BULL. AUSTRAL. MATH. SOC. 47A65; 46L30, 47A10, 47B48
VOL. 68 (2003) [169–173]

SPECTRALLY BOUNDED TRACES ON C^*-ALGEBRAS

MARTIN MATHIEU

A linear mapping T from a subspace E of a Banach algebra into another Banach algebra is called spectrally bounded if there is a constant $M \geqslant 0$ such that $r(Tx) \leqslant Mr(x)$ for all $x \in E$, where $r(\cdot)$ denotes the spectral radius. We establish the equivalence of the following properties of a unital linear mapping T from a unital C^*-algebra A into its centre:

(a) T is spectrally bounded;
(b) T is a spectrally bounded trace;
(c) T is a bounded trace.

1. INTRODUCTION AND MAIN RESULT

The results in this paper were motivated by the following two recent theorems. For an element x in a Banach algebra A, we shall denote by $\sigma(x)$ its spectrum and by $r(x)$ its spectral radius.

THEOREM 1.1. ([1]) *Let $T: A \to B$ be a surjective spectrum-preserving linear mapping between von Neumann algebras A and B, that is, $\sigma(Tx) = \sigma(x)$ for all $x \in A$. Then T is a Jordan isomorphism.*

THEOREM 1.2. ([11]) *Let $T: A \to B$ be a unital surjective spectrally bounded linear mapping from a properly infinite von Neumann algebra A onto a unital semisimple Banach algebra B. Then T is a Jordan epimorphism.*

It is fairly easy to verify that a surjective spectrum-preserving linear mapping T is injective and unital, that is, $T1 = 1$. Therefore, the main conclusion in Theorem 1.1 is that T is a Jordan homomorphism, that is, $T(x^2) = (Tx)^2$ for all $x \in A$. This remarkable result confirmed a conjecture by Kaplansky, which had been open for about 30 years, in the context of von Neumann algebras. (In fact, it is easy to see that B merely needs to be a unital semisimple Banach algebra.)

A linear mapping $T: E \to B$ defined on a subspace E of a Banach algebra A into a Banach algebra B is called *spectrally bounded* if there is a constant $M \geqslant 0$ such

Received 18th February, 2003
This paper was written during a visit to the Mathematics Departments at the University of Oregon, Eugene and Pomona College, Claremont. The author gratefully acknowledges the hospitality extended to him by his colleagues at both institutions. The paper is part of the research carried out in the EC network *Analysis and Operators* (HPRN-CT-2000-00116).

that $r(Tx) \leqslant Mr(x)$ for all $x \in E$. In this case, there is a smallest such constant, which is called the *spectral operator norm* of T and denoted by $\|T\|_\sigma$. A number of basic properties of spectrally bounded operators are discussed in [**10**]. For instance, every spectrally bounded linear functional is bounded and therefore every spectrally bounded operator into a commutative C^*-algebra is bounded. In general, spectrally bounded operators need not be continuous, turning them into a useful tool in studying derivations on Banach algebras, see [**3**] and [**7**]. On the other hand, they also play an important role in automatic continuity theory, see for example, [**2**].

Lately, the attention on spectrally bounded operators turned to their structure theory ([**4, 5, 9, 11, 13**]). Every Jordan epimorphism, that is, surjective Jordan homomorphism, preserves invertibility and hence, is spectrally bounded. Clearly, the hypothesis to preserve the spectrum is much stronger than spectral boundedness. Since commutative von Neumann algebras are finite, and in the commutative case *every* bounded operator is spectrally bounded, it is impossible to obtain an analogue of Theorem 1.2 for finite von Neumann algebras. Thus, it is the best generalisation of Theorem 1.1 that can be obtained in this setting.

Finite von Neumann algebras are characterised by the existence of a canonical centre-valued trace τ. It is known that τ is spectrally bounded with $\|\tau\|_\sigma = 1$, but it is no Jordan homomorphism. If A is a finite-dimensional factor, that is, $A = M_n(\mathbb{C})$ for some $n \in \mathbb{N}$, every surjective spectrally bounded operator on A is a linear combination of a Jordan isomorphism and the normalised trace, see [**13**]. Trivially, a decomposition into a sum of a Jordan homomorphism plus a centre-valued trace also holds for every commutative von Neumann algebra. It thus appears that traces are the essential obstruction to a direct extension of Theorem 1.2 to general von Neumann algebras. The main result of this paper, stated below, indicates that this may be the case.

