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COMPOSITION OPERATORS BETWEEN WEIGHTED BANACH SPACES OF ANALYTIC FUNCTIONS

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Abstract

We characterize those analytic self-maps φ of the unit disc which generate bounded or compact composition operators C_φ between given weighted Banach spaces H_v^∞ or H_v^0 of analytic functions with the weighted sup-norms. We characterize also those composition operators which are bounded or compact with respect to all reasonable weights v.

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0. Introduction

The aim of this paper is to study boundedness and compactness of composition operators C_φ, $C_\varphi(f) = f \circ \varphi$ on weighted Banach spaces of analytic functions, where $\varphi : \mathbb{D} \to \mathbb{D}$ is an analytic map on the unit disc $\mathbb{D}$. We are interested in complex spaces of the form

$$(1) \qquad H_v^\infty := H_v^\infty(\mathbb{D}) := \{f \in H(\mathbb{D}) : \|f\|_v := \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\},$$

$$(2) \qquad H_v^0 := H_v^0(\mathbb{D}) := \{f \in H(\mathbb{D}) : \lim_{|z|\to 1^-} v(z)|f(z)| = 0\},$$

endowed with the norm $\|\cdot\|_v$, where $H(\mathbb{D})$ denotes the space of analytic functions on $\mathbb{D}$ and $v : \mathbb{D} \to \mathbb{R}_+$ is an arbitrary *weight*, that is, bounded continuous positive (which means strictly positive throughout the paper) function.

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Composition operators were extensively studied on various 'integral type' spaces of analytic functions on the disc like Hardy spaces, Bergman spaces, Dirichlet spaces or Bloch spaces (see, for example, [CM, Sh] and[J]) and weighted spaces of continuous functions (see [SS]). The spaces H_v^∞ or H_v^0 are connected with the study of growth conditions of analytic functions and were also studied in detail (see [SW1, SW2, RS, BS, BBT, L1, L2]). Our purpose is to connect both topics.

Especially interesting are *radial* weights v, that is, $v(z) = v(|z|)$. In that case, if $v \equiv 1$, then we get $H_v^\infty = H^\infty$ the space of all bounded analytic functions and $H_v^0 = \{0\}$. If $\limsup_{|z|\to 1^-} v(z) > 0$, then obviously $H_v^\infty = H^\infty$ with an equivalent norm and also $H_v^0 = \{0\}$. Moreover, $C_\varphi : H^\infty \to H^\infty$ is always bounded and it is compact if and only if $\overline{\varphi(\mathbb{D})} \subseteq \mathbb{D}$ (see [CM, Ex. 3.3.2]). Now, if $\lim_{|z|\to 1^-} v(z) = 0$, the so-called non-increasing majorant of v, that is, the radial function $u : \mathbb{D} \to \mathbb{R}_+$, $u(r) = \sup\{v(R) : r \le R < 1\}$, is also continuous and tends to zero at the boundary. By the maximum principle $H_u^\infty = H_v^\infty$, $H_u^0 = H_v^0$, and the corresponding spaces are isometric. On account of what was just said, we call any radial, positive continuous function $v : \mathbb{D} \to \mathbb{R}_+$, which is non-increasing with respect to $|z|$ and is such that $\lim_{|z|\to 1^-} v(z) = 0$, a *typical weight*. We will be mostly interested in the radial weights but as we have seen we may assume that the weight is typical. Nevertheless, we formulate our results in a more general (even non-radial) setting whenever it is possible.

Let us explain the organization of the paper. In Section 1 we summarize preliminaries on spaces H_v^∞ and H_v^0. In Section 2 we collect results on boundedness of C_φ and in Section 3 analogous results on compactness. The last Section 4 contains some results on p-integral composition operators.

We finish the introduction with some notation. If $1 \le p < \infty$ and v is a weight, we define the Banach space

$$(3) \qquad A_v^p := A_v^p(\mathbb{D}) := \{f \in H(\mathbb{D}) : \|f\|^p := \int_{z\in D} |f(z)|^p v(z)^p dA(z) < \infty\},$$

where dA is the 2–dimensional Lebesgue measure. If $f : \mathbb{D} \to \mathbb{C}$ is an analytic function, then $M(f, r) := \sup_{|z|=r} |f(z)|$ is a log-convex non-decreasing function (see the Hadamard Three Circle Theorem [C, V.3.13]). We reserve the letters v, w for weights. We denote the natural numbers by $\mathbb{N} = \{1, 2, 3, \dots\}$. By C, C', c etcetera we denote positive constants which may vary from place to place but do not depend on indices or variables in given formulas or inequalities.

1. The spaces H_v^∞ and H_v^0

Notice that the norm topology on H_v^∞ is stronger than the topology τ_{co} of uniform convergence on the compact sets of $\mathbb{D}$. Assume for a while that $\lim_{|z|\to 1^-} v(z) = 0$.

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Since $f_r(z) := f(rz)$ tends in τ_{co} to f, the closed unit ball B_v^0 of H_v^0 is τ_{co}-dense in the unit ball B_v of H_v^∞. This implies (see [BS, Th. 1.1], compare [RS, Th. 1]) that $(H_v^0)'' = H_v^\infty$ isometrically and the embedding of H_v^0 into H_v^∞ is the canonical embedding into its bidual. Moreover, since point evaluations are continuous functionals on H_v^0, the pointwise convergence topology (denoted by τ_p) on H_v^0 is weaker than the weak topology. Looking at the representation of $(H_v^0)'$ (see [BS, Th. 1.1]), we realize immediately that if δ_z is a point evaluation at z on H_v^0, then for $f \in H_v^\infty$ we have $\langle f, \delta_z \rangle = f(z)$. Thus the pointwise convergence on H_v^∞ is weaker than its weak-star topology. Since H_v^0 is τ_p-dense in H_v^∞ and C_φ is always τ_p-continuous, $C_\varphi : H_v^\infty \to H_w^\infty$ is equal to the bi-adjoint map of $C_\varphi : H_v^0 \to H_w^0$ whenever both operators are well defined and $\lim_{|z|\to 1^-} v(z) = \lim_{|z|\to 1^-} w(z) = 0$.

In fact, from the papers of Lusky [L1, L2], we know that for radial weights and under quite general assumptions $H_v^0 \simeq c_0$ and $H_v^\infty \simeq l_\infty$.

To each weight v corresponds the so-called *growth condition* $u : \mathbb{D} \to \mathbb{R}_+, u = 1/v$ and $B_v = \{f \in H(\mathbb{D}) : |f| \leq u\}$. In [BBT] the new function $\tilde{u} : \mathbb{D} \to \mathbb{R}_+$ is defined by $\tilde{u}(z) := \sup_{f\in B_v} |f(z)|$ and the weight associated with v is defined by $\tilde{v} := 1/\tilde{u}$. It is shown there that $\tilde{u}$ and $\tilde{v}$ have the following useful properties:

(i) $0 < \tilde{u} \leq u$ and $0 < v \leq \tilde{v}$, $\tilde{v}$ is bounded;

(ii) $\tilde{u}$ and $\tilde{v}$ are continuous and, respectively, radial, non-decreasing and non-increasing whenever u and v are so;

(iii) $\|f\|_v \leq 1$ if and only if $\|f\|_{\tilde{v}} \leq 1$ for $f \in H(\mathbb{D})$;

(iv) for every $z \in \mathbb{D}$ there is $f_z \in B_v$ with $\tilde{u}(z) = |f_z(z)|$;

(v) if $\lim_{|z|\to 1^-} v(z) = 0$, then $\tilde{u}(z) = \sup_{f\in B_v^0} |f(z)|$.

