

SUPERADDITIVITY OF SOME FUNCTIONALS ASSOCIATED WITH JENSEN'S INEQUALITY FOR CONVEX FUNCTIONS ON LINEAR SPACES WITH APPLICATIONS

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Abstract

Some new results related to Jensen's celebrated inequality for convex functions defined on convex sets in linear spaces are given. Applications for norm inequalities in normed linear spaces and f -divergences in information theory are provided as well.

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1. Introduction

The Jensen inequality for convex functions plays a crucial role in the theory of inequalities due to the fact that other inequalities such as the generalized triangle inequality, the arithmetic–geometric mean inequality, Hölder and Minkowski inequalities, Ky Fan's inequality and so on can be obtained as particular cases of it.

Let C be a convex subset of the linear space X and f a convex function on C . If I denotes a finite subset of the set \mathbb{N} of natural numbers, $x_i \in C$, $p_i \geq 0$ for $i \in I$ and $P_I := \sum_{i \in I} p_i > 0$, then

$$f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right) \leq \frac{1}{P_I} \sum_{i \in I} p_i f(x_i), \quad (1.1)$$

is well known in the literature as *Jensen's inequality*.

We introduce the following notation (see also [14]):

$$\begin{aligned} F(C, \mathbb{R}) &:= \text{the linear space of all real functions on } C, \\ F^+(C, \mathbb{R}) &:= \{f \in F(C, \mathbb{R}) : f(x) > 0 \text{ for all } x \in C\}, \\ P_f(\mathbb{N}) &:= \{I \subset \mathbb{N} : I \text{ is finite}\}, \\ J(\mathbb{R}) &:= \{p = \{p_i\}_{i \in \mathbb{N}}, p_i \in \mathbb{R} \text{ are such that } P_I \neq 0 \text{ for all } I \in P_f(\mathbb{N})\}, \end{aligned}$$

$$J^+(\mathbb{R}) := \{p \in J(\mathbb{R}) : p_i \geq 0 \text{ for all } i \in \mathbb{N}\},$$

$$J_*(C) := \{x = \{x_i\}_{i \in \mathbb{N}} : x_i \in C \text{ for all } i \in \mathbb{N}\}$$

and

$$\text{Conv}(C, \mathbb{R}) := \text{the cone of all convex functions defined on } C.$$

In [14] the authors considered the following functional associated with the Jensen inequality:

$$J(f, I, p, x) := \sum_{i \in I} p_i f(x_i) - P_I f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right) \quad (1.2)$$

where $f \in F(C, \mathbb{R})$, $I \in P_f(\mathbb{N})$, $p \in J^+(\mathbb{R})$, $x \in J_*(C)$. They established some quasi-linearity and monotonicity properties and applied the results obtained to norm and mean inequalities.

The following result concerning the properties of the functional $J(f, I, \cdot, x)$ as a *function of weights* holds (see [14, Theorem 2.4]).

THEOREM 1.1. *Let $f \in \text{Conv}(C, \mathbb{R})$, $I \in P_f(\mathbb{N})$ and $x \in J_*(C)$.*

(i) *If $p, q \in J^+(\mathbb{R})$ then*

$$J(f, I, p + q, x) \geq J(f, I, p, x) + J(f, I, q, x) (\geq 0), \quad (1.3)$$

that is, $J(f, I, \cdot, x)$ is superadditive on $J^+(\mathbb{R})$.

(ii) *If $p, q \in J^+(\mathbb{R})$ with $p \geq q$, meaning that $p_i \geq q_i$ for each $i \in \mathbb{N}$, then*

$$J(f, I, p, x) \geq J(f, I, q, x) (\geq 0), \quad (1.4)$$

that is, $J(f, I, \cdot, x)$ is monotonic nondecreasing on $J^+(\mathbb{R})$.

The behavior of this functional as an *index set function* is incorporated in the following result (see [14, Theorem 2.1]).

THEOREM 1.2. *Let $f \in \text{Conv}(C, \mathbb{R})$, $p \in J^+(\mathbb{R})$ and $x \in J_*(C)$.*

(i) *If $I, H \in P_f(\mathbb{N})$ with $I \cap H = \emptyset$, then*

$$J(f, I \cup H, p, x) \geq J(f, I, p, x) + J(f, H, p, x) (\geq 0), \quad (1.5)$$

that is, $J(f, \cdot, p, x)$ is superadditive as an index set function on $P_f(\mathbb{N})$.

(ii) *If $I, H \in P_f(\mathbb{N})$ with $H \subset I$, then*

$$J(f, I, p, x) \geq J(f, H, p, x) (\geq 0), \quad (1.6)$$

that is, $J(f, \cdot, p, x)$ is monotonic nondecreasing as an index set function on $P_f(\mathbb{N})$.

As pointed out in [14], the above Theorem 1.2 is a generalization of the Vasić–Mijalković result for convex functions of a real variable obtained in [24] and therefore creates the possibility of obtaining vectorial inequalities as well.

For applications of the above results to logarithmic convex functions, to norm inequalities, in relation to the arithmetic–geometric mean inequality and to other classical results, see [6–10, 12–15, 19, 22].

Motivated by the above results, we introduce in the present paper a more general functional, establish its main properties and use it for some particular cases that provide inequalities of interest. Applications to norm inequalities in normed linear spaces and f -divergences in information theory are provided as well.

2. Some superadditivity properties for the weights

We consider the more general functional

$$D(f, I, p, x; \Phi) := P_I \Phi \left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right], \quad (2.1)$$

where $f \in \text{Conv}(C, \mathbb{R})$, $I \in P_f(\mathbb{N})$, $p \in J^+(\mathbb{R})$, $x \in J_*(C)$ and $\Phi : [0, \infty) \rightarrow \mathbb{R}$ is a function whose properties will determine the behavior of the functional D as follows. Obviously, for $\Phi(t) = t$ we recapture from D the functional J considered in [14].

