

ARTICLE

Spread-out limit of the critical points for lattice trees and lattice animals in dimensions $d > 8$

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Abstract

A spread-out lattice animal is a finite connected set of edges in $\{\{x, y\} \subset \mathbb{Z}^d : 0 < \|x - y\| \leq L\}$. A lattice tree is a lattice animal with no loops. The best estimate on the critical point p_c so far was achieved by Penrose (*J. Stat. Phys.* 77, 3–15, 1994) : $p_c = 1/e + O(L^{-2d/7} \log L)$ for both models for all $d \geq 1$. In this paper, we show that $p_c = 1/e + CL^{-d} + O(L^{-d-1})$ for all $d > 8$, where the model-dependent constant C has the random-walk representation

$$C_{\text{LT}} = \sum_{n=2}^{\infty} \frac{n+1}{2e} U^{*n}(o), \quad C_{\text{LA}} = C_{\text{LT}} - \frac{1}{2e^2} \sum_{n=3}^{\infty} U^{*n}(o),$$

where U^{*n} is the n -fold convolution of the uniform distribution on the d -dimensional ball $\{x \in \mathbb{R}^d : \|x\| \leq 1\}$. The proof is based on a novel use of the lace expansion for the 2-point function and detailed analysis of the 1-point function at a certain value of p that is designed to make the analysis extremely simple.

Keywords: Lattice trees; lattice animals; critical phenomena

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1. Introduction and the main result

Given an $L \in \mathbb{N}$, we consider spread-out lattice animals $A = (V_A, E_A)$, where the vertex set V_A is a finite subset of \mathbb{Z}^d and any pair of vertices in V_A are connected by a path of spread-out edges $E_A \subset \{\{x, y\} : 0 < \|x - y\| \leq L\}$; $\|\cdot\|$ is an arbitrary fixed norm on \mathbb{R}^d . A lattice tree is a lattice animal with no loops. Both models are statistical-mechanical models for branched polymers.

To investigate their statistical properties, we consider the following generating functions. Let

$$\Lambda = \{x \in \mathbb{Z}^d : 0 < \|x\| \leq L\}, \quad D(x) = \frac{1}{|\Lambda|} \mathbb{1}_{\{x \in \Lambda\}}, \quad (1.1)$$

where $\mathbb{1}_E$ is the indicator function of E , being 1 or 0 depending on whether or not E is true. The function D will be used as a transition probability of the underlying random walk. Then, we define the weight function for a tree T as

$$W_p(T) = \prod_{\{x, y\} \in E_T} pD(x - y) = \left(\frac{p}{|\Lambda|} \right)^{|E_T|}, \quad (1.2)$$

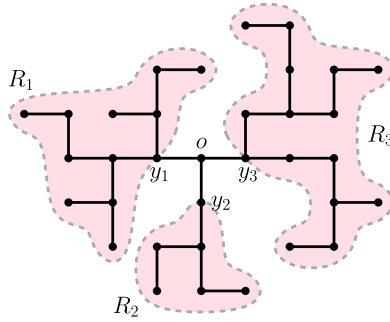


Figure 1. A sample T of \mathcal{T}_o . Removal of all edges $\{o, y_j\} \in E_T$ leaves disjoint subtrees R_j rooted at y_j : $V_T \setminus \{o\} = \bigcup_j V_{R_j}$ and $E_T \setminus \bigcup_j \{o, y_j\} = \bigcup_j E_{R_j}$.

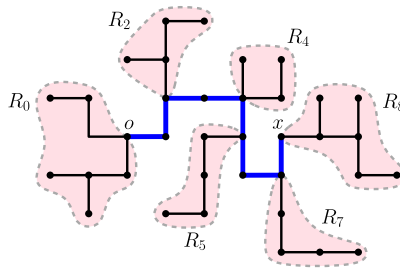


Figure 2. A sample tree in $\mathcal{T}_{o,x}$. Removal of the backbone edges (in blue) yields disjoint subtrees $\{R_j\}$, called ribs. In this example, R_1, R_3 and R_6 are single-vertex trees.

and similarly for a lattice animal A as $W_p(A)$. For a finite set $X \subset \mathbb{Z}^d$, we denote by \mathcal{T}_X (resp., \mathcal{A}_X) the set of lattice trees T with $X \subset V_T$ (resp., lattice animals A with $X \subset V_A$); if X consists of a vertex or two, we simply write, e.g., \mathcal{T}_o (for $X = \{o\}$; see Fig. 1) or $\mathcal{T}_{o,x}$ (for $X = \{o, x\}$; see Fig. 2). The generating functions we want to investigate are the 1-point and 2-point functions, defined respectively as

$$g_p = \sum_{T \in \mathcal{T}_o} W_p(T), \quad \tau_p(x) = \sum_{T \in \mathcal{T}_{o,x}} W_p(T), \quad (1.3)$$

for lattice trees, and similarly defined for lattice animals. The susceptibility χ_p is the sum of the 2-point function, defined as

$$\chi_p = \sum_{x \in \mathbb{Z}^d} \tau_p(x) = \sum_{x \in \mathbb{Z}^d} \sum_{T \in \mathcal{T}_o} \mathbb{1}_{\{x \in V_T\}} W_p(T) = \sum_{T \in \mathcal{T}_o} |V_T| W_p(T), \quad (1.4)$$

for lattice trees, and similarly for lattice animals. It has been known (see [18] and references therein) that there is a model-dependent critical point p_c such that χ_p is finite if and only if $p < p_c$ and diverges as $p \uparrow p_c$. The goal of this paper is to reveal the asymptotics of p_c as $L \uparrow \infty$ for both models.