As in [T] a weight v is called *essential* if there exists a $C > 0$ such that $v(z) \leq \tilde{v}(z) \leq Cv(z)$ for all $z \in \mathbb{D}$.

There are many known criteria for v to be essential (see [BBT], especially Proposition 3.4 there). In particular, if $v(z) = 1/M(f, |z|)$ for some analytic function $f : \mathbb{D} \to \mathbb{C}$, then $v = \tilde{v}$. It turns out that tending to zero at the boundary is preserved by the tilde operation.

PROPOSITION 1.1. *Let v be a weight on $\mathbb{D}$. Then $\lim_{|z|\to 1^-} \tilde{v}(z) = 0$, whenever $\lim_{|z|\to 1^-} v(z) = 0$. In particular, if v is a typical weight, then $\tilde{v}$ is typical as well.*

COROLLARY 1.2. *If $\lambda : \mathbb{D} \to \mathbb{R}_+$ is a radial weight then H_λ^∞ is strictly bigger than H^∞ if and only if $\lim_{|z|\to 1^-} \lambda(z) = 0$.*

PROOF OF PROPOSITION 1.1. Let us take $u = 1/v$ as usual; then the growth condition u tends to $+\infty$ at the boundary. Since there is a radial non-decreasing function $\leq u$ tending to $+\infty$ at the boundary, we may (and we will) assume that u is radial.

Let $r_n \in (0, 1)$, $u(r_n) = n$. We take $f_0 \equiv 0$ and we define inductively a sequence of functions (f_n) analytic on a neighbourhood of $\overline{\mathbb{D}}$ such that:

(a) $M(f_j, r_i) < i$ for $i \leq j$;
(b) $M(f_j, 1) = j$ and $f_j(1) = j$;
(c) $M(f_j - f_{j-1}, r_j) \leq 2^{-j}$.

Assume that $f_1, \dots, f_{n-1}$ are defined satisfying (a) – (c) for $j \leq n-1$. Then we define b_n, $0 < b_n < 2^{-n}$, such that $b_n < i - M(f_{n-1}, r_i)$ for $i = 1, \dots, n$. We define a function g, analytic on a neighbourhood of $\overline{\mathbb{D}}$ such that $g(z) := a_n/(R_n - z)^n$, where

$$a_n = \left(\frac{1-r_n}{b_n^{-1/n} - 1}\right)^n, \qquad R_n = 1 + \frac{1-r_n}{b_n^{-1/n} - 1}.$$

It is easily seen that:

$$M(g, r_n) \leq b_n, \qquad g(1) = 1, \qquad M(g, 1) = 1.$$

By the choice of b_n we obtain that $f_n = f_{n-1} + g$ satisfies conditions (a) – (c) for $j = n$. Let us define $f := \lim_{n\to\infty} f_n$. Since $M(f, r_n) \leq n = u(r_n)$ for $n \in \mathbb{N}$, we have $|f(z)| \leq u(z) + 1$ for any $z \in \mathbb{D}$ and there is a constant C such that $|f(z)| \leq Cu(z)$ for any $z \in \mathbb{D}$. Clearly, $\tilde{u}(z) \geq |f(z)|/C$ and $\lim_{|z|\to 1^-} \tilde{u}(z) = +\infty$. This completes the proof.

In any case we can substitute v by $\tilde{v}$ but unfortunately we have no easy way of calculating $\tilde{v}$ from v.

PROPOSITION 1.3. *For every weight v we have isometrically $H_v^\infty = H_{\tilde{v}}^\infty$ and, if $\lim_{|z|\to 1^-} v(z) = 0$, then $H_v^0 = H_{\tilde{v}}^0$.*

PROOF. By the property (iii), $H_v^\infty = H_{\tilde{v}}^\infty$ isometrically and $H_{\tilde{v}}^0$ is an isometric (closed) subspace of H_v^0. As we mentioned before, if $T = C_{\mathrm{id}} : H_{\tilde{v}}^0 \to H_v^0$, then $T'' : H_{\tilde{v}}^\infty \to H_v^\infty$ is also equal to $C_{\mathrm{id}} : H_{\tilde{v}}^\infty \to H_v^\infty$. Clearly, if T were not onto, then T'' would have not been onto as well.

2. Boundedness of C_φ

We give firstly necessary and sufficient conditions for the operator C_φ to be bounded on H_v^∞.

PROPOSITION 2.1. *Let v and w be weights. The following statements are equivalent:*

(i) *the operator $C_\varphi : H_v^\infty \to H_w^\infty$ is bounded;*
(ii) $\sup_{z\in\mathbb{D}} w(z)/\tilde{v}(\varphi(z)) = M < \infty$,

(iii) $\sup_{z\in\mathbb{D}} \tilde{w}(z)/\tilde{v}(\varphi(z)) = M < \infty$.

If v and w satisfy $\lim_{|z|\to 1^-} v(z) = \lim_{|z|\to 1^-} w(z) = 0$, then the above conditions are equivalent to

(iv) *the operator $C_\varphi : H_v^0 \to H_w^0$ is bounded.*

REMARK. Contrary to many cases of classical function spaces, an analytic self-map $\varphi : \mathbb{D} \to \mathbb{D}$ does not necessarily induce a bounded composition operator for general weights. For example, consider $v(z) = w(z) = e^{-(1-|z|)^{-1}}$ and $\varphi(z) = (z+1)/2$. Then $v = \tilde{v}$ and, for $z = r \in \mathbb{R}$, $v(r)/v(\varphi(r)) = e^{1/(1-r)}$ for $0 < r < 1$. Thus $v(r)/v(\varphi(r)) \to \infty$, when $r \to 1$, so C_φ is not bounded on H_v^∞.

If v is essential, then for every $z \in \mathbb{D}$ we have that $\tilde{v}(z) \le Cv(z)$, and the necessity of (4) in the next corollary follows from this.