First of all we observe that, by Jensen's inequality, the functional D is well defined and *positive homogeneous* in the third variable, that is,

$$D(f, I, \alpha p, x; \Phi) = \alpha D(f, I, p, x; \Phi),$$

for any $\alpha > 0$ and $p \in J^+(\mathbb{R})$.

The following result concerning the superadditivity and monotonicity of the functional D as a function of weights holds.

THEOREM 2.1. *Let $f \in \text{Conv}(C, \mathbb{R})$, $I \in P_f(\mathbb{N})$ and $x \in J_*(C)$. Assume that $\Phi : [0, \infty) \rightarrow \mathbb{R}$ is monotonic nondecreasing and concave.*

(i) *If $p, q \in J^+(\mathbb{R})$ then*

$$D(f, I, p + q, x; \Phi) \geq D(f, I, p, x; \Phi) + D(f, I, q, x; \Phi), \quad (2.2)$$

that is, D is superadditive as a function of weights.

(ii) *If $p, q \in J^+(\mathbb{R})$ with $p \geq q$, meaning that $p_i \geq q_i$ for each $i \in \mathbb{N}$ and $\Phi : [0, \infty) \rightarrow [0, \infty)$, then*

$$D(f, I, p, x; \Phi) \geq D(f, I, q, x; \Phi) (\geq 0), \quad (2.3)$$

that is, D is monotonic nondecreasing as a function of weights.

PROOF.

(i) Let $p, q \in J^+(\mathbb{R})$. By the convexity of the function f on C ,

$$\begin{aligned}
 & \frac{1}{P_I + Q_I} \sum_{i \in I} (p_i + q_i) f(x_i) - f\left(\frac{1}{P_I + Q_I} \sum_{i \in I} (p_i + q_i) x_i\right) \\
 &= \frac{P_I \left(\frac{1}{P_I} \sum_{i \in I} p_i f(x_i)\right) + Q_I \left(\frac{1}{Q_I} \sum_{i \in I} q_i f(x_i)\right)}{P_I + Q_I} \\
 & \quad - f\left(\frac{P_I \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right) + Q_I \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i\right)}{P_I + Q_I}\right) \\
 & \geq \frac{P_I \left(\frac{1}{P_I} \sum_{i \in I} p_i f(x_i)\right) + Q_I \left(\frac{1}{Q_I} \sum_{i \in I} q_i f(x_i)\right)}{P_I + Q_I} \\
 & \quad - \frac{P_I f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right) + Q_I f\left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i\right)}{P_I + Q_I} \\
 &= \frac{P_I \left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right)\right]}{P_I + Q_I} \\
 & \quad + \frac{Q_I \left[\frac{1}{Q_I} \sum_{i \in I} q_i f(x_i) - f\left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i\right)\right]}{P_I + Q_I}.
 \end{aligned} \tag{2.4}$$

Since Φ is monotonic nondecreasing and concave, then by (2.4),

$$\begin{aligned}
 & \Phi \left[\frac{1}{P_I + Q_I} \sum_{i \in I} (p_i + q_i) f(x_i) - f\left(\frac{1}{P_I + Q_I} \sum_{i \in I} (p_i + q_i) x_i\right) \right] \\
 & \geq \frac{P_I \Phi \left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right) \right]}{P_I + Q_I} \\
 & \quad + \frac{Q_I \Phi \left[\frac{1}{Q_I} \sum_{i \in I} q_i f(x_i) - f\left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i\right) \right]}{P_I + Q_I},
 \end{aligned}$$

which, by multiplication with $P_I + Q_I > 0$, produces the desired result (2.2).

(ii) If $p \geq q$, then by (i),

$$\begin{aligned}
 D(f, I, p, x; \Phi) &= D(f, I, (p - q) + q, x; \Phi) \\
 &\geq D(f, I, p - q, x; \Phi) + D(f, I, p, x; \Phi) \\
 &\geq D(f, I, p, x; \Phi)
 \end{aligned}$$

since $D(f, I, p - q, x; \Phi) \geq 0$. □

COROLLARY 2.2. *Let $f \in \text{Conv}(C, \mathbb{R})$, $I \in P_f(\mathbb{N})$ and $x \in J_*(C)$. Assume that $\Phi : [0, \infty) \rightarrow [0, \infty)$ is monotonic nondecreasing and concave.*

If there exist the numbers $M \geq m \geq 0$ such that $Mq \geq p \geq mq$, then

$$\begin{aligned} & M Q_I \Phi \left[\frac{1}{Q_I} \sum_{i \in I} q_i f(x_i) - f \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right] \\ & \geq P_I \Phi \left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right] \\ & \geq m Q_I \Phi \left[\frac{1}{Q_I} \sum_{i \in I} q_i f(x_i) - f \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right]. \end{aligned} \tag{2.5}$$

In particular,

$$\begin{aligned} & \frac{M}{m} \Phi \left[\frac{1}{Q_I} \sum_{i \in I} q_i f(x_i) - f \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right] \\ & \geq \Phi \left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right] \\ & \geq \frac{m}{M} \Phi \left[\frac{1}{Q_I} \sum_{i \in I} q_i f(x_i) - f \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right]. \end{aligned} \tag{2.6}$$

Now, if we write

$$S(\mathbf{1}) := \{p \in J^+(\mathbb{R}) : p_i \leq 1 \text{ for all } i \in \mathbb{N}\},$$

then we can state the following result as well.