The best estimate so far on p_c for the spread-out model was achieved by Penrose [23]. He investigated the growth constant, which is defined by the $n \uparrow \infty$ limit of the n th root of the number $t_n = \frac{1}{n} \sum_{T \in \mathcal{T}_o} \mathbb{1}_{\{|V_T|=n\}}$ of n -vertex unrooted lattice trees. Since $\{t_n\}_{n \in \mathbb{N}}$ is a supermultiplicative sequence, i.e., $t_{n+m} \geq t_n t_m$ (see, e.g., [16]), $\lim_{n \uparrow \infty} t_n^{1/n}$ exists and is asymptotically $e|\Lambda| + O(|\Lambda|^{5/7} \log |\Lambda|)$ as $|\Lambda| \uparrow \infty$ [23]. Since $|E_T| = |V_T| - 1$ for each lattice tree, we can

rewrite χ_p as

$$\chi_p \stackrel{(1.4)}{=} \sum_{n=1}^{\infty} n \sum_{T \in \mathcal{T}_o} \mathbb{1}_{\{|V_T|=n\}} W_p(T) = \sum_{n=1}^{\infty} n^2 \left(\frac{p}{|\Lambda|} \right)^{n-1} t_n. \tag{1.5}$$

Therefore, for large $|\Lambda|$,

$$p_c = \lim_{n \uparrow \infty} \left(\frac{n^2}{|\Lambda|^{n-1}} t_n \right)^{-1/n} = \frac{|\Lambda|}{e|\Lambda| + O(|\Lambda|^{5/7} \log |\Lambda|)} = \frac{1}{e} + O(|\Lambda|^{-2/7} \log |\Lambda|), \tag{1.6}$$

which is true for all dimensions $d \geq 1$. Penrose also claimed in [23, Section 3.1] that p_c for lattice animals obeys the same bound, due to the result of Klarner [15].

A weaker estimate, $p_c = 1/e + o(1)$ as $L \uparrow \infty$ for all d bigger than the critical dimension $d_c = 8$, was obtained by Miranda and Slade [20]. In fact, their main concern was to obtain $1/d$ expansions of p_c for the nearest-neighbor models. In [19, 21], they showed that,

$$p_c = \frac{1}{e} + \frac{3}{2e} |\Lambda|^{-1} + \begin{cases} \frac{115}{24e} |\Lambda|^{-2} + o(|\Lambda|^{-2}) & \text{[lattice trees],} \\ \left(\frac{115}{24e} - \frac{1}{2e^2} \right) |\Lambda|^{-2} + o(|\Lambda|^{-2}) & \text{[lattice animals],} \end{cases} \tag{1.7}$$

as $|\Lambda| = 2d \uparrow \infty$. The proof is based on the lace expansion for the 2-point function $\tau_p(x)$ and an expansion for the 1-point function g_p based on inclusion-exclusion. Notice that the model-dependence appears only from the $O(|\Lambda|^{-2})$ term. This is due to unit squares that are in g_p for lattice animals, but not in g_p for lattice trees. The lace expansion has been successful in showing mean-field critical behaviour in high dimensions for various models, including lattice trees and lattice animals for $d > 8$ (e.g., [6, 8, 10]). The other models are self-avoiding walk for $d > 4$ (e.g., [2, 6, 9]), percolation for $d > 6$ (e.g., [6, 7]), oriented percolation and the contact process for the spatial dimension $d > 4$ (e.g., [22, 24]), and the Ising and φ^4 models for $d > 4$ (e.g., [1, 25–27]).

For the nearest-neighbor lattice trees and lattice animals, in particular, Hara and Slade [6, 8, 10] show mean-field behaviour for both models in dimensions higher than an unspecified number bigger than 8. In contrast, Fitzner and van der Hofstad [5] prove that $d > 16$ is enough for lattice trees, and $d > 17$ for lattice animals to exhibit mean-field behaviour. The proof is based on the so-called non-backtracking lace expansion (NoBLE) that is different from the standard lace expansion by Hara and Slade.

In [12], van der Hofstad and the second-named author of the current paper applied the lace expansion to the spread-out models [defined by D in equation (1.1)] of self-avoiding walk, percolation, oriented percolation and the contact process, and showed that, for all d bigger than the respective critical dimension d_c ,

$$p_c = 1 + CL^{-d} + O(L^{-d-1}), \tag{1.8}$$

as $L \uparrow \infty$, where 1 is the mean-field value, and the model-dependent constant C has the following random-walk representation:

$$C = \begin{cases} \sum_{n=2}^{\infty} U^{*n}(o) & \text{[self-avoiding walk, the contact process],} \\ \frac{1}{2} \sum_{n=2}^{\infty} U^{*2n}(o) & \text{[oriented percolation],} \\ U^{*2}(o) + \sum_{n=3}^{\infty} \frac{n+1}{2} U^{*n}(o) & \text{[percolation],} \end{cases} \tag{1.9}$$

where U^{*n} is the n -fold convolution in \mathbb{R}^d of the uniform probability distribution U on $\{x \in \mathbb{R}^d : \|x\| \leq 1\}$. For example, if $\|x\| = \|x\|_\infty := \max_j |x_j|$, then, for all $n \in \mathbb{N}$,

$$U(x) = \frac{\mathbb{1}_{\{\|x\|_\infty \leq 1\}}}{2^d}, \quad U^{*(n+1)}(x) = \int_{\mathbb{R}^d} U^{*n}(y) U(x - y) \, d^d y. \tag{1.10}$$

These quantities are the spread-out limit of the underlying random walk generated by D [12, Section 4]. For example, for $d > 4$,

$$\sum_{n=2}^\infty \frac{n+1}{2} D^{*n}(o) = L^{-d} \sum_{n=2}^\infty \frac{n+1}{2} U^{*n}(o) + O(L^{-d-1}), \tag{1.11}$$

where we have used the same notation $*$ to represent convolutions on \mathbb{Z}^d as well. The error term $O(L^{-d-1})$ is due to Riemann-sum approximation.

We want to achieve a similar result for lattice trees and lattice animals, i.e., a random-walk representation for the difference between p_c and its mean-field value $1/e$, and see how the model-dependence arises in it. In the rest of the paper, we will show the following:

Theorem 1.1. *For both lattice trees and lattice animals with $d > 8$ and $L \uparrow \infty$,*

$$p_c = \frac{1}{e} + CL^{-d} + O(L^{-d-1}), \tag{1.12}$$

where the model-dependent constant C has the following random-walk representation:

$$C_{LT} = \sum_{n=2}^\infty \frac{n+1}{2e} U^{*n}(o), \quad C_{LA} = C_{LT} - \frac{1}{2e^2} \sum_{n=3}^\infty U^{*n}(o), \tag{1.13}$$

The difference in p_c already shows up in the first error term of order L^{-d} for the spread-out models, while it appears in (1.7) from the second error term of order d^{-2} for the nearest-neighbor models, as mentioned earlier. This is due to closed loops of length bigger than 2 in g_p for lattice animals. The smallest among such loops for the spread-out model is of length 3 and of order L^{-d} , while that for the nearest-neighbor model is of length 4 and of order d^{-2} (see Lemma 4.1 below). Identifying coefficients of the higher-order terms for the spread-out models may need more work since they are absorbed in the error term $O(L^{-d-1})$ in (1.12), which is inherent in Riemann-sum approximation, just as mentioned below (1.11).