THEOREM 1.3. *Let $T: A \to Z(A)$ be a unital linear mapping on a unital C^*-algebra A into the centre $Z(A)$ of A. Then the following conditions on T are equivalent.*

(a) *T is spectrally bounded;*
(b) *T is a spectrally bounded trace;*
(c) *T is a bounded trace.*

Here, and in the sequel, we call a linear mapping T a *trace* if $T(xy) = T(yx)$ for all x, y in the domain. Note that, in this case, the mapping T^* defined by $T^*(x) = (Tx^*)^*$ is another trace, and we say that T is *self-adjoint* if $T = T^*$. This clearly amounts to the requirement that T preserves self-adjoint elements. We say that a trace T on a unital C^*-algebra A is *normalised* if $T1 = 1$. A *tracial state* on a unital C^*-algebra is a normalised trace functional of norm 1.

2. Proof of the Main Result

In this section we give the proof of Theorem 1.3. We start with two auxiliary lemmas.

Lemma 2.1. *Let $T\colon A \to B$ be a spectrally bounded operator from a Banach algebra A into a commutative C^*-algebra B. Then $T(xy) = T(yx)$ for all $x, y \in A$.*

Proof: Take $x, y \in A$. For each $\lambda \in \mathbb{C}$, let $g(\lambda) = T(e^{\lambda x} y e^{-\lambda x})$. As T is bounded, g is an entire function into B. Since B is commutative, the assumption on T implies that

$$\|T(e^{\lambda x} y e^{-\lambda x})\| = r(T(e^{\lambda x} y e^{-\lambda x})) \leqslant \|T\|_\sigma r(y)$$

and thus g is bounded. By Liouville's theorem it follows that g is constant. Therefore,

$$Ty = T(e^{\lambda x} y e^{-\lambda x}) = Ty + \lambda T(xy - yx) + \text{higher terms} \qquad (\lambda \in \mathbb{C})$$

entailing that $T(xy) = T(yx)$, as claimed. ∎

Lemma 2.2. *Let T be a normalised bounded self-adjoint centre-valued trace on a unital C^*-algebra A. Then T is spectrally bounded with $\|T\|_\sigma = \|T\|$.*

Proof: Suppose first that T is positive; then the proof given for example in [**8**] in the case of a finite von Neumann algebra can easily be adapted to the situation of a general unital C^*-algebra to obtain that $\|T\|_\sigma = 1 = \|T\|$. We therefore have to reduce the assertion to this case.

Let φ be a state on $Z(A)$. Then $\varphi \circ T$ is a normalised bounded self-adjoint trace functional on A. By [**6**, Proposition 2.8] there exist positive trace functionals $(\varphi \circ T)^+$ and $(\varphi \circ T)^-$ on A such that $\varphi \circ T = (\varphi \circ T)^+ - (\varphi \circ T)^-$ and $\|\varphi \circ T\| = \|(\varphi \circ T)^+\| + \|(\varphi \circ T)^-\|$. Evaluating at 1, it follows that $(\varphi \circ T)^+ \neq 0$. Since a positive functional attains its norm at 1, we can thus write $\varphi \circ T = \alpha\varphi_1 - \beta\varphi_2$ for two tracial states φ_i and $\alpha = \|(\varphi \circ T)^+\| > 0$, $\beta = \|(\varphi \circ T)^-\|$. By the first paragraph of the proof, we conclude that

$$|(\varphi \circ T)(x)| \leqslant (\alpha + \beta) r(x) = \|\varphi \circ T\| r(x) \qquad (x \in A).$$

Consequently, T is spectrally bounded with $\|T\|_\sigma \leqslant \|T\|$ and the reverse estimate follows from [**10**, Proposition 2.9]. ∎

We are now in a position to give the proof of the main result.

Proof of Theorem 1.3: The implication (a) $\Rightarrow$ (b) is immediate from Lemma 2.1, and the implication (b) $\Rightarrow$ (c) follows from the fact that norm and spectral radius coincide in a commutative C^*-algebra.

In order to show that (c) $\Rightarrow$ (a) we at first observe that, whenever $T_1, T_2: A \to Z(A)$ are spectrally bounded, every linear combination $\alpha T_1 + \beta T_2$ is spectrally bounded (with $\|\alpha T_1 + \beta T_2\|_\sigma \leqslant |\alpha|\|T_1\|_\sigma + |\beta|\|T_2\|_\sigma$). Let $\mathrm{ReT} = (1/2)(T + T^*)$ and $\mathrm{ImT} = (1/2i)(T - T^*)$ be the real and the imaginary part of T, respectively. Since T is a normalised bounded trace, it follows that both ReT and ImT are bounded traces and that $\mathrm{ReT}\,(1) = 1$ whereas $\mathrm{ImT}\,(1) = 0$. Consequently, $\mathrm{ReT} + \mathrm{ImT}$ is a normalised trace as well. By Lemma 2.2, ReT and $\mathrm{ReT} + \mathrm{ImT}$ are both spectrally bounded. Therefore, $\mathrm{ImT} = \mathrm{ReT} + \mathrm{ImT} - \mathrm{ReT}$ is spectrally bounded. As $T = \mathrm{ReT} + i\,\mathrm{ImT}$ we finally conclude that T itself is spectrally bounded (with $\|T\|_\sigma \leqslant 4\|T\|$). This completes the proof. ∎