COROLLARY 2.3. *Let $f \in \text{Conv}(C, \mathbb{R})$, $I \in P_f(\mathbb{N})$ and $x \in J_*(C)$. Assume that $\Phi : [0, \infty) \rightarrow [0, \infty)$ is monotonic nondecreasing and concave. Then we have the bound*

$$\begin{aligned} & \sup_{p \in S(\mathbf{1})} \left\{ P_I \Phi \left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right] \right\} \\ & = \text{card}(I) \Phi \left[\frac{1}{\text{card}(I)} \sum_{i \in I} f(x_i) - f \left(\frac{1}{\text{card}(I)} \sum_{i \in I} x_i \right) \right], \end{aligned} \tag{2.7}$$

where $\text{card}(I)$ denotes the cardinal of the finite set I .

REMARK 2.4. If we consider the concave and monotonic increasing function $\Phi(t) = \ln t$ and assume that $f \in \text{Conv}(C, \mathbb{R})$ and $x \in J_*(C)$ are selected such that

$$\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) > f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right)$$

for any $I \in P_f(\mathbb{N})$ with $\text{card}(I) \geq 2$ and $p \in J^+(\mathbb{R})$ (notice that it is enough to assume that f is strictly convex and x is not constant) then by the superadditivity of the functional

$$D(f, I, p, x; \ln) := P_I \ln \left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right] = \ln K(f, I, p, x),$$

where

$$K(f, I, p, x) := \left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right]^{P_I}, \tag{2.8}$$

we deduce that $K(f, I, \cdot, x)$ is supermultiplicative, that is, it satisfies the property

$$K(f, I, p + q, x) \geq K(f, I, p, x) K(f, I, q, x) \tag{2.9}$$

for any $p, q \in J^+(\mathbb{R})$.

The proof is obvious by the monotonicity and the positive homogeneity of the functional $D(f, I, \cdot, x; \ln)$.

Note that inequality (2.9) has been obtained in a different way by Agarwal and Dragomir in [1].

Another important example of a concave and monotonic increasing function is $\Phi(t) = t^s$ with $s \in (0, 1]$. In this situation the functional

$$D_s(f, I, p, x) := \left[P_I^{s-1} \sum_{i \in I} p_i f(x_i) - P_I^s f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right]^s \geq 0 \tag{2.10}$$

is superadditive and monotonic nondecreasing as a function of the weights p .

It might be useful for applications to observe that the superadditivity property is translated into the following version of Jensen's inequality:

$$\begin{aligned} & \left[(P_I + Q_I)^{s-1} \sum_{i \in I} (p_i + q_i) f(x_i) - (P_I + Q_I)^s f \left(\frac{\sum_{i \in I} (p_i + q_i) x_i}{P_I + Q_I} \right) \right]^s \\ & \geq \left[P_I^{s-1} \sum_{i \in I} p_i f(x_i) - P_I^s f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right]^s \\ & \quad + \left[Q_I^{s-1} \sum_{i \in I} q_i f(x_i) - Q_I^s f \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right]^s (\geq 0), \end{aligned} \tag{2.11}$$

where $p, q \in J^+(\mathbb{R})$.

The property of monotonicity provides the following double inequality for $p, q \in J^+(\mathbb{R})$ such that $Mq \geq p \geq mq$ and $M \geq m \geq 0$:

$$\begin{aligned}
 & M \left[Q_I^{s-1} \sum_{i \in I} q_i f(x_i) - Q_I^s f \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right]^s \\
 & \geq \left[P_I^{s-1} \sum_{i \in I} p_i f(x_i) - P_I^s f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right]^s \\
 & \geq m \left[Q_I^{s-1} \sum_{i \in I} q_i f(x_i) - Q_I^s f \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right]^s.
 \end{aligned}
 \tag{2.12}$$

This inequality has the equivalent form

$$\begin{aligned}
 & M^{1/s} \left[Q_I^{s-1} \sum_{i \in I} q_i f(x_i) - Q_I^s f \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right] \\
 & \geq P_I^{s-1} \sum_{i \in I} p_i f(x_i) - P_I^s f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \\
 & \geq m^{1/s} \left[Q_I^{s-1} \sum_{i \in I} q_i f(x_i) - Q_I^s f \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right].
 \end{aligned}
 \tag{2.13}$$

Finally, from the Corollary 2.3 we also have the bound

$$\begin{aligned}
 & \sup_{p \in S(\mathbf{1})} \left\{ P_I^{s-1} \sum_{i \in I} p_i f(x_i) - P_I^s f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right\} \\
 & = [\text{card}(I)]^{s-1} \sum_{i \in I} f(x_i) - [\text{card}(I)]^s f \left(\frac{1}{\text{card}(I)} \sum_{i \in I} x_i \right).
 \end{aligned}
 \tag{2.14}$$

For a function $\Psi : (0, \infty) \rightarrow (0, \infty)$ we now consider the functional

$$\begin{aligned}
 D(f, I, p, x; \Phi, \Psi) := & \sum_{i \in I} \Psi(p_i) \Phi \left[\frac{1}{\sum_{i \in I} \Psi(p_i)} \sum_{i \in I} \Psi(p_i) f(x_i) \right. \\
 & \left. - f \left(\frac{1}{\sum_{i \in I} \Psi(p_i)} \sum_{i \in I} \Psi(p_i) x_i \right) \right]
 \end{aligned}
 \tag{2.15}$$

where $f \in \text{Conv}(C, \mathbb{R})$, $I \in P_f(\mathbb{N})$, $p \in J^+(\mathbb{R})$, $x \in J_*(C)$. Now, if we denote by $\Psi(p)$ the sequence $\{\Psi(p_i)\}_{i \in \mathbb{N}}$, then we observe that

$$D(f, I, p, x; \Phi, \Psi) = D(f, I, \Psi(p), x; \Phi).$$

The following result may be stated.