The proof of the above theorem is based on the lace expansion for the 2-point function and detailed analysis of the 1-point function, similarly to the previous work by Miranda and Slade [21]. The key to our analysis is to introduce a new base point p_1 defined in (2.1) below, as $p_1 g_{p_1} = 1$. It is to estimate various generating functions in terms of massless random walks. For the spread-out models of self-avoiding walk, percolation, oriented percolation and the contact process, van der Hofstad and Sakai [12] use the base point $p_1 = 1$, because of the unity of the 1-point function for those models. Since the analysis in terms of the underlying random walks is very simple, we do not have to know in detail the lace expansion; the exception is in Lemma 2.1 below, where we investigate the first lace-expansion coefficient $\hat{\pi}_p^{(1)}$ to prove $p_c - p_1 = O(L^{-2d})$. However, the basic facts (summarized in Proposition 2.2 below) and a *minimum* definition about the lace-expansion coefficients should be enough to read the proof, which we hope makes this paper more accessible to wider audience.

Our method can be applied to the nearest-neighbor models as well to identify the coefficient of $(2d)^{-1}$, as we can use the same method (i.e., Lemma 2.1 below) to conclude $p_c - p_1 = O(d^{-2})$, but this limits the accuracy our method can achieve. Therefore, to identify the higher-order coefficients, we may need investigate the lace-expansion coefficients at p_c more carefully as in Miranda and Slade [21].

The rest of the paper is organized as follows. In Section 2, we show that p_c is close (up to order L^{-2d}) to p_1 that satisfies the identity $p_1 g_{p_1} = 1$, which is heavily used in the analysis in Sections 3 and 4. Section 3 is devoted to evaluating g_{p_1} for lattice trees. The 1-point function is split into two parts, G and H , which are investigated in Sections 3.1 and 3.2, respectively. Finally, in Section 4, we demonstrate how to evaluate the difference between lattice trees and lattice animals.

2. Results due to the lace expansion

In this section, we approximate p_c by p_1 that is defined for both models by the identity

$$p_1 g_{p_1} = 1. \tag{2.1}$$

From now on, we frequently use

$$\beta = L^{-d}. \tag{2.2}$$

Lemma 2.1. *For both lattice trees and lattice animals with $d > 8$ and $L \uparrow \infty$,*

$$0 < p_c - p_1 = O(\beta^2). \tag{2.3}$$

The key to the proof is the following collection of the lace-expansion results [6, 17], in which we use

$$h_p(x) = \begin{cases} 0 & \text{[lattice trees],} \\ (1 - \delta_{o,x}) \sum_{A \in \mathcal{A}_o} \mathbb{1}_{\{o \longleftrightarrow x\}} W_p(A) & \text{[lattice animals],} \end{cases} \tag{2.4}$$

where $o \longleftrightarrow x$ means that ($o = x$ or) there is at least one pair of edge-disjoint paths from o to x in an animal A . Let \hat{h}_p denote the sum of $h_p(x)$ over $x \in \mathbb{Z}^d$:

$$\hat{h}_p = \sum_{x \in \mathbb{Z}^d} h_p(x). \tag{2.5}$$

Proposition 2.2 ([6, 17]). *For both lattice trees and lattice animals with $d > 8$, there is a model-dependent $L_0 < \infty$ such that, for all $L \geq L_0$, the following holds for all $p \leq p_c$:*

1. *The 1-point function is bounded away from zero and infinity. In fact,*

$$1 \leq g_p \leq 4. \tag{2.6}$$

2. *There are nonnegative lace-expansion coefficients $\pi_p^{(n)}(x)$, $n \in \mathbb{N}$, such that*

$$\exists K < \infty, \quad \forall x \in \mathbb{Z}^d, \quad \pi_p^{(n)}(x) \leq \frac{KL^{-6}(K\beta)^{n-1}}{(\|x\| \vee L)^{2d-6}}, \tag{2.7}$$

and that, by defining $\pi_p(x) = \sum_{n \in \mathbb{N}} (-1)^n \pi_p^{(n)}(x)$, the recursion equation

$$\begin{aligned} \tau_p(x) &= g_p \delta_{o,x} + h_p(x) + \pi_p(x) \\ &\quad + \sum_{u,v} \left(g_p \delta_{o,u} + h_p(u) + \pi_p(u) \right) pD(v-u) \tau_p(x-v) \end{aligned} \tag{2.8}$$

holds for all $x \in \mathbb{Z}^d$.

Consequently, there is a $K' < \infty$ such that

$$\forall x \neq o, \quad \tau_{p_c}(x) \leq \frac{K'L^{-2}}{(\|x\| \vee L)^{d-2}}, \quad \chi_p \underset{p \uparrow p_c}{\asymp} (p_c - p)^{-1/2}, \tag{2.9}$$

where the latter means $\chi_p/(p_c - p)^{-1/2}$ is bounded away from 0 and ∞ as $p \uparrow p_c$, and

$$p_c = \frac{1}{g_{p_c} + \hat{h}_{p_c} + \hat{\pi}_{p_c}} = \left(g_{p_c} + \sum_{x \neq o} h_{p_c}(x) + \sum_x \pi_{p_c}(x) \right)^{-1}. \tag{2.10}$$

The above results for lattice trees are proven in [17] by following the same line of proof as in [6] and using the convolution bounds in [4, Lemma 3.2] instead of the weaker ones in [6, Proposition 1.7]. The same strategy applies to lattice animals, and we refrain from showing details.

