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ORTHOGONAL RECURRENCE POLYNOMIALS AND HAMBURGER MOMENTS

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1. **Introduction and Summary.** A three-term recurrence

$$(1)\qquad \begin{gathered} P_{-1}(x) \equiv 0, \qquad P_0(x) \equiv 1, \\ P_{n+1}(x) = (A_n x + B_n)P_n(x) - C_n P_{n-1}(x), \quad n \geq 0, \end{gathered}$$

where A_n $(n \geq 0)$, B_n $(n \geq 0)$ and C_n $(n \geq 0)$ are real numbers for which $A_n C_{n+1} \neq 0$ $(n \geq 0)$, generates a sequence $\{P_n\}$ of real polynomials in which P_n has degree exactly n. Some (but not all) sequences so generated consist of orthogonal polynomials associated with a distribution $d\psi(x)$ over some interval $[a, b]$; that is, they are polynomials for which there exists an integrator $\psi(x)$ such that

$$(2)\qquad \int_a^b P_i(x)P_j(x)\, d\psi(x) = 0, \quad i \neq j,$$

where $\psi(x)$ is bounded, is nondecreasing and assumes infinitely many different values over $[a, b]$(1).

It is first shown below that, for recurrence polynomials P_n, the orthogonality conditions (2) are simply

$$(3)\qquad \begin{gathered} \int_a^b x^{n-1} P_n(x)\, d\psi(x) = 0, \quad n \geq 1 \\ \text{and} \\ \int_a^b x^{n-2} P_n(x)\, d\psi(x) = 0, \quad n \geq 2, \end{gathered}$$

although (2) and (3) are generally not equivalent for a polynomial family $\{P_n\}$ which does *not* satisfy some recurrence (1).

Now, let $\{P_n\}$ be any family of real polynomials in which P_n has degree n. Then a corresponding sequence $\{\mu_n\}$ of *quasi-moments* can be constructed successively (with relations (3) as the guide) as follows: for

$$P_n(x) = \sum_{j=0}^{n} a_{nj} x^j,$$

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(1) A necessary and sufficient condition for such orthogonality is [2]: $C_n / A_n A_{n-1} > 0$ for $n \geq 1$.

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let

(4)
$$\mu_0 = 1,$$
$$\mu_{2n-1} = -\frac{1}{a_{nn}} \sum_{j=0}^{n-1} a_{nj}\mu_{n+j-1} \quad (n \geq 1)$$
and
$$\mu_{2n-2} = -\frac{1}{a_{nn}} \sum_{j=0}^{n-1} a_{nj}\mu_{n+j-2} \quad (n \geq 2).$$

Thus, the Hamburger Moment Problem associated with the sequence $\{P_n\}$ is the problem of determining when the corresponding quasi-moments are actually moments of some distribution $d\psi(x)$ over some interval $[a, b]$(2). It is shown here that recurrence polynomials P_n are orthogonal if, and only if, the corresponding quasi-moments are moments.

2. **Equivalent Form of Orthogonality.** It is generally not true, for an arbitrary sequence $\{P_n\}$ of polynomials, that (2) and (3) are equivalent. For example, the polynomials

(5)
$$P_0(x) = 1, \qquad P_1(x) = x - \tfrac{1}{2},$$
$$P_n(x) = x^n - x^{n-1} + \frac{1}{4n-2} \quad (n \geq 2)$$

satisfy
$$\int_0^1 x^{n-1} P_n(x)\, dx = 0 \quad (\mathrm{n} \geq 1)$$
and
$$\int_0^1 x^{n-2} P_n(x)\, dx = 0 \quad (n \geq 2),$$
but
$$\int_0^1 P_0(x) P_n(x)\, dx \neq 0 \quad (n > 2).$$

As a matter of fact, it is easy to verify directly that the polynomials (5) do not satisfy any recurrence of the form (1). Consequently [3] they cannot be orthogonal polynomials associated with any distribution over any interval.

The following lemma shows the equivalence of conditions (2) and (3) for a recurrence family $\{P_n\}$. Thus, *any* family of polynomials P_n with properties (3) will be orthogonal when, and only when, the P_n satisfy a recurrence (1).

LEMMA. *Let* $\{P_n\}$ *be a sequence of polynomials generated by a recurrence* (1). *Then* (2) *and* (3) *are equivalent.*

Proof. For convenience of notation, let
$$\langle f(x), g(x)\rangle \equiv \int_a^b f(x) g(x)\, d\psi(x).$$

(2) That is, $\mu_n = \int_a^b x^n\, d\psi(x) (n \geq 0)$ for some (normalized) distribution $d\psi(x)$ over some interval $[a, b]$.

