEOREM. Let

$$q(z) = \sum_{j=1}^{k'-1} a_j z^{n_j} + a_{k'}. \tag{15}$$

For $k' \leqq 2$ the Theorem is obvious. Suppose that it is true for $k' < k$. We shall prove it for $k' = k$. Put in Lemma 3

$$\mathbf{n} = (n_1, \ldots, n_{k-1}) \in \mathbf{Z}^{k-1}$$

where

$$n_1 = \deg(g) > n_2 > \cdots > n_{k-1} \geqq 1,$$

$$\psi(x,\ y) = 10.8216\, k^5 4^k (2x)^{x+1} y^{x+2} \quad \text{and} \quad K = k - 1. \tag{16}$$

We get

$$N \leqq k-1, \quad \mathbf{r} \in \mathbf{Z}^N, \quad \mathbf{M} \in \mathcal{E}, \quad \mathbf{n} = \mathbf{rM}, \tag{17}$$

such that $\mathbf{vr} = 0$ with $\mathbf{v} \in \mathbf{Z}^N$ and $\mathbf{v} \neq 0$ implies $h(\mathbf{v}) > \psi(N, h(\mathbf{M}))$. In view of the result of C. J. Smyth [6] we can suppose that $g(z)$ is a product of reciprocal polynomials and possibly $z-1$. Hence for $k = 3$ we find $g(z) = z^{2n} + az^n + 1$, which implies

$$M(q) \geqq \frac{3+\sqrt{5}}{2}$$

and for $k = 4$, $g(z) = z^{n_1} + bz^{n_2} + bz^{n_3} + 1$ with $n_1 = n_2 + n_3$. In this case

$$(n_1,\ n_2,\ n_3) = (n_1,\ n_2)\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

and $h_2(\mathbf{M}) = 1$.

In the general case, from (6) we get

$$\log h_m = \log m + (m+2)\log 2 + \log c + (m+3)\log(m+1) + (m+4)\log h_{m+1}$$

where $c = 10.8216\, k^5 4^k$.

By putting $h_k = 1$ and developping this relation we obtain

$$\log h_2 \leqq [\log(K-1) + (K+1)\log 2 + \log c + (K+2)\log K] \times ((K-2)(K-1)\cdots 6)\left(1 + \frac{1}{K+2} + \frac{1}{(K+2)(K+1)} + \cdots + \frac{1}{(K+2)(K+1)\cdots 6}\right)$$

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For $K = k - 1 \geqq 4$ this gives

$$\log h_2 \leqq (K+2) \log K \left[\frac{\log 3}{6 \log 4} + \frac{5 \log 2}{6 \log 4} + \frac{5 \log 5 + 5 \log 4 + \log 10.8216}{6 \log 4} + 1 \right] \frac{(K+2)!}{5!} \frac{7}{6}$$

Hence

$$h_2 \leqq \exp \{0.0354\,(k+1)!(k+1) \log (k-1)\}, \quad k \geqq 4, \tag{18}$$

and

$$\log h_1 \leqq 7.234 + 1.387k + 5 \log k + 5 \log h_2, \quad k \geqq 4. \tag{19}$$

Clearly $M(g) = M(g_1)$ and deg $(g_1) = h(\mathbf{M}) \leqq h_1$. Hence from (19) and Theorem 1 of [2] we check by computation that

$$M(g) \geqq 1 + \frac{1}{1200} \left(\frac{\log \log h_1}{\log h_1} \right)^3, \tag{20}$$

which is greater than the bound in the Theorem.