COROLLARY 2.5. *Let $f \in \text{Conv}(C, \mathbb{R})$, $I \in P_f(\mathbb{N})$ and $x \in J_*(C)$. Assume that $\Phi : [0, \infty) \rightarrow [0, \infty)$ is monotonic nondecreasing and concave. If $\Psi : (0, \infty) \rightarrow (0, \infty)$ is concave, then $D(f, I, \cdot, x; \Phi, \Psi)$ is also concave on $J^+(\mathbb{R})$.*

PROOF. Utilizing the properties of monotonicity, superadditivity and positive homogeneity of the functional $D(f, I, \cdot, x; \Phi)$, we may successively write

$$\begin{aligned} D(f, I, tp + (1 - t)q, x; \Phi, \Psi) &= D(f, I, \Psi(tp + (1 - t)q), x; \Phi) \\ &\geq D(f, I, t\Psi(p) + (1 - t)\Psi(q), x; \Phi) \\ &\geq D(f, I, t\Psi(p), x; \Phi) + D(f, I, (1 - t)\Psi(q), x; \Phi) \\ &= tD(f, I, \Psi(p), x; \Phi) + (1 - t)D(f, I, \Psi(q), x; \Phi) \\ &= tD(f, I, p, x; \Phi, \Psi) + (1 - t)D(f, I, q, x; \Phi, \Psi) \end{aligned}$$

for any $p, q \in J^+(\mathbb{R})$ and $t \in [0, 1]$, which proves the statement. □

3. Some superadditivity properties for the index

The following result concerning the superadditivity and monotonicity of the functional D as an index set function holds.

THEOREM 3.1. *Let $f \in \text{Conv}(C, \mathbb{R})$, $p \in J^+(\mathbb{R})$ and $x \in J_*(C)$. Assume that $\Phi : [0, \infty) \rightarrow \mathbb{R}$ is monotonic nondecreasing and concave.*

(i) *If $I, H \in P_f(\mathbb{N})$ with $I \cap H = \emptyset$, then*

$$D(f, I \cup H, p, x; \Phi) \geq D(f, I, p, x; \Phi) + D(f, H, p, x; \Phi), \tag{3.1}$$

that is, $D(f, \cdot, p, x; \Phi)$ is superadditive as an index set function on $P_f(\mathbb{N})$.

(ii) *If $I, H \in P_f(\mathbb{N})$ with $H \subset I$ and $\Phi : [0, \infty) \rightarrow [0, \infty)$, then*

$$D(f, I, p, x; \Phi) \geq D(f, H, p, x; \Phi) (\geq 0), \tag{3.2}$$

that is, $D(f, \cdot, p, x; \Phi)$ is monotonic nondecreasing as an index set function on $P_f(\mathbb{N})$.

PROOF.

(i) Let $I, H \in P_f(\mathbb{N})$ with $I \cap H = \emptyset$. By the convexity of the function f on C we have

$$\begin{aligned} &\frac{1}{P_{I \cup H}} \sum_{k \in I \cup H} p_k f(x_k) - f\left(\frac{1}{P_{I \cup H}} \sum_{k \in I \cup H} p_k x_k\right) \\ &= \frac{P_I \left(\frac{1}{P_I} \sum_{i \in I} p_i f(x_i)\right) + P_H \left(\frac{1}{P_H} \sum_{j \in H} p_j f(x_j)\right)}{P_I + P_H} \\ &\quad - f\left(\frac{P_I \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right) + P_H \left(\frac{1}{P_H} \sum_{j \in H} p_j x_j\right)}{P_I + P_H}\right) \\ &\geq \frac{P_I \left(\frac{1}{P_I} \sum_{i \in I} p_i f(x_i)\right) + P_H \left(\frac{1}{P_H} \sum_{j \in H} p_j f(x_j)\right)}{P_I + P_H} \end{aligned}$$

$$\begin{aligned}
 & - \frac{P_I f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right) + P_H f\left(\frac{1}{P_H} \sum_{j \in H} p_j x_j\right)}{P_I + P_H} \\
 & = \frac{P_I \left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right) \right]}{P_I + P_H} \\
 & + \frac{P_H \left[\frac{1}{P_H} \sum_{j \in H} p_j f(x_j) - f\left(\frac{1}{P_H} \sum_{j \in H} p_j x_j\right) \right]}{P_I + P_H}.
 \end{aligned} \tag{3.3}$$

Since Φ is monotonic nondecreasing and concave, then by (3.3),

$$\begin{aligned}
 & \Phi \left[\frac{1}{P_{I \cup H}} \sum_{k \in I \cup H} p_k f(x_k) - f\left(\frac{1}{P_{I \cup H}} \sum_{k \in I \cup H} p_k x_k\right) \right] \\
 & \geq \frac{P_I \Phi \left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right) \right]}{P_I + P_H} \\
 & + \frac{P_H \Phi \left[\frac{1}{P_H} \sum_{j \in H} p_j f(x_j) - f\left(\frac{1}{P_H} \sum_{j \in H} p_j x_j\right) \right]}{P_I + P_H},
 \end{aligned}$$

which, by multiplication with $P_I + P_H > 0$, produces the desired result (3.2).

(ii) If $I, H \in P_f(\mathbb{N})$ with $H \subset I$, then

$$\begin{aligned}
 D(f, I, p, x; \Phi) & = D(f, (I \setminus H) \cup H, p, x; \Phi) \\
 & \geq D(f, I \setminus H, p, x; \Phi) + D(f, H, p, x; \Phi) \\
 & \geq D(f, H, p, x; \Phi) (\geq 0)
 \end{aligned}$$

since $D(f, I \setminus H, p, x; \Phi) \geq 0$. □

For the special case $I = I_n := \{1, \dots, n\}$ we write $D_n(f, p, x; \Phi)$ instead of $D(f, I_n, p, x; \Phi)$, that is,

$$D_n(f, p, x; \Phi) = P_n \Phi \left[\frac{1}{P_n} \sum_{i=1}^n p_i f(x_i) - f\left(\frac{1}{P_n} \sum_{i=1}^n p_i x_i\right) \right] \tag{3.4}$$

where $P_n = P_{I_n} = \sum_{i=1}^n p_i > 0$.