Consequently, for any $p \leq p_c$,

$$\hat{\pi}_p^{(n)} = \sum_{x \in \mathbb{Z}^d} \pi_p^{(n)}(x) \stackrel{(2.7)}{\leq} K(K\beta)^{n-1} \left(\sum_{x: \|x\| \leq L} L^{-2d} + \sum_{x: \|x\| > L} \frac{L^{-6}}{\|x\|^{2d-6}} \right) = O(\beta)^n. \tag{2.11}$$

Moreover, by subadditivity (i.e., forgetting edge-disjointness among paths from o to x),

$$\hat{h}_p \leq \sum_{x \neq o} \tau_p(x)^2 \stackrel{(2.9)}{\leq} (K'L^{-2})^2 \left(\sum_{x: \|x\| \leq L} L^{2(2-d)} + \sum_{x: \|x\| > L} \|x\|^{2(2-d)} \right) = O(\beta). \tag{2.12}$$

The identity (2.10) is obtained by summing (2.8) over $x \in \mathbb{Z}^d$, solving the resulting equation for χ_p and then using the fact that χ_p diverges as $p \uparrow p_c$. Substituting (2.11)–(2.12) to (2.10) yields¹

$$p_c = \frac{1}{g_{p_c}} \left(1 + \frac{\hat{h}_{p_c} - \hat{\pi}_{p_c}^{(1)}}{g_{p_c}} + O(\beta^2) \right)^{-1} = \frac{1}{g_{p_c}} \left(1 + \frac{\hat{\pi}_{p_c}^{(1)} - \hat{h}_{p_c}}{g_{p_c}} \right) + O(\beta^2), \tag{2.14}$$

which is the starting point of the analysis.

Proof of Lemma 2.1. First we show $p_1 < p_c$. Since pg_p is increasing in p with $p_1g_{p_1} = 1$, it suffices to show $p_c g_{p_c} > 1$. By (2.6) and (2.14), it then suffices to show that $\hat{\pi}_{p_c}^{(1)} - \hat{h}_{p_c}$ is bounded from below by β times a positive constant for large L . Here, and only here, we use the actual definition of the lace-expansion coefficient $\hat{\pi}_p^{(1)}$ (see, e.g., [8]). We can easily check that $\hat{\pi}_p^{(1)}$ for both models is larger than the sum of triangles consisting only of three distinct edges: $\hat{\pi}_p^{(1)} \geq |\Lambda|(|\Lambda| - 1)(p/|\Lambda|)^3$, which is enough for lattice trees because $\hat{h}_p \equiv 0$. For lattice animals, we show below $\hat{h}_p \leq \frac{1}{4}\hat{\pi}_p^{(1)} + O(\beta^2)$ for $p \leq p_c$ in high dimensions $d > 8$. The aforementioned sufficient condition for $p_c g_{p_c} > 1$ is now verified.

Next we show $p_c - p_1 = O(\beta^2)$ for lattice animals by induction. The same induction also works for lattice trees with $A = T$ and $\hat{h}_{p_c} \equiv 0$. Let $\{\ell_n\}_{n \in \mathbb{N}}$ be the following increasing sequence bounded above by 2:

$$\ell_1 = 1, \quad \ell_{j+1} = 1 + \frac{\ell_j}{2} \quad [j \in \mathbb{N}]. \tag{2.15}$$

Since $p_c = O(1)$ (see, e.g., (1.6) or [6, Proposition 2.2]) and $p_1g_{p_1} = p_c(g_{p_c} + \hat{h}_{p_c} + \hat{\pi}_{p_c}) = 1$, we have

$$0 < 1 - \frac{p_1}{p_c} = 1 - \frac{g_{p_c} + \hat{h}_{p_c} + \hat{\pi}_{p_c}}{g_{p_1}} = - \underbrace{\frac{g_{p_c} - g_{p_1}}{g_{p_1}}}_{\geq 0} - \frac{\hat{h}_{p_c} + \hat{\pi}_{p_c}}{g_{p_1}}, \tag{2.16}$$

which is bounded above by $-\hat{\pi}_{p_c}/g_{p_1} = O(\beta)$ [due to (2.11)], confirming $p_c - p_1 = O(\beta^{\ell_1})$.

¹In [17], Liang investigated $\hat{\pi}_{p_c}^{(1)}$ in (2.14) for lattice trees and showed that, for all $d > 8$, $p_c g_{p_c}$ rather than p_c exhibits

$$p_c g_{p_c} = 1 + \frac{\beta}{e} \sum_{n=2}^{\infty} \binom{n+1}{2} U^{*n}(o) + O(\beta/L) \quad \text{as } L \uparrow \infty. \tag{2.13}$$

This may be a bit of surprise, as the coefficient of β is much larger than that in (1.12)–(1.13).

Now we suppose $p_c - p_1 = O(\beta^{\ell_j})$. Notice that $g_{p_c} - g_{p_1}$ can be rewritten as

$$g_{p_c} - g_{p_1} = \sum_{A \in \mathcal{A}_o} \left(1 - \left(\frac{p_1}{p_c}\right)^{|E_A|}\right) W_{p_c}(A) = \left(1 - \frac{p_1}{p_c}\right) \underbrace{\sum_{A \in \mathcal{A}_o} \sum_{n=0}^{|E_A|-1} \left(\frac{p_1}{p_c}\right)^n W_{p_c}(A)}_{=:F} \stackrel{(2.16)}{=} \left(-\frac{g_{p_c} - g_{p_1}}{g_{p_1}} - \frac{\hat{h}_{p_c} + \hat{\pi}_{p_c}}{g_{p_1}}\right) F. \tag{2.17}$$

Solving this for $g_{p_c} - g_{p_1}$ yields

$$g_{p_c} - g_{p_1} = -\frac{\hat{h}_{p_c} + \hat{\pi}_{p_c}}{g_{p_1} + F} F, \tag{2.18}$$

which is bounded above by $-\hat{\pi}_{p_c} = O(\beta)$ [due to (2.11)] for both models. By substituting (2.18) to (2.16), we obtain

$$p_c - p_1 = p_c \left(\frac{1}{g_{p_1}} \frac{\hat{h}_{p_c} + \hat{\pi}_{p_c}}{g_{p_1} + F} F - \frac{\hat{h}_{p_c} + \hat{\pi}_{p_c}}{g_{p_1}}\right) = -p_c \frac{\hat{h}_{p_c} + \hat{\pi}_{p_c}}{g_{p_1} + F} \stackrel{(2.11)}{=} p_c \frac{\hat{\pi}_{p_c}^{(1)} - \hat{h}_{p_c}}{g_{p_1} + F} + O(\beta^2). \tag{2.19}$$

Recall the definition of F in (2.17). Since $(p_1/p_c)^n W_{p_c}(A) = (p_c/p_1)^{|E_A|-n} W_{p_1}(A)$, which is also true for lattice trees, we have

$$F = \sum_{A \in \mathcal{A}_o} \sum_{n=1}^{|E_A|} \left(\frac{p_c}{p_1}\right)^n W_{p_1}(A) \stackrel{p_1 < p_c}{\geq} \sum_{A \in \mathcal{A}_o} |E_A| W_{p_1}(A) \stackrel{|V_A| \leq 2|E_A|}{\geq} \frac{\chi_{p_1}}{2}. \tag{2.20}$$