If $N \geqq 2$ then we put

$$G(\mathbf{z}) = \sum_{j=1}^{k-1} a_j z_i^{m_{ij}} \ldots z_N^{m_{Nj}} + a_k, \tag{21}$$

where the coefficients are given by (15). The exponents m_{ij} can be negative, so to avoid this we define

$$F(\mathbf{z}) = IG(\mathbf{z}). \tag{22}$$

Thus

$$IF(z^{r_1}, \ldots, z^{r_N}) = G_{\mathbf{r}}(z) = g(z). \tag{23}$$

This means that the subsequent substitution of variables in (15) does not cause any cancellation of terms. If we express F in the form (1), then $J = |\mathcal{J}| = k$. We factorize $F = F_1 F_2$ where F_1 is the product of all extended cyclotomic polynomials dividing F and F_2 is not divisible by any of them. The assumption on g implies that F_2 is not a constant. We shall find a bound for Δ / Δ_2 where

$$\Delta = \deg IF_{\mathbf{r}}(z) \quad \text{and} \quad \Delta_2 = \deg IF_{2,\mathbf{r}}(z).$$

For this we put $\mathbf{a} = \mathbf{r}$ and $B = 2(1 + 1/N)h(\,)$ in Lemma 4. We get

$$\mathbf{r} = \mathbf{r}' + q\mathbf{c}, \quad q \in Q_+, \quad \mathbf{r}' \in Q^N \quad \text{etc} \ldots$$

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and if $q = l/t$ with coprime l and t then

$$\begin{aligned} &t\mathbf{r} = l\mathbf{c} + \mathbf{r}''; \; l,\; t \in \mathbf{Z}_+; \; \mathbf{c},\; \mathbf{r}'' \in \mathbf{Z}^N; \\ &0 < l(\mathbf{c}) < (BN)^N + B^{-1}, \quad l > Bl(\mathbf{r}''). \end{aligned} \tag{24}$$

We put

$$H(y_1,\, y_2) = G(y_1^{c_1} y_2^{r_1''}, \ldots, y_1^{c_N} y_2^{r_N''}) \tag{25}$$

so that

$$lF_{\mathbf{r}}(y^t) = G_{\mathbf{r}}(y^t) = H(y^t,\, y) = g(y^t) \text{ and } t\Delta = \deg g(y^t). \tag{26}$$

We express $F_i, i = 1, 2$ in the form

$$F_i(\mathbf{z}) = \sum_{\mathbf{j} \in \mathcal{J}_i} b(\mathbf{j})^{(i)} \mathbf{z}^{\mathbf{j}}$$

We choose $\mathbf{j}_0^{(i)}$ such that $\mathbf{j}_0^{(i)}\mathbf{r}$ is minimal and define

$$G_i(\mathbf{z}) = \sum_{\mathbf{j} \in \mathcal{J}_i} b(\mathbf{j})^{(i)} \mathbf{z}^{\mathbf{j} - \mathbf{j}_0^{(i)}}. \tag{27}$$

and

$$H_i(y_1,\, y_2) = G_i(y_1^{c_1} y_2^{r_1''}, \ldots, y_1^{c_N} y_2^{r_N''}).$$

Then

$$t\Delta_2 = \deg H_2(y^l,\, y). \tag{28}$$

We shall show that

$$y_1 \text{ always occurs in the expression of } H(y_1,\, y_2). \tag{29}$$

By the remark following the formula (23) it suffices to show that y_1 occurs in one of the terms which appear by substitution of $\mathbf{z} = (y_1^{c_1} y_2^{r_1''}, \ldots, y_1^{N} y_2^{C_N''})$ in (21). Let i_0 be the smallest number such that $c_{i_0} \neq 0$ and let $j_0 = j_{\mathbf{M}}(i_0)$ be defined by (5). Then $H(y_1, y_2)$ contains a term of the form $a_{j_0} y_1^a y_2^b$ with $a = \sum_{i=1}^{i_0} c_i m_{i_{j0}} = c_{i_0} m_{i_{0_{j0}}} \neq 0$, by the definition of i_0 and j_0.

Concerning $H_2(y_1, y_2)$ two cases may happen:

CASE I. y_1 occurs in the expression of $H_2(y_1, y_2)$.