The following particular case is of interest.

COROLLARY 3.2. *Let $f \in \text{Conv}(C, \mathbb{R})$, $p \in J^+(\mathbb{R})$ and $x \in J_*(C)$. Assume that $\Phi : [0, \infty) \rightarrow [0, \infty)$ is monotonic nondecreasing and concave. Then*

$$\max_{I \subseteq I_n} D(f, I, p, x; \Phi) = D_n(f, p, x; \Phi) \geq 0, \tag{3.5}$$

$$\begin{aligned}
 & D_n(f, p, x; \Phi) \\
 & \geq \max_{1 \leq i < j \leq n} \left\{ (p_i + p_j) \Phi \left[\frac{p_i f(x_i) + p_j f(x_j)}{p_i + p_j} - f\left(\frac{p_i x_i + p_j x_j}{p_i + p_j}\right) \right] \right\} \geq 0 \tag{3.6}
 \end{aligned}$$

and

$$D_n(f, p, x; \Phi) \geq D_{n-1}(f, p, x; \Phi) \geq \dots \geq D_2(f, p, x; \Phi) \geq 0. \tag{3.7}$$

This is obvious by the monotonicity property of the functional $D(f, \cdot, p, x; \Phi)$ as an index set function.

If we use the superadditivity property, then we can state the following result as well.

COROLLARY 3.3. *Let $f \in \text{Conv}(C, \mathbb{R})$, $p \in J^+(\mathbb{R})$ and $x \in J_*(C)$. Assume that $\Phi : [0, \infty) \rightarrow \mathbb{R}$ is monotonic nondecreasing and concave. Then*

$$\begin{aligned} & P_{2n}\Phi \left[\frac{1}{P_{2n}} \sum_{i=1}^{2n} p_i f(x_i) - f \left(\frac{1}{P_{2n}} \sum_{i=1}^{2n} p_i x_i \right) \right] \\ & \geq \sum_{i=1}^n p_{2i} \Phi \left[\frac{1}{\sum_{i=1}^n p_{2i}} \sum_{i=1}^n p_{2i} f(x_{2i}) - f \left(\frac{1}{\sum_{i=1}^n p_{2i}} \sum_{i=1}^n p_{2i} x_{2i} \right) \right] \\ & \quad + \sum_{i=1}^n p_{2i-1} \Phi \left[\frac{1}{\sum_{i=1}^n p_{2i-1}} \sum_{i=1}^n p_{2i-1} f(x_{2i-1}) \right. \\ & \quad \left. - f \left(\frac{1}{\sum_{i=1}^n p_{2i-1}} \sum_{i=1}^n p_{2i-1} x_{2i-1} \right) \right] \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} & P_{2n+1}\Phi \left[\frac{1}{P_{2n+1}} \sum_{i=1}^{2n+1} p_i f(x_i) - f \left(\frac{1}{P_{2n+1}} \sum_{i=1}^{2n+1} p_i x_i \right) \right] \\ & \geq \sum_{i=1}^n p_{2i} \Phi \left[\frac{1}{\sum_{i=1}^n p_{2i}} \sum_{i=1}^n p_{2i} f(x_{2i}) - f \left(\frac{1}{\sum_{i=1}^n p_{2i}} \sum_{i=1}^n p_{2i} x_{2i} \right) \right] \\ & \quad + \sum_{i=1}^n p_{2i+1} \Phi \left[\frac{1}{\sum_{i=1}^n p_{2i+1}} \sum_{i=1}^n p_{2i+1} f(x_{2i+1}) \right. \\ & \quad \left. - f \left(\frac{1}{\sum_{i=1}^n p_{2i+1}} \sum_{i=1}^n p_{2i+1} x_{2i+1} \right) \right]. \end{aligned} \tag{3.9}$$

REMARK 3.4. If we consider the functional defined in (2.7), namely

$$K(f, I, p, x) := \left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right]^{P_I},$$

then by Theorem 3.1,

$$K(f, I \cup H, p, x) \geq K(f, I, p, x) \cdot K(f, H, p, x) \tag{3.10}$$

for any $I, H \in P_f(\mathbb{N})$ with $I \cap H = \emptyset$, meaning that the functional $K(f, \cdot, p, x)$ is supermultiplicative as an index set mapping.

This fact obviously implies the following multiplicative inequalities of interest:

$$\begin{aligned} & \left[\frac{1}{P_{2n}} \sum_{i=1}^{2n} p_i f(x_i) - f\left(\frac{1}{P_{2n}} \sum_{i=1}^{2n} p_i x_i\right) \right]^{P_{2n}} \\ & \geq \left[\frac{1}{\sum_{i=1}^n p_{2i}} \sum_{i=1}^n p_{2i} f(x_{2i}) - f\left(\frac{1}{\sum_{i=1}^n p_{2i}} \sum_{i=1}^n p_{2i} x_{2i}\right) \right]^{\sum_{i=1}^n p_{2i}} \\ & \quad \times \left[\frac{1}{\sum_{i=1}^n p_{2i-1}} \sum_{i=1}^n p_{2i-1} f(x_{2i-1}) \right. \\ & \quad \left. - f\left(\frac{1}{\sum_{i=1}^n p_{2i-1}} \sum_{i=1}^n p_{2i-1} x_{2i-1}\right) \right]^{\sum_{i=1}^n p_{2i-1}} \end{aligned} \quad (3.11)$$

where $f \in \text{Conv}(C, \mathbb{R})$, $p \in J^+(\mathbb{R})$ and $x \in J_*(C)$.