COROLLARY 2.2. *Assume that v is an essential weight. The operator $C_\varphi : H_v^\infty \to H_w^\infty$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} w(z)/v(\varphi(z)) < \infty. \tag{4}$$

If v and w satisfy $\lim_{|z|\to 1^-} v(z) = \lim_{|z|\to 1^-} w(z) = 0$, then $C_\varphi : H_v^0 \to H_w^0$ is bounded if and only if (4) *holds.*

REMARKS. (1) If $H_v^\infty = H^\infty$, then $C_\varphi : H_v^\infty \to H_w^\infty$ is continuous for all weights w and, if w tends to zero at the boundary $C_\varphi(H_v^\infty) \subseteq H_w^0$. On the other hand, by Proposition 1.1 and by properties of associated weights, one can prove that if $(z_n) \subseteq \mathbb{D}$, $\lim_{n\to\infty} |z_n| = 1$ and v is a weight on $\mathbb{D}$ tending to zero at the boundary, then there is a sequence of functions $f_n \in B_v$ such that $|f_n(z_n)| \to \infty$ as $n \to \infty$. Thus, if $H_w^\infty = H^\infty$ and $\partial\varphi(\mathbb{D}) \cap \partial\mathbb{D} \ne \emptyset$ then $C_\varphi f_n$ is unbounded in H_w^∞. Finally, in that case $C_\varphi : H_v^\infty \to H_w^\infty$ is bounded if and only if $\overline{\varphi(\mathbb{D})} \subseteq \mathbb{D}$.

(2) The condition in the corollary is no more necessary for boundedness of C_φ whenever v is not essential. Let us take an arbitrary non-essential weight v and $w = \tilde{v}$. Then, clearly $\limsup_{|z|\to 1^-} w(z)/v(z) = +\infty$ but $C_{\mathrm{id}} : H_v^\infty \to H_w^\infty$ is an isometry.

PROOF OF PROPOSITION 2.1. (iii) implies (ii) is trivial as $w \le \tilde{w}$.
(ii) implies (i): By assumption, we have $w(z) \le M\tilde{v}(\varphi(z))$ for all $z \in \mathbb{D}$. Thus

$$w(z)|f(\varphi(z))| = \frac{w(z)}{\tilde{v}(\varphi(z))}\tilde{v}(\varphi(z))|f(\varphi(z))| \le M\|f\|_{\tilde{v}}$$

(i) implies (iii) and (iv) implies (iii): If not, then there is a sequence $(z_n) \subset \mathbb{D}$, with $\tilde{w}(z_n) > n\tilde{v}(\varphi(z_n))$ for all n. For all n, there exists $f_n \in B_v$ (which can be chosen in

B_v^0, whenever $\lim_{|z|\to 1^-} v(z) = 0$, by τ_p-density of B_v^0 in B_v) such that $|f_n(\varphi(z_n))| > \tilde{u}(\varphi(z_n))/2$. By (i) or (iv), $(f_n \circ \varphi)$ is bounded in $H_w^\infty(D) = H_{\tilde{w}}^\infty(D)$, so there is $C > 0$ such that $|f_n(\varphi(z))|\tilde{w}(z) \le C$ for all $z \in D$ and all $n \in \mathbb{N}$. On the other hand,

$$|f_n(\varphi(z_n))|\tilde{w}(z_n) = |f_n(\varphi(z_n))|\tilde{v}(\varphi(z_n))\tilde{w}(z_n)/\tilde{v}(\varphi(z_n)) > n/2$$

for all n, so we have a contradiction.

(iii) implies (iv): By (iii) implies (i) and Proposition 1.3, it suffices to show that $C_\varphi(f) \in H_{\tilde{w}}^0$ for each $f \in H_{\tilde{v}}^0$.

Take $f \in H_{\tilde{v}}^0$. Given $\varepsilon > 0$, there is $r_1 \in]0, 1[$, such that $\tilde{v}(z)|f(z)| < \varepsilon/M$ for $|z| > r_1$. For $|z| > r_1$ we consider two cases: If $|\varphi(z)| > r_1$, then

$$\tilde{w}(z)|f(\varphi(z))| = \tilde{v}(\varphi(z))|f(\varphi(z))| \frac{\tilde{w}(z)}{\tilde{v}(\varphi(z))} < \varepsilon.$$

For $|\varphi(z)| \le r_1$, we have that there is $r_2 \ge r_1$, $0 < r_2 < 1$, such that

$$\tilde{w}(z)|f(\varphi(z))| \le \tilde{w}(z)\sup_{|z|\le r_1}|f(z)| < \varepsilon \quad \text{for} \quad |z| > r_2.$$

Thus $\tilde{w}(z)|f(\varphi(z))| < \varepsilon$ for $|z| > r_2$.

THEOREM 2.3. *Let v be a typical weight. The following assertions are equivalent:*

(i) *all operators $C_\varphi : H_v^\infty \to H_v^\infty$ are bounded;*
(ii) *all operators $C_\varphi : H_v^0 \to H_v^0$ are bounded;*
(iii) *the following inequality holds:*

$$\inf_{n\in\mathbb{N}} \frac{\tilde{v}(1-2^{-n-1})}{\tilde{v}(1-2^{-n})} > 0. \tag{5}$$

REMARK. For example, an essential weight $v(z) = (1-|z|)^\alpha$, $\alpha > 0$, satisfies the conditions of Theorem 2.3 The condition (5) was used by Lusky [L2, p. 310], when he studied the isomorphism $H_v^0 \simeq c_0$. Also see [SW2].

PROOF. By Proposition 2.1, (i) holds if and only if (ii) does.

By the Schwarz lemma, if $\varphi(0) = 0$, then $|\varphi(z)| \le |z|$ for every $z \in \mathbb{D}$ and therefore $C_\varphi : H_v^\infty \to H_v^\infty$ is bounded. For $p \in \mathbb{D}$, let

$$\alpha_p(z) = (p - z)/(1 - \bar{p}z),$$

that takes $\mathbb{D}$ onto itself. If each C_{α_p} is bounded, then all C_φ are bounded. Indeed, each $\varphi = \alpha_p \circ \psi$, where $\psi = \alpha_p \circ \varphi$, $p = \varphi(0)$ and $\psi(0) = 0$. We have to show that for any $p \in \mathbb{D}$, C_{α_p} is bounded on $H_v^\infty(\mathbb{D})$ if and only if $\tilde{v}$ satisfies (5).

(i) implies (iii): Let us assume that all C_{α_p} are bounded. Then, by Proposition 2.1, for every $p \in \mathbb{D}$ there exist $M_p > 0$ such that $\tilde{v}(z) < M_p \tilde{v}(\alpha_p(z))$ for all $z \in \mathbb{D}$. Since it is easily seen that $\sup_{|z|=r} |\alpha_p(z)| = (|p| + r)/(1 + |p|r)$, we get that $\tilde{v}(z) < M_p \tilde{v}((|p| + r)/(1 + |p|r))$ for all $|z| = r$. Let us define $l(r) = \tilde{v}(1 - r)$, $s = 1 - r$. Since $1 - (|p| + 1 - s)/(1 + |p|(1 - s)) = s(1 - |p|)/(1 + |p| - |p|s)$, we obtain for $s < 1/2$ that

$$l\left(s\frac{1-|p|}{1+|p|}\right) \leq l\left(1 - \frac{|p|+1-s}{1+|p|(1-s)}\right) \leq l\left(s\frac{(1-|p|)}{1+|p|/2}\right). \tag{6}$$

Finally, for $p = 2/5$, we use the second inequality in (6) to get $M > 0$ and $s_0 > 0$ such that $l(s) \leq M l(\frac{s}{2})$ for all $s \in]0, s_0[$. This immediately implies (5).