By (2.19) and (2.20), we can estimate $p_c - p_1$ as

$$p_c - p_1 = p_c \frac{\hat{\pi}_{p_c}^{(1)} - \hat{h}_{p_c}}{g_{p_1} F^{-1} + 1} F^{-1} + O(\beta^2) = O(\beta) \chi_p^{-1} + O(\beta^2) = O(\beta)(p_c - p_1)^{\frac{1}{2}} + O(\beta^2), \tag{2.21}$$

where, for the last inequality, we use $\chi_{p_1} \asymp (p_c - p_1)^{-1/2}$ for both models in dimensions $d > 8$. Applying the inductive hypothesis $p_c - p_1 = O(\beta^{\ell_j})$ to (2.21), we obtain $p_c - p_1 = O(\beta^{\ell_j+1})$, which completes the induction. Since $\lim_{j \uparrow \infty} \ell_j = 2$, this proves $p_c - p_1 = O(\beta^2)$, as required. \square

Proof of $\hat{h}_p \leq \frac{1}{4} \hat{\pi}_p^{(1)} + O(\beta^2)$ for lattice animals. First we recall that

$$\hat{h}_p = \sum_{x \neq o} \sum_{A \in \mathcal{A}_o} \left(\frac{p}{|\Lambda|}\right)^{|E_A|} \mathbb{1}_{\{o \longleftrightarrow x\}}. \tag{2.22}$$

We split the sum into two depending on whether or not there are distinct vertices $y, z \in V_A$ such that $o \longleftrightarrow y, y \longleftrightarrow x, o \longleftrightarrow z, z \longleftrightarrow x$ and $y \longleftrightarrow z$ occur in A edge-disjointly, i.e., those connections occur in distinct sets of E_A . (We note that, if $y = o$, for example, then we should interpret this as $o \longleftrightarrow x, o \longleftrightarrow z$ and $z \longleftrightarrow x$ occurring edge-disjointly.) Intuitively,

$$\bigcup_{\substack{y, z \in V_A \\ (y \neq z)}} \begin{array}{c} z \\ \circ \quad \circ \\ \backslash \quad / \\ \circ \quad \circ \\ y \end{array} x. \tag{2.23}$$

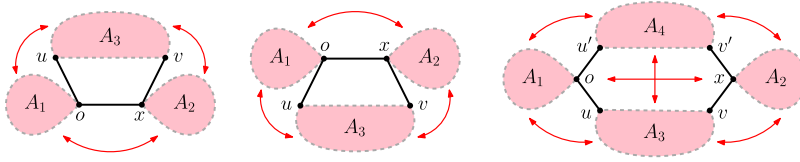


Figure 3. Schematic representations of the three terms in (2.25). The black line segments are pivotal for $o \iff x$ in A . Removal of those edges results in the animals $\{A_j\}_{j=1}^3$ or $\{A_j\}_{j=1}^4$ that are mutually avoiding, as indicated by the red arrows. The vertices in Λ are ordered in an arbitrary way (counter-clockwise in the above 2-dimensional figures).

Using submultiplicativity and the x -space bound in (2.9), we can show that the contribution from this case is $O(\beta^2)$. On the other hand, if there are no such vertices $y, z \in V_A$, i.e.,

$$\begin{aligned}
 & \text{Diagram of } o \text{ and } x \text{ with two edges} \setminus \bigcup_{\substack{y, z \in V_A \\ (y \neq z)}} \text{Diagram of } o \text{ and } x \text{ with three edges } y, z, x, \\
 & \hspace{15em} (2.24)
 \end{aligned}$$

then there are exactly two edge-disjoint connections between o and x , with two pivotal edges from o , say $\{o, u\}, \{o, u'\}$, and two from x , say $\{v, x\}, \{v', x\}$, one of which may coincide with either $\{o, u\}$ or $\{o, u'\}$, for the double connection $o \iff x$ in A . Suppose that there is order among vertices in Λ . If $u \in \Lambda$ is earlier than $u' \in \Lambda$ in this order, we write $u < u'$. Let $\Lambda(x) = \{v \in V : v - x \in \Lambda\}$. Then, the contribution to \hat{h}_p from (2.24) is bounded above by (see Fig. 3)

$$\begin{aligned}
 & \sum_{x \neq o} \sum_{\substack{A_1 \in \mathcal{A}_o \\ A_2 \in \mathcal{A}_x}} W_p(A_1) W_p(A_2) \left(\mathbb{1}_{\{x \in \Lambda\}} \frac{p}{|\Lambda|} \sum_{\substack{u \in \Lambda \\ (x < u)}} \sum_{\substack{v \in \Lambda(x) \\ (v \neq o)}} \left(\frac{p}{|\Lambda|} \right)^2 \sum_{A_3 \in \mathcal{A}_{u,v}} W_p(A_3) \right. \\
 & + \mathbb{1}_{\{x \in \Lambda\}} \frac{p}{|\Lambda|} \sum_{\substack{u \in \Lambda \\ (u < x)}} \sum_{\substack{v \in \Lambda(x) \\ (v \neq o)}} \left(\frac{p}{|\Lambda|} \right)^2 \sum_{A_3 \in \mathcal{A}_{u,v}} W_p(A_3) \\
 & \left. + \sum_{\substack{u, u' \in \Lambda \\ (u < u')}} \sum_{\substack{v, v' \in \Lambda(x) \\ (v \neq v')}} \left(\frac{p}{|\Lambda|} \right)^4 \sum_{\substack{A_3 \in \mathcal{A}_{u,v} \\ A_4 \in \mathcal{A}_{u',v'}}} W_p(A_3) W_p(A_4) \right) \prod_{i \neq j} \mathbb{1}_{\{V_{A_i} \cap V_{A_j} = \emptyset\}}. \quad (2.25)
 \end{aligned}$$