Moreover, if we consider the functional defined in (2.10) by

$$D_s(f, I, p, x) := \left[P_I^{s-1} \sum_{i \in I} p_i f(x_i) - P_I^s f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right) \right]^s \geq 0$$

where $s \in (0, 1]$ and introduce the associated functional

$$F_s(f, I, p, x) := P_I^{s-1} \sum_{i \in I} p_i f(x_i) - P_I^s f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right), \quad (3.12)$$

then by denoting

$$F_{s,n}(f, p, x) := F_s(f, I_n, p, x) = P_n^{s-1} \sum_{i=1}^n p_i f(x_i) - P_n^s f\left(\frac{1}{P_n} \sum_{i=1}^n p_i x_i\right) \quad (3.13)$$

where $I_n = \{1, \dots, n\}$, we have that the sequence $\{F_{s,n}(f, p, x)\}_{n \geq 2}$ is nondecreasing and the following bounds are valid:

$$\max_{I \subseteq I_n} F_s(f, I, p, x) = F_{s,n}(f, p, x) \quad (3.14)$$

and

$$\begin{aligned} F_{s,n}(f, p, x) & \geq \max_{1 \leq i < j \leq n} \left\{ \frac{p_i f(x_i) + p_j f(x_j)}{(p_i + p_j)^{1-s}} - (p_i + p_j)^s f\left(\frac{p_i x_i + p_j x_j}{p_i + p_j}\right) \right\} \\ & \geq 0. \end{aligned} \quad (3.15)$$

4. Applications for norm inequalities

Let $(X, \|\cdot\|)$ be a real or complex normed linear space. It is well known that the function $f_p : X \rightarrow \mathbb{R}, f_p(x) = \|x\|^p, p \geq 1$, is convex on X . Assume that $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$ are probability distributions with all q_j nonzero. In [10] we obtained the following refinements of the generalized triangle inequality:

$$\begin{aligned} \max_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \left[\sum_{j=1}^n q_j \|x_j\|^p - \left\| \sum_{j=1}^n q_j x_j \right\|^p \right] &\geq \sum_{j=1}^n p_j \|x_j\|^p - \left\| \sum_{j=1}^n p_j x_j \right\|^p \\ &\geq \min_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \left[\sum_{j=1}^n q_j \|x_j\|^p - \left\| \sum_{j=1}^n q_j x_j \right\|^p \right] (\geq 0) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \max_{1 \leq i \leq n} \{p_i\} \left[\sum_{j=1}^n \|x_j\|^p - n^{1-p} \left\| \sum_{j=1}^n x_j \right\|^p \right] &\geq \sum_{j=1}^n p_j \|x_j\|^p - \left\| \sum_{j=1}^n p_j x_j \right\|^p \\ &\geq \min_{1 \leq i \leq n} \{p_i\} \left[\sum_{j=1}^n \|x_j\|^p - n^{1-p} \left\| \sum_{j=1}^n x_j \right\|^p \right] (\geq 0) \end{aligned} \tag{4.2}$$

for all $p \geq 1$.

We remark that, for $p = 1$ one may obtain from the previous results the following inequalities that are intimately related with the generalized triangle inequality in normed spaces:

$$\begin{aligned} \max_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \left[\sum_{j=1}^n q_j \|x_j\| - \left\| \sum_{j=1}^n q_j x_j \right\| \right] &\geq \sum_{j=1}^n p_j \|x_j\| - \left\| \sum_{j=1}^n p_j x_j \right\| \\ &\geq \min_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \left[\sum_{j=1}^n q_j \|x_j\| - \left\| \sum_{j=1}^n q_j x_j \right\| \right] (\geq 0), \end{aligned} \tag{4.3}$$

$$\begin{aligned} \max_{1 \leq i \leq n} \{p_i\} \left[\sum_{j=1}^n \|x_j\| - \left\| \sum_{j=1}^n x_j \right\| \right] &\geq \sum_{j=1}^n p_j \|x_j\| - \left\| \sum_{j=1}^n p_j x_j \right\| \\ &\geq \min_{1 \leq i \leq n} \{p_i\} \left[\sum_{j=1}^n \|x_j\| - \left\| \sum_{j=1}^n x_j \right\| \right] (\geq 0). \end{aligned} \tag{4.4}$$

If in (4.4) we take

$$p_j := \frac{1/\|x_j\|}{\sum_{k=1}^n (1/\|x_k\|)}, \quad \text{with } x_j \neq 0 \forall j \in \{1, \dots, n\},$$

then, by rearranging the inequality, we get the result:

$$\begin{aligned} \max_{1 \leq j \leq n} \{\|x_j\|\} \left[n - \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| \right] &\geq \sum_{j=1}^n \|x_j\| - \left\| \sum_{j=1}^n x_j \right\| \\ &\geq \min_{1 \leq j \leq n} \{\|x_j\|\} \left[n - \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| \right]. \end{aligned} \quad (4.5)$$

We note that inequality (4.5) has been obtained in a different way by Kato *et al.* in [16] where an analysis of the equality case for strictly convex spaces has been performed as well.

We can state the following result which provides a generalization of inequality (4.1).