In this case we get from (27), (28) and (24)

$$\begin{aligned} t\Delta_2 &= \deg H_2(y^l, y) = \max_{\mathbf{j} \in \mathcal{J}_2} ((\mathbf{j} - \mathbf{j}_0^{(2)}) \cdot t\mathbf{r}) = (\mathbf{j}'\mathbf{c})l + \mathbf{j}'\mathbf{r}'' \geqq \\ &\geqq l - h(\mathbf{j}')l(\mathbf{r}'') \geqq l(1 - 2h(\mathbf{M})B^{-1}), \end{aligned} \tag{30}$$

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where $\mathbf{j}'$ realizes the maximum. On the other hand from (21), (26), the fact that $\mathbf{M} \in \mathcal{E}$, and (24) we have

$$t\Delta = m_{11}(c_1 l + r_1'') \leqq h(\mathbf{M}) l((BN)^N + 2B^{-1}). \tag{31}$$

From (30) and (31) with our choice of B we get

$$\Delta / \Delta_2 \leqq 2.7054(2N)^{N+1} h(\mathbf{M})^{N+1}. \tag{32}$$

and

$$2PJ^5 D\Delta / \Delta_2 \leqq 10.8215 \cdot 4^k k^5 (2N)^{N+1} h(\mathbf{M})^{N+2},$$

Hence by (17) we conclude from Lemma 2 that the sum of degrees of all cyclotomic factors of $IF_{2,\mathbf{r}}(z)$ counted with their multiplicities does not exceed $1/2\ \Delta_2$.

Now we can proceed as in the proof of the Theorem in [3] pp. 142-144. We put $g(z)$ in place of $f(z)$. In our case $J = k$. If we denote by P, P_1, P_2 the product of all cyclotomic factors of $g(z)$, $IF_{1,\mathbf{r}}(z)$, and $IF_{2,\mathbf{r}}(z)$, respectively, then

$$P = P_1 P_2, \quad P_1(z) = IF_{1,\mathbf{r}}(z),$$

$$P_2(z) \mid IF_{2,\mathbf{r}}(z) \quad \text{and} \quad \deg P_2(z) \leqq \frac{1}{2} \deg IF_{2,\mathbf{r}}(z).$$

If $g(z) = P(z) f_1(z) f_2(z)$ as on page 143 of [3], then

$$\frac{\deg f_1}{\deg c} \geqq \frac{1}{4} \frac{\Delta_2}{\Delta}, \quad \text{for} \quad \deg f_1 \geqq \deg f_2,$$

and

$$\frac{\deg f_z}{\deg c} \geqq \frac{1}{4} \frac{\Delta_2}{\Delta}, \quad \text{for} \quad \deg f_2 \geqq \deg f_1.$$

where c is defined as in [3] but with g in place of f. Following the original proof we obtain

$$M(g) \geqq \min\{M(f_1), M(f_2)\} \geqq 1 + \frac{1}{4\Delta/\Delta_2 \exp(2k^k + 3)},$$

We check that the minimal value is attained for $N = 2$. From (18) and (32) by tedious computation we find the bound of the Theorem.

CASE II.

$$y_1 \text{ does not occur in the expression of } H_2(y_1, y_2). \tag{33}$$

We put

$$h_2(y_2) = H_2(y_1, y_2).$$

Then

$$M(g) = M(IF_{1,\mathbf{r}}(z) IF_{2,\mathbf{r}}(z)) = M(Ih_2(z)).$$

However

$$\int_0^1 \int_0^1 \log |IH_1(e^{2\pi_i\theta_1}, e^{2\pi_i\theta_2})| d\theta_1 d\theta_2 = 0$$

and

$$\log M(g) = \int_0^1 \log |Ih_2(e^{2\pi_i\theta})|\, d\theta = \int_0^1 \int_0^1 \log |IH_2|\, d\theta_1 d\theta_2,$$

by Jensen's formula, the definition of H_1 and (33). Hence

$$\begin{aligned} \log M(g) &= \int_0^1 \int_0^1 \log |IH_1 IH_2|\, d\theta_1 d\theta_2 = \int_0^1 \int_0^1 \log |IH|\, d\theta_1 d\theta_2 \\ &= \int_0^1 \log |IH(0,\ e^{2\pi_i\theta})|\, d\theta . \end{aligned} \tag{34}$$

Finally (29), (21) and the fact that $g(0) \neq 0$ implies that y_1 occurs in the expression of $IH(y_1, y_2)$. Hence $IH(0, z)$ has at most $k - 1$ terms and the Theorem follows by the induction hypothesis. ∎

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