Since Λ is symmetric with respect to the underlying lattice symmetry, the first and second terms are the same. Due to the same reason, the third term remains unchanged when the restriction $u < u'$ is replaced by $u' < u$. Therefore, (2.25) equals

$$\begin{aligned}
 & \sum_{x \neq o} \sum_{\substack{A_1 \in \mathcal{A}_o \\ A_2 \in \mathcal{A}_x}} W_p(A_1) W_p(A_2) \left(\mathbb{1}_{\{x \in \Lambda\}} \frac{p}{|\Lambda|} \sum_{\substack{u \in \Lambda \\ (u \neq x)}} \sum_{\substack{v \in \Lambda(x) \\ (v \neq o)}} \left(\frac{p}{|\Lambda|} \right)^2 \sum_{A_3 \in \mathcal{A}_{u,v}} W_p(A_3) \right. \\
 & \left. + \frac{1}{2} \sum_{\substack{u, u' \in \Lambda \\ (u \neq u')}} \sum_{\substack{v, v' \in \Lambda(x) \\ (v \neq v')}} \left(\frac{p}{|\Lambda|} \right)^4 \sum_{\substack{A_3 \in \mathcal{A}_{u,v} \\ A_4 \in \mathcal{A}_{u',v'}}} W_p(A_3) W_p(A_4) \right) \prod_{i \neq j} \mathbb{1}_{\{V_{A_i} \cap V_{A_j} = \emptyset\}}. \quad (2.26)
 \end{aligned}$$

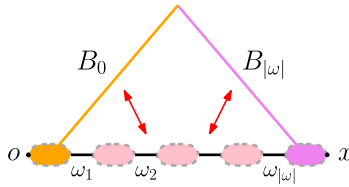
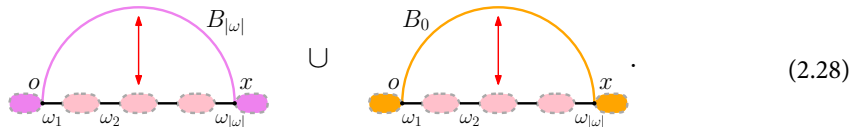


Figure 4. Schematic representation of $\pi_p^{(1)}(x)$. The sequence of edges $\omega_1, \dots, \omega_{|\omega|}$ joined by the animals $B_0, \dots, B_{|\omega|}$ form the backbone from o to x in A . The terminal animals B_0 and $B_{|\omega|}$ share a vertex [due to $\mathbb{1}_{B_0 \cap B_{|\omega|} \neq \emptyset}$ in (2.27)], otherwise those animals are mutually avoiding [due to the product of indicators in (2.27)]. Each animal B_j contains a double connection between $\bar{\omega}_j$ and $\underline{\omega}_{j+1}$.

Now we compare (2.26) with $\hat{\pi}_p^{(1)}$ for lattice animals, which is defined as (see Fig. 4)

$$\begin{aligned} \hat{\pi}_p^{(1)} = & \sum_x \sum_{\substack{\omega = \{\underline{\omega}_i, \bar{\omega}_i\}_{i=1}^{|\omega|} \\ (|\omega| \geq 1)}} \left(\frac{p}{|\Lambda|}\right)^{|\omega|} \prod_{j=0}^{|\omega|} \sum_{B_j \in \mathcal{A}_{\bar{\omega}_j, \underline{\omega}_{j+1}}} W_p(B_j) \mathbb{1}_{\{\bar{\omega}_j \leftrightarrow \underline{\omega}_{j+1} \text{ in } B_j\}} \\ & \times \mathbb{1}_{\{B_0 \cap B_{|\omega|} \neq \emptyset\}} \prod_{\substack{0 \leq k < l \leq |\omega| \\ ((k, l) \neq (0, |\omega|))}} \mathbb{1}_{\{B_k \cap B_l = \emptyset\}}, \end{aligned} \tag{2.27}$$

where we have abused the notation $\bar{\omega}_0 = o$ and $\underline{\omega}_{|\omega|+1} = x$. This can be bounded below by restricting the sum over ω to those satisfying $\underline{\omega}_1 = o$ and $\bar{\omega}_{|\omega|} = x$ (so that $\mathcal{A}_{\bar{\omega}_0, \underline{\omega}_1} = \mathcal{A}_o$ and $\mathcal{A}_{\bar{\omega}_{|\omega|}, \underline{\omega}_{|\omega|+1}} = \mathcal{A}_x$) and then by restricting the sum over $B_0 \in \mathcal{A}_o$ to $B_0 = \{o\}$ (so that $\mathbb{1}_{\{B_0 \cap B_{|\omega|} \neq \emptyset\}} = \mathbb{1}_{\{o \in B_{|\omega|}\}}$) or restricting the sum over $B_{|\omega|} \in \mathcal{A}_x$ to $B_{|\omega|} = \{x\}$ (so that $\mathbb{1}_{\{B_0 \cap B_{|\omega|} \neq \emptyset\}} = \mathbb{1}_{\{x \in B_0\}}$):



Those two terms are basically the same. Splitting the sum over ω into two depending on whether $|\omega| = 1$ (so that $\omega = \{\omega_1\}$, where $\omega_1 = (o, x)$) or $|\omega| \geq 2$ and then, for the latter, by summing over the animals $B_1, \dots, B_{|\omega|-1}$ (to form an animal $A_3 \in \mathcal{A}_{\bar{\omega}_1, \underline{\omega}_{|\omega|}}$), we obtain

$$\begin{aligned} \hat{\pi}_p^{(1)} \geq & \sum_{x \neq o} \sum_{B \in \mathcal{A}_{o,x}} W_p(B) \left(\mathbb{1}_{\{(o,x) \notin E_B\}} \mathbb{1}_{\{x \in \Lambda\}} \frac{p}{|\Lambda|} \right. \\ & \left. + \sum_{\substack{u \in \Lambda \\ (u \neq x)}} \sum_{\substack{v \in \Lambda(x) \\ (v \neq o)}} \left(\frac{p}{|\Lambda|}\right)^2 \sum_{A_3 \in \mathcal{A}_{u,v}} W_p(A_3) \mathbb{1}_{\{V_B \cap V_{A_3} = \emptyset\}} \right). \end{aligned} \tag{2.29}$$

We further bound this below by restricting the sum over $B \in \mathcal{A}_{o,x}$ to smaller animals $B = (V_B, E_B)$ with either

- (i) $V_B = V_{A_1} \cup V_{A_2}, E_B = E_{A_1} \cup \{(o, x)\} \cup E_{A_2}$ for some $A_1 \in \mathcal{A}_o, A_2 \in \mathcal{A}_x$ (as in the left and middle figures of Fig. 3), or
- (ii) $V_B = V_{A_1} \cup V_{A_2} \cup V_{A_4}, E_B = E_{A_1} \cup \{(o, u')\} \cup E_{A_4} \cup \{(v', x)\} \cup E_{A_2}$ for some $A_1 \in \mathcal{A}_o, A_2 \in \mathcal{A}_x, u' \in \Lambda, v' \in \Lambda(x), A_4 \in \mathcal{A}_{u',v'}$ (as in the right figure of Fig. 3).