PROPOSITION 4.1. *Let $(X, \|\cdot\|)$ be a normed linear space, $x = (x_1, \dots, x_n)$ an n -tuple of vectors in X , and $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$ probability distributions with all q_j nonzero. If $t \geq 1$ and $\Phi : [0, \infty) \rightarrow [0, \infty)$ is monotonic nondecreasing and concave, then*

$$\begin{aligned} \max_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \Phi \left[\sum_{i=1}^n q_i \|x_i\|^t - \left\| \sum_{i=1}^n q_i x_i \right\|^t \right] \\ \geq \Phi \left[\sum_{i=1}^n p_i \|x_i\|^t - \left\| \sum_{i=1}^n p_i x_i \right\|^t \right] \\ \geq \min_{1 \leq i \leq n} \left\{ \frac{p_i}{q_i} \right\} \Phi \left[\sum_{i=1}^n q_i \|x_i\|^t - \left\| \sum_{i=1}^n q_i x_i \right\|^t \right] \end{aligned} \quad (4.6)$$

and, in particular,

$$\begin{aligned} n \max_{1 \leq i \leq n} \{p_i\} \Phi \left[n^{-1} \sum_{i=1}^n \|x_i\|^t - n^{-t} \left\| \sum_{i=1}^n x_i \right\|^t \right] \\ \geq \Phi \left[\sum_{i=1}^n p_i \|x_i\|^t - \left\| \sum_{i=1}^n q_i x_i \right\|^t \right] \\ \geq n \min_{1 \leq i \leq n} \{p_i\} \Phi \left[n^{-1} \sum_{i=1}^n \|x_i\|^t - n^{-t} \left\| \sum_{i=1}^n x_i \right\|^t \right]. \end{aligned} \quad (4.7)$$

The proof follows from Corollary 2.2 and the details are omitted.

Now, if $p = (p_1, \dots, p_n)$ are positive weights with $P_n = \sum_{i=1}^n p_i > 0$ and $x = (x_1, \dots, x_n)$ is an n -tuple of vectors in X , then by defining the functional

$$D_n(t, \|\cdot\|, p, x; \Phi) = P_n \Phi \left[P_n^{-1} \sum_{i=1}^n p_i \|x_i\|^t - P_n^{-t} \left\| \sum_{i=1}^n p_i x_i \right\|^t \right] \quad (4.8)$$

we can state the following result as well.

PROPOSITION 4.2. *If $t \geq 1$ and $\Phi : [0, \infty) \rightarrow [0, \infty)$ is monotonic nondecreasing and concave, then*

$$D_n(t, \|\cdot\|, p, x; \Phi) \geq \max_{1 \leq i < j \leq n} \left\{ (p_i + p_j) \Phi \left[\frac{p_i \|x_i\|^t + p_j \|x_j\|^t}{p_i + p_j} - \left\| \frac{p_i x_i + p_j x_j}{p_i + p_j} \right\|^t \right] \right\} \geq 0 \quad (4.9)$$

and

$$D_n(t, \|\cdot\|, p, x; \Phi) \geq D_{n-1}(t, \|\cdot\|, p, x; \Phi) \geq \cdots \geq D_2(t, \|\cdot\|, p, x; \Phi) \geq 0. \quad (4.10)$$

The proof follows from Corollary 3.2 and the details are omitted.

5. Applications for f -divergences

Given a convex function $f : [0, \infty) \rightarrow \mathbb{R}$, the f -divergence functional

$$I_f(p, q) = \sum_{i=1}^n q_i f\left(\frac{p_i}{q_i}\right) \quad (5.1)$$

was introduced by Csiszár [3] as a generalized measure of information, a ‘distance function’ on the set of probability distributions \mathbb{P}^n . The restriction here to discrete distributions is only for convenience; similar results hold for general distributions. As in Csiszár [3], we interpret undefined expressions as follows:

$$f(0) = \lim_{t \rightarrow 0^+} f(t), \quad 0 f\left(\frac{0}{0}\right) = 0, \\ 0 f\left(\frac{a}{0}\right) = \lim_{\varepsilon \rightarrow 0^+} \varepsilon f\left(\frac{a}{\varepsilon}\right) = a \lim_{t \rightarrow \infty} \frac{f(t)}{t}, \quad a > 0.$$

The following results were essentially given by Csiszár and Körner [4].

PROPOSITION 5.1 (Joint convexity). *If $f : [0, \infty) \rightarrow \mathbb{R}$ is convex, then $I_f(p, q)$ is jointly convex in p and q .*

PROPOSITION 5.2 (Jensen’s inequality). *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be convex. Then for any $p, q \in [0, \infty)^n$ with $P_n := \sum_{i=1}^n p_i > 0$, $Q_n := \sum_{i=1}^n q_i > 0$, we have the inequality*

$$I_f(p, q) \geq Q_n f\left(\frac{P_n}{Q_n}\right). \quad (5.2)$$

If f is strictly convex, equality holds in (5.2) if and only if

$$\frac{p_1}{q_1} = \frac{p_2}{q_2} = \cdots = \frac{p_n}{q_n}.$$

It is natural to consider the following corollary.

COROLLARY 5.3 (Nonnegativity). *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be convex and normalized, that is,*

$$f(1) = 0. \quad (5.3)$$

Then for any $p, q \in [0, \infty)^n$ with $P_n = Q_n$, we have the inequality

$$I_f(p, q) \geq 0. \quad (5.4)$$

If f is strictly convex, equality holds in (5.4) if and only if

$$p_i = q_i \text{ for all } i \in \{1, \dots, n\}.$$

In particular, if p, q are probability vectors, then Corollary 5.3 shows, for strictly convex and normalized $f : [0, \infty) \rightarrow \mathbb{R}$, that

$$I_f(p, q) \geq 0 \text{ and } I_f(p, q) = 0 \text{ if and only if } p = q. \quad (5.5)$$

We now give some examples of divergence measures in information theory which are particular cases of f -divergences.