(iii) implies (i): If (5) is satisfied, then l defined as above has the property that there are $M > 0$ and $t_0 \in]0, 1[$ with $l(t) \leq M l(t/2)$ for all $t > t_0$. Then for any $c < \infty$ we find $n \in \mathbb{N}$ such that $c < 2^n$ and hence $l(t) \leq M^n l(\frac{t}{c})$. We take $c = (1+|p|)/(1-|p|)$. Then by the first inequality in (6), for all $p \in \mathbb{D}$ there is $M_p > 0$ with

$$l(t) \leq M_p l\left(1 - \frac{|p|+1-t}{1+|p|(1-t)}\right),$$

for all $t > t_0$. Clearly, this implies that for all $p \in \mathbb{D}$ there exists $M_p > 0$ such that for every $|z| = r$ we have that $\tilde{v}(z) < M_p \tilde{v}(\alpha_p(z))$ by the argument above.

THEOREM 2.4. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map. The following statements are equivalent:*

(i) *for any typical weight v the map C_φ is bounded on H_v^∞;*

(ii) *for every $\theta \in]-\pi, \pi]$, $\alpha_\theta \circ \varphi$ fixes a point in $\mathbb{D}$, where $\alpha_\theta(z) = e^{i\theta} z$;*

(iii) *either φ is a rotation or for any $\theta \in]-\pi, \pi]$, $\psi = \alpha_\theta \circ \varphi$ fixes an attracting point $p_\theta \in \mathbb{D}$, that is, $\psi(p_\theta) = p_\theta$ and $\psi_n(z) \to p_\theta$ uniformly on compact sets, where $\psi_n := \psi \circ \cdots \circ \psi$, ($n$ times);*

(iv) *either φ is a rotation or* $\liminf_{|z|\to 1^-} (1 - |\varphi(z)|)/(1 - |z|) > 1$*;*

(v) *there is an $r_0 \in]0, 1[$ such that $|\varphi(z)| \leq |z|$ for every $z \in \mathbb{D}$ with $|z| \geq r_0$.*

For the proof of Theorem 2.4 we need the following result.

LEMMA 2.5. *For any two increasing sequences $r_n \to 1$ and $R_n \to 1$ such that $r_0 < R_0 < r_1 < R_1 < r_3 < R_3 < \cdots$ there is an analytic map $f : \mathbb{D} \to \mathbb{C}$ with*

$$\lim_{n\to\infty} \frac{M(f, R_n)}{M(f, r_n)} = \infty,$$

where $M(f, r) = \sup_{|z|=r} |f(z)|$.

PROOF. We define $f_0 \equiv 1$. Assume that we have already found polynomials $f_1, f_2, \dots, f_{n-1}$ such that

$$\text{(a)}\quad f_k|_{r_k\mathbb{D}} < 1/2^k, \qquad \text{(b)}\quad \frac{M(f_k, R_k)}{M(\sum_{i=1}^{k-1} f_i, r_k)} > k \quad \text{for} \quad k = 1, \dots, n-1.$$

Then put $M := M\left(\sum_{i=1}^{n-1} f_i, r_n\right)$ and $\tilde{f}_n(z) = A/((1-\varepsilon)R_n - z)$, where A, ε are chosen positive and

$$\text{(c)}\quad \frac{A}{(1-\varepsilon)R_n - r_n} = \frac{1}{2^{2n}}, \qquad \text{(d)}\quad \frac{A}{\varepsilon R_n} = 2Mn.$$

In fact, it suffices to take

$$A = \frac{2Mn(R_n - r_n)}{(2^{2n+1}Mn + 1)} \qquad \text{and} \qquad \varepsilon = \frac{(R_n - r_n)}{R_n(2^{2n+1}Mn + 1)}.$$

Clearly $\tilde{f}_n|_{r_n\mathbb{D}} < 1/2^{2n}$ because of (c). Moreover, by (d), $\tilde{f}_n(R_n) = 2nM$. Now, by Runge's Theorem, we can approximate $\tilde{f}_n$ on $\overline{R_n\mathbb{D}}$ by a polynomial f_n satisfying (a) and (b). We define $f = \sum_{i=1}^{\infty} f_i$. The series converges almost uniformly. The condition $\lim_n M(f, R_n)/M(f, r_n) = \infty$ follows from (b).

PROOF OF THEOREM 2.4. (i) implies (v): Assume that there is a sequence (z_n) in $\mathbb{D}$ with $|z_n| \to 1$ such that $|\varphi(z_n)| > |z_n|$ for all n. Define $r_n = |z_n|$, $R_n = |\varphi(z_n)|$ and without loss of generality we may assume that $r_0 < R_0 < r_1 < R_1 < r_2 < R_2 < \cdots$. By Lemma 2.5, there exists $f \in H(\mathbb{D})$ such that $\lim_{n\to\infty} M(f, R_n)/M(f, r_n) = \infty$. Take $v(z) = 1/M(f, |z|)$ and assume that $C_\varphi : H_v^\infty \to H_v^\infty$ is bounded. Then there is $c > 0$ such that $\|C_\varphi\| \le c$. Moreover, with $\alpha_\theta(z) = e^{i\theta}z$, $\|C_{\alpha_\theta}\| = 1$ and $\|f\|_v = 1$, so

$$\|C_{\alpha_\theta \circ \varphi}(f)\|_v \le c \quad \text{for all} \quad \theta \in]-\pi, \pi].$$

Now,

$$\sup_{z\in D} |f \circ \alpha_\theta \circ \varphi(z)| \frac{1}{M(f, |z|)} \ge |f \circ \alpha_\theta \circ \varphi(z_n)| \frac{1}{M(f, |z_n|)}$$

for all n and all θ. Choosing θ_n in a suitable way, we get $|f \circ \alpha_{\theta_n} \circ \varphi(z_n)| = M(f, |\varphi(z_n)|) = M(f, R_n)$. Thus, $\|C_{\alpha_{\theta_n} \circ \varphi}(f)\|_v \ge M(f, R_n)/M(f, r_n)$ for all n, and we have a contradiction.
(v) implies (i) is obvious.

(i) implies (ii): Let us assume firstly that φ is an automorphism. Then either φ is a rotation and (ii) is satisfied or $\varphi(z) = \alpha_\theta((p-z)/(1-\bar{p}z))$, where $p \neq 0$. Since $\sup_{|z|=r} |\varphi(z)| = (|p|+r)/(1+|p|r) > r$ for $r < 1$, it follows that (v) is not satisfied. We have proved that if φ is an automorphism, then C_φ satisfies (i) only for rotations, and then (ii) is satisfied as well.