The contribution from (i) to the right-hand side of (2.29) is

$$2 \sum_{x \neq o} \sum_{\substack{A_1 \in \mathcal{A}_o \\ A_2 \in \mathcal{A}_x}} W_p(A_1) W_p(A_2) \mathbb{1}_{\{x \in \Lambda\}} \frac{p}{|\Lambda|} \sum_{\substack{u \in \Lambda \\ (u \neq x)}} \sum_{\substack{v \in \Lambda(x) \\ (v \neq o)}} \left(\frac{p}{|\Lambda|}\right)^2 \sum_{A_3 \in \mathcal{A}_{u,v}} W_p(A_3) \prod_{i \neq j} \mathbb{1}_{\{V_{A_i} \cap V_{A_j} = \emptyset\}}, \tag{2.30}$$

while the contribution from (ii) is

$$2 \sum_{x \neq o} \sum_{\substack{A_1 \in \mathcal{A}_o \\ A_2 \in \mathcal{A}_x}} W_p(A_1) W_p(A_2) \left(\mathbb{1}_{\{x \in \Lambda\}} \frac{p}{|\Lambda|} \sum_{\substack{u \in \Lambda \\ (u \neq x)}} \sum_{\substack{v \in \Lambda(x) \\ (v \neq o)}} \left(\frac{p}{|\Lambda|}\right)^2 \sum_{A_3 \in \mathcal{A}_{u,v}} W_p(A_3) \right. \\ \left. + \sum_{\substack{u, u' \in \Lambda \\ (u \neq u')}} \sum_{\substack{v, v' \in \Lambda(x) \\ (v \neq v')}} \left(\frac{p}{|\Lambda|}\right)^4 \sum_{\substack{A_3 \in \mathcal{A}_{u,v} \\ A_4 \in \mathcal{A}_{u',v'}}} W_p(A_3) W_p(A_4) \right) \prod_{i \neq j} \mathbb{1}_{\{V_{A_i} \cap V_{A_j} = \emptyset\}}. \tag{2.31}$$

Notice that the sum of (2.30) and (2.31) is four times as large as (2.26). This completes the proof of $\hat{h}_p \leq \frac{1}{4} \hat{\pi}_p^{(1)} + O(\beta^2)$. \square

3. Detailed analysis of the 1-point function for lattice trees

To complete the proof of Theorem 1.1, it remains to investigate $p_1 = 1/g_{p_1}$ (due to (2.1) and Lemma 2.1). In this section, we concentrate our attention to lattice trees and show the following:

Lemma 3.1. *For lattice trees with $d > 8$ and $L \uparrow \infty$,*

$$g_{p_1} = e \left(1 - \sum_{n=2}^{\infty} \frac{n+1}{2} D^{*n}(o) \right) + O(\beta^2). \tag{3.1}$$

Consequently,

$$p_1 = \frac{1}{e} + \sum_{n=2}^{\infty} \frac{n+1}{2e} D^{*n}(o) + O(\beta^2). \tag{3.2}$$

To prove Lemma 3.1, we first rewrite g_{p_1} by identifying the connected neighbours Y of the origin as

$$g_{p_1} = \sum_{T \in \mathcal{T}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_T|} \\ = 1 + \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 1)}} \sum_{T \in \mathcal{T}_{Y \cup \{o\}}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_T|} \\ = 1 + \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 1)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y|} \prod_{y \in Y} \sum_{R_y \in \mathcal{T}_y \setminus \mathcal{T}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|} \prod_{\substack{u, v \in Y \\ (u \neq v)}} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} = \emptyset\}}, \tag{3.3}$$

where, and from now on, $\sum_{Y \subset \Lambda}$ is the sum over sets Y of distinct vertices of Λ (we recall that o is not included in Λ) and

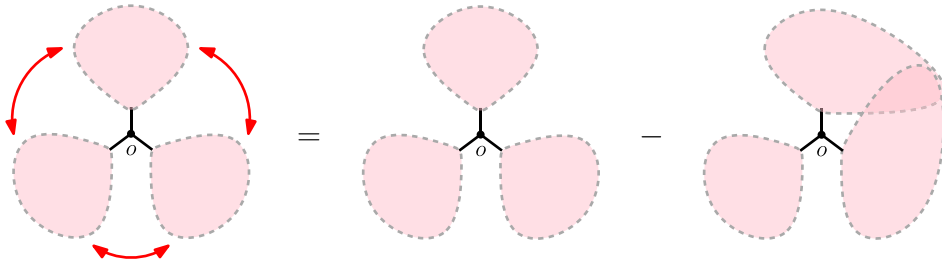


Figure 5. Intuitive explanation of (3.5). The double-headed arrows on the left ($= \overline{g_{p_1}}$) represents mutual avoidance among subtrees. In the first term on the right ($= G$), those subtrees are independently summed over $\mathcal{T}_y \setminus \mathcal{T}_o, y \in Y$, where Y is the set of connected neighbours of the origin. In the second term on the right ($= H$), there is at least one pair of subtrees that share vertices.

$$\prod_{y \in Y} \sum_{R_y \in \mathcal{T}_y \setminus \mathcal{T}_o} (p_1/|\Lambda|)^{|E_{R_y}|} = \sum_{R_{y_1} \in \mathcal{T}_{y_1} \setminus \mathcal{T}_o} (p_1/|\Lambda|)^{|E_{R_{y_1}}|} \dots \sum_{R_{y_n} \in \mathcal{T}_{y_n} \setminus \mathcal{T}_o} (p_1/|\Lambda|)^{|E_{R_{y_n}}|} \tag{3.4}$$

for $Y = \{y_1, \dots, y_n\}$. By convention, $\prod_{u \neq v} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} = \emptyset\}}$ is regarded as 1 when $|Y| = 1$. Let (see Fig. 5)

$$g_{p_1} = G - H, \tag{3.5}$$

where

$$G = 1 + \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 1)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y|} \prod_{y \in Y} \sum_{R_y \in \mathcal{T}_y \setminus \mathcal{T}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|}, \tag{3.6}$$

$$H = \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 2)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y|} \prod_{y \in Y} \sum_{R_y \in \mathcal{T}_y \setminus \mathcal{T}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|} \left(1 - \prod_{\substack{u, v \in Y \\ (u \neq v)}} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} = \emptyset\}}\right). \tag{3.7}$$

We investigate those G and H in Sections 3.1 and 3.2, respectively (cf., Lemmas 3.2 and 3.4 below).