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Any x^j is a linear combination of the $P_i(x)$ for $0 \le i \le j$; hence, (2) implies (3). Suppose, then, that $\{P_n\}$ is a family of polynomials, given by a recurrence (1), for which conditions (3) hold. Corresponding to an integer $k \ge 1$, let T_k denote the statement:

$$\text{for each } m = 1, 2, 3, \ldots, k, \qquad \langle P_n(x), x^{n-m} \rangle = 0 \quad \text{for all } n \ge m. \tag{6}$$

The remainder of the proof follows easily once (6) is established (by induction). Conditions (3) surely give T_2 and, thus, T_1. To show that T_k implies T_{k+1}, it will be sufficient to conclude that $\langle P_n(x), x^{n-(k+1)} \rangle = 0$ for all $n \ge (k+1)$. Pick any integer $n \ge (k+1)$; multiplication throughout the recurrence (1) by $x^{n-(k+1)}$, followed by an integration, yields:

$$\begin{aligned} \langle P_n(x), x^{n-(k+1)} \rangle = {} & A_{n-1}\langle P_{n-1}(x), x^{n-k} \rangle + B_{n-1}\langle P_{n-1}(x), x^{n-(k+1)} \rangle \\ & - C_{n-1}\langle P_{n-2}(x), x^{n-(k+1)} \rangle. \end{aligned} \tag{7}$$

Now, $\langle P_{n-1}(x), x^{n-k} \rangle = \langle P_{n-1}(x), x^{(n-1)-m} \rangle$ where $m = k-1$; this vanishes (by the induction hypothesis, since $m < k$) whenever $(n-1) \ge m$. That is, $\langle P_{n-1}(x), x^{n-k} \rangle = 0$ for all $n \ge k$; but this implies that the coefficient of A_{n-1} in (7) is zero for any $n \ge (k+1)$. A similar argument shows that the coefficients of B_{n-1} and C_{n-1} in (7) also are zero, which completes the induction. For the remainder of the proof, let j be any positive integer; the statement T_j shows in particular that $\langle P_j(x), x^{j-m} \rangle = 0$ for $m = 1, 2, 3, \ldots, j$—whence $\langle P_j(x), P_i(x) \rangle = 0$ whenever $i < j$.

3. **Moments for Recurrence Polynomials.** Let $\{P_n\}$ be any family of real polynomials in which P_n has degree exactly n. The corresponding quasi-moments $\{\mu_n\}$ (as prescribed in (4)) might, in fact, be *moments* even though the polynomials are not orthogonal. For example, the moments

$$\mu_n = \int_0^1 x^n \, dx = \frac{1}{n+1}, \quad n \ge 0,$$

are the quasi-moments corresponding to the nonorthogonal family (5). It is shown below that this situation cannot occur when the P_n are recurrence polynomials.

THEOREM. *Let $\{P_n\}$ be a sequence of polynomials generated by a recurrence* (1), *and let $\{\mu_n\}$ be the corresponding sequence of quasi-moments. Then the μ_n are moments, if, and only if, the P_n are orthogonal.*

Proof. Suppose, first, that the P_n are orthogonal polynomials associated with a distribution $d\psi(x)$ over an interval $[a, b]$. Let $\{\nu_n\}$ $(\nu_0 = 1)$ be the sequence of moments of $d\psi(x)$ over $[a, b]$. Surely $\{\nu_n\}$, as well as $\{\mu_n\}$, satisfies (4); then $\{\mu_n - \nu_n\}$ satisfies a difference scheme of the form (4), but with initial condition zero. Hence $\mu_n = \nu_n$ for all n—that is, the μ_n are moments of $d\psi(x)$ over $[a, b]$.

For the converse, suppose the μ_n are moments of some distribution $d\psi(x)$ over some interval $[a, b]$. The relations (4) are precisely relations (3); hence, by the preceding lemma, the P_n are orthogonal polynomials associated with $d\psi(x)$ over $[a, b]$.

References

1. J. Favard, *Sur les polynomes de Tchebicheff*, Comptes Rendus, Acad. Sci. Paris, **200** (1935), 2052–2053.

2. A. G. Law, *Solutions of some countable systems of ordinary differential equations*, Doctoral Dissertation, Georgia Inst. Tech., 1968.

3. G. Szegö, *Orthogonal polynomials*, 3rd ed., Colloq. Publ., Vol. XXIII, Amer. Math. Soc., Providence, R.I., 1967.

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