Now let φ be a non-automorphism. Then every $\alpha_\theta \circ \varphi$ is also a non-automorphism. Hence, by [Sh, Section 5.4], either $\alpha_\theta \circ \varphi$ has a fixed point or $\psi_n := \psi \circ \cdots \circ \psi$, ($n$ times), where $\psi = \alpha_\theta \circ \varphi$, satisfies $|\psi_n(z)| \to 1$ uniformly on compact sets. Suppose that some $\alpha_\theta \circ \varphi$ has no fixed point. Let $K := \{z : |z| \leq r_0\}$, where r_0 is given in (v). By the Maximum Modulus Theorem, $\sup_{z\in K} |\varphi(z)| = \sup_{z\in\partial K} |\varphi(z)|$ and therefore it follows from (v) that $\sup_{z\in\partial K} |\psi_n(z)| \leq r_0$. Since $|\psi_n(z)| \to 1$ uniformly on compact sets, we obtain a contradiction.

(ii) implies (iii): Assume that φ is an automorphism. Hence

$$\varphi(z) = e^{i\theta}\frac{p-z}{1-\bar{p}z} \quad \text{for some} \quad p \in \mathbb{D},\ \theta \in]-\pi,\pi].$$

If $p \neq 0$, then for a suitable chosen θ_0, $|p| = \cos(\theta_0/2)$ and by [Sh, Section 0.5.4], φ is a parabolic automorphism, meaning (see [Sh, p. 5]) that φ has a single fixed point lying on the boundary of $\mathbb{D}$. If $p = 0$, then φ is a rotation.

Now, let φ be a non-automorphism. Then every $\alpha_\theta \circ \varphi$ is also a non-automorphism. Hence the assumption and [Sh, Section 5.2.1], give that every $\alpha_\theta \circ \varphi$ fixes an attracting point in $\mathbb{D}$.

(iii) implies (iv): Let us assume that each $\alpha_\theta \circ \varphi$ has an attracting fixed point p_θ in $\mathbb{D}$. If $\liminf_{|z|\to 1^-}(1-|\varphi(z)|)/(1-|z|) \leq 1$, then there is a sequence $(z_n) \subset \mathbb{D}$, $|z_n| \to 1$ and $\lim_n(1-|\varphi(z_n)|)/(1-|z_n|) = \delta \leq 1$. Without loss of generality, we may assume that $z_n \to w \in \partial\mathbb{D}$. Clearly, $|\varphi(z_n)| \to 1$, and we may assume that $\varphi(z_n) \to \eta \in \partial\mathbb{D}$. Choosing $\theta \in]-\pi,\pi]$ suitable, we get $\alpha_\theta \circ \varphi(z_n) \to w$. Now, by Julia's theorem [Sh, p. 63],

$$\alpha_\theta \circ \varphi\left(\frac{\lambda}{1+\lambda}\mathbb{D} + \frac{w}{1+\lambda}\right) \subset \frac{\lambda}{1+\lambda}\mathbb{D} + \frac{w}{1+\lambda} \quad \text{for all} \quad \lambda > 0.$$

Since p_θ is an attracting point of $\alpha_\theta \circ \varphi$, it is in any disc $(\lambda/(1+\lambda))\mathbb{D} + w/(1+\lambda)$, which gives a contradiction by taking $\lambda > 0$ small enough.

(iv) implies (v): The condition is obviously satisfied for rotations. Now, if $\liminf_{|z|\to 1^-}(1-|\varphi(z)|)/(1-|z|) > 1$, then there is $r_0 \in]0,1[$ such that $(1-|\varphi(z)|)/(1-|z|) > 1$ for $|z| > r_0$. Clearly, then $|\varphi(z)| < |z|$, and we are done.

3. Compactness of C_φ

To deal with the compactness we need the following form of the Weak Compactness Theorem. The proof is similar to the case of H^2. The reader is asked to refer to [Sh, Section 2.4].

LEMMA 3.1. *Let X and Y be H_v^∞ and H_w^∞ or H_v^0 and H_w^0 respectively. A bounded operator $C_\varphi : X \to Y$ is compact, if and only if, given any bounded sequence $(f_n) \subset X$ which converges to 0 uniformly on the compact subsets of $\mathbb{D}$, also the sequence $(C_\varphi(f_n))$ converges to 0 in Y.*

COROLLARY 3.2. *Let v and w be weights. If there exists an r, $0 < r < 1$, such that $|\varphi(z)| \le r$ for all $z \in \mathbb{D}$, then $C_\varphi : H_v^\infty \to H_w^\infty$ is compact.*

THEOREM 3.3. *Let v and w be weights. The following assertions are equivalent:*

(i) *the operator $C_\varphi : H_v^\infty \to H_w^\infty$ is compact;*
(ii) $\lim_{r\to 1^-} \sup_{|\varphi(z)|>r} w(z)/\tilde{v}(\varphi(z)) = 0$ *or* $\overline{\varphi(\mathbb{D})} \subseteq \mathbb{D}$.

If v and w satisfy $\lim_{|z|\to 1^-} v(z) = \lim_{|z|\to 1^-} w(z) = 0$, then the above conditions are also equivalent to the following ones:

(iii) *the operator $C_\varphi : H_v^0 \to H_w^0$ is compact;*
(iv) $\lim_{|z|\to 1^-} w(z)/\tilde{v}(\varphi(z)) = 0$,
(v) $\lim_{|z|\to 1^-} \tilde{w}(z)/\tilde{v}(\varphi(z)) = 0$.

REMARKS. (1) It is not difficult to show that if $H_v^\infty = H^\infty$, then $C_\varphi : H_v^\infty \to H_w^\infty$ is compact for any weight w tending to zero at the boundary. On the other hand, if $H_w^\infty = H^\infty$, then $C_\varphi : H_v^\infty \to H_w^\infty$ is compact if and only if $\overline{\varphi(\mathbb{D})} \subseteq \mathbb{D}$ (compare the remark after Proposition 2.1).

(2) If for a fixed typical weight v, $C_\varphi : H_v^\infty \to H_v^\infty$ is compact, then there is $r < 1$ such that $|\varphi(z)| < |z|$ for $|z| > r$. Thus $C_\varphi : H_w^\infty \to H_w^\infty$ is bounded for any typical weight w.

(3) If for some weights v, w tending to zero at the boundary, $C_\varphi : H_v^\infty \to H_w^\infty$ is compact, then it is weakly compact and $C_\varphi : H_v^\infty \to H_w^0$.

PROOF. We assume first that v and w satisfy $\lim_{|z|\to 1^-} v(z) = \lim_{|z|\to 1^-} w(z) = 0$.

(v) implies (iv) is obvious.

(iv) implies (i): By Proposition 2.1, C_φ is bounded. We shall use Lemma 3.1. Take a bounded sequence $(f_n) \subset B_v$ and assume that $f_n \to 0$ in the τ_{co}-topology. For given $\varepsilon > 0$ there is $r_0 \in]0, 1[$ with $w(z) < \varepsilon\tilde{v}(\varphi(z))/2$ for all $|z| > r_0$. Put

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$C' := \sup_{z\in\mathbb{D}} w(z) < \infty$. For n big enough, we have that $\sup_{|z|\le r_0} |f_n(\varphi(z))|C' < \varepsilon/2$ and thus

$$\begin{aligned}\|C_\varphi(f_n)\|_w &\le \sup_{|z|\le r_0}|f_n(\varphi(z))|w(z) + \sup_{|z|>r_0}|f_n(\varphi(z))|w(z)\\ &\le \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon\|f_n\|_{\tilde{v}} \le \varepsilon,\end{aligned}$$

so (i) follows.