Kullback–Leibler distance [18]. The *Kullback–Leibler distance* $D(\cdot, \cdot)$ is defined by

$$D(p, q) := \sum_{i=1}^n p_i \log \left(\frac{p_i}{q_i} \right).$$

If we choose $f(t) = t \ln t$, $t > 0$, then obviously

$$I_f(p, q) = D(p, q).$$

Variational distance (l_1 -distance). The *variational distance* $V(\cdot, \cdot)$ is defined by

$$V(p, q) := \sum_{i=1}^n |p_i - q_i|.$$

If we choose $f(t) = |t - 1|$, $t \in [0, \infty)$, then

$$I_f(p, q) = V(p, q).$$

Hellinger discrimination [2]. The *Hellinger discrimination* is defined by $\sqrt{2h^2(\cdot, \cdot)}$, where $h^2(\cdot, \cdot)$ is given by

$$h^2(p, q) := \frac{1}{2} \sum_{i=1}^n (\sqrt{p_i} - \sqrt{q_i})^2.$$

It is obvious that if $f(t) = \frac{1}{2}(\sqrt{t} - 1)^2$, then

$$I_f(p, q) = h^2(p, q).$$

Triangular discrimination [22]. We define *triangular discrimination* between p and q by

$$\Delta(p, q) = \sum_{i=1}^n \frac{|p_i - q_i|^2}{p_i + q_i}.$$

It is obvious that if $f(t) = (t - 1)^2/(t + 1)$, $t \in (0, \infty)$, then

$$I_f(p, q) = \Delta(p, q).$$

Note that $\sqrt{\Delta(p, q)}$ is known in the literature as the Le Cam distance.

Chi-square distance. We define the *chi-square* (χ^2) *distance* by

$$D_{\chi^2}(p, q) := \sum_{i=1}^n \frac{(p_i - q_i)^2}{q_i}.$$

It is clear that if $f(t) = (t - 1)^2$, $t \in [0, \infty)$, then

$$I_f(p, q) = D_{\chi^2}(p, q).$$

Rényi's divergences [21]. For $\alpha \in \mathbb{R} \setminus \{0, 1\}$, consider

$$\rho_\alpha(p, q) := \sum_{i=1}^n p_i^\alpha q_i^{1-\alpha}.$$

It is obvious that if $f(t) = t^\alpha$ ($t \in (0, \infty)$), then

$$I_f(p, q) = \rho_\alpha(p, q).$$

Rényi's divergences $R_\alpha(p, q) := 1/(\alpha(\alpha - 1))\ln[\rho_\alpha(p, q)]$ have been introduced for all real orders $\alpha \neq 0$, $\alpha \neq 1$ (and continuously extended for $\alpha = 0$ and $\alpha = 1$) in [19], where the reader may find many inequalities valid for these divergences, without, as well as with, restrictions for p and q .

For other examples of divergence measures, see the paper [15] and the books [19, 23], where further references are given.

For a function $f : (0, \infty) \rightarrow \mathbb{R}$ we denote by $f^\#$ the function defined on $(0, \infty)$ by the equation $f^\#(x) := f(1/x)$. With this notation,

$$I_{f^\#}(p, q) = \sum_{i=1}^n q_i f^\# \left(\frac{p_i}{q_i} \right) = \sum_{i=1}^n q_i f \left(\frac{q_i}{p_i} \right). \tag{5.6}$$

Using Corollary 2.2, we can state the following result for f -divergences.

PROPOSITION 5.4. *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be convex and normalized and p, q two probability distributions such that $R := \max_{i \in \{1, \dots, n\}} \{p_i/q_i\} < \infty$ and $r := \min_{i \in \{1, \dots, n\}} \{p_i/q_i\} > 0$. If $\Phi : [0, \infty) \rightarrow [0, \infty)$ is monotonic nondecreasing and concave, then*

$$\begin{aligned} R\Phi[I_{f^\#}(p, q) - f(D_{\chi^2}(q, p) + 1)] &\geq \Phi[I_f(q, p)] \\ &\geq r\Phi[I_{f^\#}(p, q) - f(D_{\chi^2}(q, p) + 1)]. \end{aligned} \tag{5.7}$$

PROOF. Utilizing the inequality (2.5),

$$\begin{aligned} R\Phi \left[\sum_{i=1}^n q_i f\left(\frac{q_i}{p_i}\right) - f\left(\sum_{i=1}^n \frac{q_i^2}{p_i}\right) \right] &\geq \Phi \left[\sum_{i=1}^n p_i f\left(\frac{q_i}{p_i}\right) - f(1) \right] \\ &\geq r\Phi \left[\sum_{i=1}^n q_i f\left(\frac{q_i}{p_i}\right) - f\left(\sum_{i=1}^n \frac{q_i^2}{p_i}\right) \right]. \end{aligned} \quad (5.8)$$

Since

$$\sum_{i=1}^n \frac{q_i^2}{p_i} = D_{\chi^2}(q, p) + 1,$$

then by (5.8) we deduce the desired result (5.7). \square

Finally, by means of Corollary 3.2 we also obtain the following lower bound for the f -divergence.

PROPOSITION 5.5. *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be convex and normalized and p, q two probability distributions. If $\Phi : [0, \infty) \rightarrow [0, \infty)$ is monotonic nondecreasing and concave, then:*

$$\Phi[I_f(q, p)] \geq \max_{1 \leq i < j \leq n} \left\{ (p_i + p_j) \Phi \left[\frac{p_i f\left(\frac{q_i}{p_i}\right) + p_j f\left(\frac{q_j}{p_j}\right)}{p_i + p_j} - f\left(\frac{q_i + q_j}{p_i + p_j}\right) \right] \right\} \geq 0. \quad (5.9)$$

REMARK 5.6. If one chooses different examples of convex functions generating the particular divergences mentioned at the beginning of the section, that one can obtain various inequalities of interest. However, the details are not presented here.

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