Examples of unital C^*-algebras without tracial states are properly infinite C^*-algebras (that is, there exist no pair of orthogonal projections which are both equivalent to the identity). A C^*-algebra A has no tracial states if and only if its universal enveloping von Neumann algebra A'' is properly infinite. Putting a recent result by Pop together with Theorem 1.3 we obtain the following consequence.

Corollary 2.3. *Let A be a unital C^*-algebra without tracial states. Then there is no non-zero spectrally bounded operator from A into a commutative C^*-algebra.*

Proof: Let $T: A \to B$ be a spectrally bounded operator from A into the commutative C^*-algebra B. By Theorem 1.3(b) (which does not need the assumption $T1 = 1$), T vanishes on all commutators. By the main result in [**12**], every element in A can be written as a finite sum of commutators. Therefore, $T = 0$. ∎

The *spectral dual* E^σ of a subspace E of a Banach algebra A is the Banach space of all spectrally bounded linear functionals on E equipped with the spectral operator norm, see [**10**]. From Corollary 2.3 we readily deduce that $A^\sigma = \{0\}$ for every unital C^*-algebra A without tracial states. This extends Corollary 3.9 in [**11**], which contains the same conclusion for properly infinite von Neumann algebras. On the other hand, suppose that A has a unique tracial state φ. By [**6**], the linear span of all commutators is dense in the kernel of φ. By the above, every spectrally bounded linear functional has to vanish on $\ker\varphi$. As a result, $A^\sigma = \mathbb{C}$.

In [**9**] we obtain Theorem 1.2 under the assumption that A is a purely infinite simple unital C^*-algebra. However, no description of spectrally bounded operators is as yet available on any infinite-dimensional C^*-algebra which allows a non-trivial trace, such as a finite simple unital C^*-algebra and in particular not for any II_1 factor.

References

[1] B. Aupetit, 'Spectrum-preserving linear mappings between Banach algebras or Jordan-Banach algebras', *J. London Math. Soc.* **62** (2000), 917–924.

[2] B. Aupetit and M. Mathieu, 'The continuity of Lie homomorphisms', *Studia Math.* **138** (2000), 193–199.

[3] M. Brešar and M. Mathieu, 'Derivations mapping into the radical, III', *J. Funct. Anal.* **133** (1995), 21–29.

[4] M. Brešar and P. Šemrl, 'Linear maps preserving the spectral radius', *J. Funct. Anal.* **142** (1996), 360–368.

[5] J. Cui and J. Hou, 'The spectrally bounded linear maps on operator algebras', *Studia Math.* **150** (2002), 261–271.

[6] J. Cuntz and G.K. Pedersen, 'Equivalence and traces on C^*-algebras', *J. Funct. Anal.* **33** (1979), 135–164.

[7] R.E. Curto and M. Mathieu, 'Spectrally bounded generalized inner derivations', *Proc. Amer. Math. Soc.* **123** (1995), 2431–2434.

[8] J. Dixmier, *Von Neumann algebras* (North-Holland, Amsterdam, 1969).

[9] M. Mathieu, 'Spectrally bounded operators on simple C^*-algebras', *Proc. Amer. Math. Soc.* (to appear).

[10] M. Mathieu and G.J. Schick, 'First results on spectrally bounded operators', *Studia Math.* **152** (2002), 187–199.

[11] M. Mathieu and G.J. Schick, 'Spectrally bounded operators from von Neumann algebras', *J. Operator Theory* **49** (2003) (to appear).

[12] C. Pop, 'Finite sums of commutators', *Proc. Amer. Math. Soc.* **130** (2002), 3039–3041.

[13] P. Šemrl, 'Spectrally bounded linear maps on $B(H)$', *Quart. J. Math. Oxford* **49** (1998), 87–92.

Department of Pure Mathematics
Queen's University Belfast
Belfast BT7 1NN
Northern Ireland
e-mail: m.m@qub.ac.uk