(i) implies (v): If not, then there are $c > 0$ and a sequence $(z_n) \subset \mathbb{D}$, $|z_n| \to 1$, with $\tilde{w}(z_n) > c\tilde{v}(\varphi(z_n))$ for all n. For all n, there exists $f_n \in B_v$ such that $|f_n(\varphi(z_n))| = \tilde{u}(\varphi(z_n))$. By going to a subsequence we can assume that $\varphi(z_n) \to z_0$ for some $z_0 \in \overline{\mathbb{D}}$, when $n \to \infty$. If $|z_0| \ne 1$, then by assumption,

$$0 = \lim_n \tilde{w}(z_n) \ge c \lim_n \tilde{v}(\varphi(z_n)) = c\tilde{v}(z_0) > 0,$$

which is a contradiction. Thus $|z_0| = 1$. Now, since $|\varphi(z_n)| \to 1$, there exist natural numbers $\alpha(n)$ with $\lim_n \alpha(n) = \infty$ and such that $|\varphi(z_n)|^{\alpha(n)} \ge 1/2$ for all n. We define the analytic functions $g_n(z) := z^{\alpha(n)} f_n(z)$ for all n. Clearly (g_n) is a bounded sequence in H_v^∞. It converges pointwise to 0 because of the factor $z^{\alpha(n)}$. Hence, supposing that C_φ is compact, Lemma 3.1 implies that $\|C_\varphi(g_n)\|_w \to 0$ as $n \to \infty$. On the other hand, we get for all n,

$$\begin{aligned}\|C_\varphi(g_n)\|_{\tilde{w}} &\ge |g_n(\varphi(z_n))|\tilde{w}(z_n) = |\varphi(z_n)|^{\alpha(n)}|f_n(\varphi(z_n))|\tilde{w}(z_n)\\ &\ge \tfrac{1}{2}|f_n(\varphi(z_n))|c\tilde{v}(\varphi(z_n)) \ge \tfrac{1}{2}c,\end{aligned}$$

which is a contradiction.

(i) if and only if (iii): By Proposition 2.1, if one of the considered operators is compact, then both are bounded and $C_\varphi : H_v^\infty \to H_w^\infty$ is the bi-adjoint of $C_\varphi : H_v^0 \to H_w^0$ (see Section 1). Apply the Schauder Theorem.

Now, in the general case, (i) implies (ii) is similar to (i) $\Rightarrow$ (v) and (ii) implies (i) is similar to (iv) implies (i).

COROLLARY 3.4. *Let v and w be essential weights and assume that* $\lim_{|z|\to 1} w(z) = 0$. *Then $C_\varphi : H_v^\infty \to H_w^\infty$ is compact (or, equivalently, $C_\varphi : H_v^0 \to H_w^0$ is compact), if and only if*

$$\lim_{|z|\to 1^-} \frac{w(z)}{v(\varphi(z))} = 0. \tag{7}$$

We see below that for general weights our simple characterization of compactness in 3.4 fails. Of course, in that case v is no more essential and this fact gives another motivation for the concept of an essential weight.

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EXAMPLE 3.5. There exist a typical weight w and a compact composition operator $C_\varphi : H_w^\infty \to H_w^\infty$ which does not satisfy (7).

CONSTRUCTION. Let us define increasing sequences (r_n), (p_n) tending to 1 as follows:

$$r_0 = \frac{1}{2}, \qquad r_{n+1} = \frac{2r_n}{1+r_n}, \qquad p_0 = 0, \qquad p_n < r_n < p_{n+1} \quad \text{for } n \in \mathbb{N}.$$

Choose an increasing sequence (a_n) of natural numbers such that

$$a_n(\log r_{n+1} - \log r_n) \geq n. \tag{8}$$

We define three non-decreasing, unbounded and continuous functions $\eta, \theta, \tilde{\theta} : \mathbb{R}_- \to \mathbb{R}_+$ as follows:

(i) $\eta \equiv 0$ on $(-\infty, \log r_0]$ and for each $n \in \mathbb{N}$ the function η is affine on $[\log r_n, \log r_{n+1}]$ with the derivative $\equiv a_n$;

(ii) for n even $\theta \equiv \tilde{\theta} \equiv \eta$ on $[\log p_n, \log p_{n+1}]$;

(iii) for n odd $\tilde{\theta}$ is affine on $[\log p_n, \log p_{n+1}]$;

(iv) for n odd θ is affine on $[\log p_n, \log r_n]$ and constant on $[\log r_n, \log p_{n+1}]$.

We could choose (p_n) in such a way that:

(v) $|\eta(\log p_n) - \eta(\log r_n)| \leq 1$ and $|\tilde{\theta}(s) - \eta(s)| \leq 1$ for all $s \in \mathbb{R}_-$.

Now, we can define our typical weights:

$$v(z) := e^{-\eta(\log|z|)}, \qquad w(z) := e^{-\theta(\log|z|)}.$$

Clearly, our assumptions imply that $\theta(s) \geq \tilde{\theta}(s) \geq \eta(s)$ for $s \in \mathbb{R}_-$. By the Hadamard Three Circle Theorem and (v),

$$\tilde{w}(z) \geq e^{-\tilde{\theta}(\log|z|)} \geq e^{-1}v(z).$$

On the other hand, on the annulus $\{z : r_n \leq |z| \leq r_{n+1}\}$, we have $v(z) = b_n|z|^{-a_n}$ with a_n natural. Moreover, $1/v(z)$ is a supremum of a sequence $M(b_n^{-1}z^{a_n}, |z|)$, which means that $v = \tilde{v}$. Taking the above into account, we obtain that $\tilde{w}$ is equivalent to $v = \tilde{v}$.

We will show that $C_\varphi : H_v^\infty = H_w^\infty \to H_w^\infty = H_v^\infty$ is compact for $\varphi(z) := z/(2-z)$ but still $\lim_{|z|\to 1^-} w(z)/w(\varphi(z)) \neq 0$.

Observe, that $|\varphi(z)| \leq \varphi(|z|) = |z|/(2-|z|)$, thus if $|z| \in (r_n, r_{n+1})$, then $|\varphi(z)| \leq \varphi(|z|) \in (r_{n-1}, r_n)$. Hence

$$\frac{v(z)}{v(\varphi(z))} \leq \frac{v(z)}{v(\varphi(|z|))} = C\,\frac{(2-|z|)^{a_{n-1}}}{|z|^{a_{n+1}-a_n}}$$

and the function on the right hand side is decreasing with respect to $|z|$. Summarizing, by (8) for $|z| \in (r_n, r_{n+1})$,

$$\frac{v(z)}{v(\varphi(z))} \leq \frac{v(r_n)}{v(\varphi(r_n))} = \frac{v(r_n)}{v(r_{n-1})} \leq e^{-(n-1)}.$$

By Corollary 3.4, C_φ is compact.

On the other hand, for n even

$$\frac{w(r_n)}{w(\varphi(r_n))} = \frac{w(r_n)}{w(r_{n-1})} = \frac{v(r_n)}{v(p_n)} \geq e^{-1}.$$

This completes the proof.

Theorem 3.3 yields a direct method to deduce the compactness of C_φ in some weighted spaces once we know the compactness with respect to other weights. Precisely, assume that v is essential and that $C_\varphi : H_v^\infty \to H_w^\infty$ is compact. If the weights ν and ω on $\mathbb{D}$ satisfy

$$\frac{\omega}{\nu \circ \varphi} \leq C \frac{w}{v \circ \varphi} \tag{9}$$

for some constant $C > 0$, then $C_\varphi : H_\nu^\infty \to H_\omega^\infty$ is compact. Namely, the assumptions imply that (ii) of Theorem 3.3 is satisfied for v and w, hence, by (9), it is satisfied by ν and ω as well. We especially have the following corollary.

COROLLARY 3.6. *Assume that v and w are typical weights, v is essential, v/w is increasing as $|z| \to 1$, $\varphi(0) = 0$ and that $C_\varphi : H_v^\infty \to H_v^\infty$ is compact. Then also $C_\varphi : H_w^\infty \to H_w^\infty$ is compact.*

PROOF. In this case (9) is equivalent to

$$\frac{w}{w \circ \varphi} \leq C \frac{v}{v \circ \varphi}, \quad \text{or to} \quad \frac{v}{w} \circ \varphi \leq C \frac{v}{w}.$$

But this holds with $C = 1$ by the Schwarz lemma, since $v(z)/w(z)$ is radial and increasing as $|z| \to 1$.

THEOREM 3.7. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map such that $\partial\varphi(\mathbb{D}) \cap \partial\mathbb{D} \neq \emptyset$. Then there exists an essential typical weight v such that $C_\varphi : H_v^\infty \to H_v^\infty$ is bounded but not compact.*

COROLLARY 3.8. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map. The following assertions are equivalent:*

(i) *$C_\varphi : H_v^\infty \to H_v^\infty$ is compact for any (typical) weight v;*

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(ii) $C_\varphi : H_v^0 \to H_v^0$ *is compact for any weight* v *tending to zero at the boundary;*
(iii) $C_\varphi : H^\infty \to H^\infty$ *is compact;*
(iv) $\varphi(\mathbb{D}) \subseteq r\mathbb{D}$ *for some* $r < 1$.

PROOF OF THEOREM 3.7. The weight $w(z) = (1 - |z|)$ is typical and essential and $C_\varphi : H_w^\infty \to H_w^\infty$ is always bounded by Theorem 2.3. Let us assume that C_φ is compact in that case.

By Theorem 3.3, there is $r_0 < 1$ such that for any $r > r_0$:

$$1 - |z| < \tfrac{1}{2}(1 - |\varphi(z)|) \quad \text{for} \quad |z| = r.$$

Let us define $M(r) := \sup_{|z|=r} |\varphi(z)|$, then we have

$$1 - r < \tfrac{1}{2}(1 - M(r)) \quad \text{for} \quad r > r_0. \tag{10}$$

The function $M(r)$ is non-decreasing, tends to 1 as $r \to 1^-$ and, by the Hadamard Three Circle Theorem, it is logarithmically convex. In particular, $\log M(r)/\log r$ is non-decreasing as $r \to 1^-$. By (10), there is $\delta > 1$ such that for $r \geq r_0$ we have $\log M(r)/\log r > \delta$. We define inductively a sequence $(r_n)_{n\in\mathbb{N}}$ such that $r_{n-1} = M(r_n)$. Obviously,

$$r_{n-1} < r_n^\delta \quad \text{for} \quad n \in \mathbb{N} \tag{11}$$

and $\lim_{n\to\infty} r_n = 1$. We define an increasing function $u : [0, 1) \to \mathbb{R}_+$ which is equal to 1 on $[0, r_0]$, $u(r_n) = 2^n$ and it is affine on each interval $[r_{n-1}, r_n]$. The weight we are looking for is defined as $v(z) = 1/u(|z|)$.

Firstly, we show that v is essential. By [BBT, Proposition 3.4], it suffices to prove the so-called condition (U) for v, that is, to find $\alpha > 0$ and $C > 0$ such that

$$u(y)(1 - y)^\alpha \leq Cu(x)(1 - x)^\alpha \quad \text{for all} \quad 0 \leq x < y < 1. \tag{12}$$

We take $C = 4$ and arbitrary α such that $\left((1 - r_0)/(1 - r_0^{1/\delta})\right)^\alpha > 2$. Indeed, for any x, y either $u(y) \leq 2u(x)$ or we find $n \leq k$ such that

$$r_n \leq y, \; u(y) \leq 2u(r_n) \quad \text{and} \quad x \leq r_k, \; u(r_k) \leq 2u(x).$$

Thus

$$\frac{u(y)}{u(x)} \leq 4\frac{u(r_n)}{u(r_k)} < 4\left(\frac{1 - r_0}{1 - r_0^{1/\delta}}\right)^{\alpha(n-k)}. \tag{13}$$

Since $g(t) := (1 - t)/(1 - t^{1/\delta})$ is an increasing function,

$$\frac{1 - r_0}{1 - r_0^{1/\delta}} \leq \frac{1 - r_i}{1 - r_i^{1/\delta}} \quad \text{for any} \quad i \geq 0$$

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and, by (11), we get

$$\frac{1-r_0}{1-r_0^{1/\delta}} \leq \frac{1-r_i}{1-r_{i+1}}.$$

Combining with (13), we get

$$\frac{u(y)}{u(x)} \leq 4\prod_{i=k}^{n-1}\left(\frac{1-r_i}{1-r_{i+1}}\right)^{\alpha} = 4\left(\frac{1-r_k}{1-r_n}\right)^{\alpha} \leq 4\left(\frac{1-x}{1-y}\right)^{\alpha},$$

which gives (12) and v is essential.

Now, for $|z| > r_0$ we find $n \in \mathbb{N}$ such that $r_n \leq |z| < r_{n+1}$, then

$$\frac{v(z)}{v(\varphi(z))} \leq \frac{u(M(|z|))}{u(|z|)} \leq \frac{u(M(r_{n+1}))}{u(r_n)} = 1,$$

by the definition of (r_n). Thus $C_\varphi : H_v^\infty \to H_v^\infty$ is bounded by Proposition 2.1.

On the other hand, for any $n \in \mathbb{N}$ there is z_n, $|z_n| = r_n$ such that $|\varphi(z_n)| = r_{n-1}$. Clearly,

$$\frac{v(z_n)}{v(\varphi(z_n))} = \frac{u(r_{n-1})}{u(r_n)} = \frac{1}{2}$$

and (since $|z_n| \to 1$ as $n \to \infty$) $C_\varphi : H_v^\infty \to H_v^\infty$ is not compact by Corollary 3.4.

4. Integral operators C_φ

Unfortunately we are not able to give a characterization of nuclear, integral or absolutely summing composition operators for general weights. It is, however, not too complicated to find sufficient conditions for example as follows.

PROPOSITION 4.1. *Let $1 \leq p < \infty$, $1/p + 1/q = 1$, let v and w be radial weights and let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. The operator $C_\varphi : H_v^\infty \to H_w^\infty$ is p-integral, if*

$$\sup_{z\in\mathbb{D}} \int_{\xi\in\mathbb{D}} \frac{w(z)^q}{v\circ\varphi(\xi)^q |1 - z\bar{\xi}|^{2q} |\varphi'(\xi)|^{2q/p}} dA(\xi) < \infty. \tag{14}$$

In the case $p = 1$ the integral is replaced by a supremum in the usual way.

PROOF. Since the identity operator $H_v^\infty \to A_v^p \subset L^p(v(z)dA(z))$ is order bounded, it is enough to show that $C_\varphi : H_v^\infty \to H_w^\infty$ factorizes through the above defined map; see [DJT, Propositions 5.18 and 5.5]. To this end it suffices to prove that

$C_\varphi : A^p_v \to H^\infty_w$ is bounded. Indeed, using the Bergman reproducing kernel ([Z, Section 4.1]) and the dominated convergence theorem,

$$\sup_{z\in\mathbb{D}} |f\circ\varphi(z)|w(z) = \sup_{z\in\mathbb{D}} \left| \int_{\mathbb{D}} \frac{f\circ\varphi(\xi)}{(1-z\bar{\xi})^2} dA(\xi) \right| w(z)$$
$$\leq \sup_{z\in\mathbb{D}} \left(\int_{\xi\in\mathbb{D}} \frac{w(z)^q}{v\circ\varphi(\xi)^q |1-z\bar{\xi}|^{2q}} |J_\varphi(\xi)|^{-q/p} dA(\xi) \right)^{1/q}$$
$$\times \left(\int_{\mathbb{D}} |f\circ\varphi(\xi)|^p v\circ\varphi(\xi)^p |J_\varphi(\xi)| dA(\xi) \right)^{1/p},$$

where $|J_\varphi|$ is the 2-dimensional Jacobian determinant of φ. The identity $|J_\varphi| = |\varphi'|^2$ and (14) permit us to conclude.

EXAMPLE 4.2. An example of integral composition operators.

CONSTRUCTION. Let $v(z) := (1-|z|)^\alpha$ and $w(z) := (1-|z|)^\beta$, where $\alpha, \beta > 0$. Let $\Omega \subset \mathbb{D}$ be an open subset such that $\partial\Omega \cap \partial\mathbb{D} = \{1\}$. We also assume that $\partial\Omega$ is Dini-smooth ([Po, Theorem 3.5]) except at the point 1 where it has a Dini-smooth corner of opening $\gamma\pi$, where $\gamma < 1$, in the sense of [Po, p. 51]; we make the technical assumption that for some $c > 0$ the inequality

$$|\operatorname{Im}(z)| \leq c(1-\operatorname{Re}(z)) \tag{15}$$

is satisfied for every $z \in \Omega$. Let φ be a Riemann conformal mapping $\mathbb{D} \to \Omega$ such that $\lim_{z\to 1}\varphi(z) = 1$, $\varphi(0) = 0$ ([Po, Theorem 2.6]); for example $\varphi(z) := 1 - a(1-z)^\gamma$ for a suitable $a \in \mathbb{C}$.

We show that (14) is satisfied for $p = 1$ and the operator C_φ is thus 1-integral, if

(a) $\beta \geq 2 \geq (\alpha+2)\gamma$, or
(b) $\beta \geq (\alpha+2)\gamma \geq 2$.

By [Po, Theorem 3.9], there exists a neighbourhood U of 1 such that for some constants $c, C > 0$,

$$c < \left| \frac{1-\varphi(z)}{(1-z)^\gamma} \right| < C \quad \text{for } z \in U, \text{ and} \tag{16}$$

$$c < \left| \frac{\varphi'(z)}{(1-z)^{\gamma-1}} \right| < C \quad \text{for } z \in U, \tag{17}$$

and the same estimates also hold for numbers z outside U, because of $\overline{\varphi(\mathbb{D} \setminus U)} \subset \mathbb{D}$ and the Dini–smoothness assumption on $\partial\Omega$. From (17) we deduce that $|J_\varphi| = |\varphi'|^2$ is bounded from below and above by a positive constant times $|1 - z|^{2(\gamma-1)}$. Moreover, (15) implies that there exists a $C' > 0$ such that $|1 - \xi| \le C'(1 - |\xi|)$ for $\xi \in \Omega$; this and (16) imply that for $c' > 0$, $1 - |\varphi(z)| \ge c'|1 - z|^\gamma$ for $z \in \mathbb{D}$. Hence, for $c'' > 0$

$$v \circ \varphi(z) = (1 - |\varphi(z)|)^\alpha \ge c''|1 - z|^{\gamma\alpha}. \tag{18}$$

Hence,

$$\sup_{z,\xi\in\mathbb{D}} \frac{w(z)}{v \circ \varphi(\xi)|1 - z\bar{\xi}|^2|\varphi'(\xi)|} \le C \sup_{z,\xi\in\mathbb{D}} \frac{(1 - |z|)^\beta}{|1 - \xi|^{(\alpha+2)\gamma-2}|1 - z\bar{\xi}|^2}. \tag{19}$$

We now distinguish between the two cases (a) and (b) mentioned above. If (a) holds, (19) is not larger than

$$C \sup_{z,\xi\in\mathbb{D}} \frac{(1 - |z|)^\beta}{|1 - z\bar{\xi}|^2} \le C \sup_{z\in\mathbb{D}} \frac{(1 - |z|)^\beta}{(1 - |z|)^2} < \infty.$$

In the case (b) (19) can be estimated by

$$C \sup_{z,\xi\in\mathbb{D}} \frac{(1 - |z|)^\beta}{(1 - |\xi|)^{(\alpha+2)\gamma-2}|1 - z\bar{\xi}|^2}.$$

Clearly we can here take the sup only over the values $0 < z,\ \xi < 1$. The partial derivative of

$$\frac{(1 - z)^\beta}{(1 - \xi)^{(\alpha+2)\gamma-2}(1 - z\xi)^2}, \qquad 0 < z, \xi < 1, \tag{20}$$

with respect to ξ vanishes at the point $\xi = (((\alpha + 2)\gamma - 2 + 2z)/((\alpha + 2)\gamma z))^\beta$, and at this point (20) becomes equal to a constant times $(1 - z)^{\beta-(\alpha+2)\gamma} z^{(\alpha+2)\gamma-2}$. This expression is also finite, if the condition (b) is satisfied.

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