3.1 Detailed analysis of G

From now on, we frequently use

$$S_{\geq t}(x) = \sum_{n=t}^{\infty} D^{*n}(x), \tag{3.8}$$

where $D^{*0}(x) = \delta_{o,x}$ by convention. The following is what we are going to show in this section:

Lemma 3.2. For lattice trees with $d > 8$ and $L \uparrow \infty$,

$$G = e\left(1 - \frac{1}{2}D^{*2}(o) - S_{\geq 2}(o)\right) + O(\beta^2). \tag{3.9}$$

Proof. Since $p_1 g_{p_1} = 1$, we can rewrite G as

$$\begin{aligned}
 G &= 1 + \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 1)}} \left(\frac{p_1}{|\Lambda|} \right)^{|Y|} \prod_{y \in Y} \left(\underbrace{\sum_{R_y \in \mathcal{T}_y} \left(\frac{p_1}{|\Lambda|} \right)^{|E_{R_y}|}}_{g_{p_1}} - \underbrace{\sum_{R_y \in \mathcal{T}_{o,y}} \left(\frac{p_1}{|\Lambda|} \right)^{|E_{R_y}|}}_{\tau_{p_1}(y)} \right) \\
 &= 1 + \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 1)}} \left(\frac{1}{|\Lambda|} \right)^{|Y|} \prod_{y \in Y} \left(1 - \frac{\tau_{p_1}(y)}{g_{p_1}} \right). \tag{3.10}
 \end{aligned}$$

If we replace $\prod_{y \in Y} (1 - \tau_{p_1}(y)/g_{p_1})$ by 1, then we obtain

$$G_0 := 1 + \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 1)}} \left(\frac{1}{|\Lambda|} \right)^{|Y|} = \left(1 + \frac{1}{|\Lambda|} \right)^{|\Lambda|}. \tag{3.11}$$

Since $k \log(1 + 1/k) = 1 - 1/(2k) + O(k^{-2})$ as $k \uparrow \infty$, and since $|\Lambda|^{-1} = D^{*2}(o)$, we obtain

$$G_0 = e \left(1 - \frac{1}{2|\Lambda|} \right) + O(|\Lambda|^{-2}) = e \left(1 - \frac{1}{2} D^{*2}(o) \right) + O(\beta^2). \tag{3.12}$$

Next we consider the remainder, which is

$$\begin{aligned}
 G - G_0 &= \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 1)}} \left(\frac{1}{|\Lambda|} \right)^{|Y|} \left(\prod_{y \in Y} \left(1 - \frac{\tau_{p_1}(y)}{g_{p_1}} \right) - 1 \right) \\
 &= \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 1)}} \left(\frac{1}{|\Lambda|} \right)^{|Y|} \sum_{\substack{Z \subset Y \\ (|Z| \geq 1)}} \prod_{y \in Z} \frac{-\tau_{p_1}(y)}{g_{p_1}}. \tag{3.13}
 \end{aligned}$$

Changing the order of sums yields

$$\begin{aligned}
 G - G_0 &= \sum_{\substack{Z \subset \Lambda \\ (|Z| \geq 1)}} \prod_{y \in Z} \frac{-\tau_{p_1}(y)}{g_{p_1}} \left(\frac{1}{|\Lambda|} \right)^{|Z|} \sum_{Z \subset Y \subset \Lambda} \left(\frac{1}{|\Lambda|} \right)^{|Y \setminus Z|} \\
 &= \sum_{\substack{Z \subset \Lambda \\ (|Z| \geq 1)}} \prod_{y \in Z} \frac{-\tau_{p_1}(y) D(y)}{g_{p_1}} \left(1 + \frac{1}{|\Lambda|} \right)^{|\Lambda \setminus Z|} \\
 &= G_1 + G_2, \tag{3.14}
 \end{aligned}$$

where

$$G_1 = \sum_{\substack{Z \subset \Lambda \\ (|Z|=1)}} \prod_{y \in Z} \frac{-\tau_{p_1}(y) D(y)}{g_{p_1}} \left(1 + \frac{1}{|\Lambda|} \right)^{|\Lambda \setminus Z|} = \frac{-(\tau_{p_1} * D)(o)}{g_{p_1}} \frac{G_0}{(1 + 1/|\Lambda|)}, \tag{3.15}$$

$$G_2 = \sum_{\substack{Z \subset \Lambda \\ (|Z| \geq 2)}} \prod_{y \in Z} \frac{-\tau_{p_1}(y) D(y)}{g_{p_1}} \left(1 + \frac{1}{|\Lambda|} \right)^{|\Lambda \setminus Z|}. \tag{3.16}$$

We have made use of reflection symmetry in (3.14)–(3.16), such as $\tau_p(-y) = \tau_p(y)$ and $D(-y) = D(y)$; we will frequently use the symmetry without mentioning each time. To estimate G_1 and G_2 , we use the following lemma, which will be proven after the proof of Lemma 3.2 is completed. \square

Lemma 3.3. For any $d > 2$ and $x \neq o$, the lattice tree 2-point function satisfies

$$0 \leq S_{\geq 1}(x) - \frac{\tau_{p_1}(x)}{g_{p_1}} \leq \sum_{\substack{y,z \in \mathbb{Z}^d \\ (y \neq z)}} S_{\geq 0}^{*2}(z-y) S_{\geq 0}(y) S_{\geq 1}(z-y) S_{\geq 0}(x-z). \tag{3.17}$$

Remark. The right-hand side of (3.17) is diagrammatically represented by

$$\sum_{\substack{y,z \in \mathbb{Z}^d \\ (y \neq z)}} S_{\geq 0}^{*2}(z-y) S_{\geq 0}(y) S_{\geq 1}(z-y) S_{\geq 0}(x-z) = o \text{---} \triangle \text{---} x, \tag{3.18}$$

where an unslashed (resp., slashed) line represents $S_{\geq 0}$ (resp., $S_{\geq 1}$) and an unlabelled vertex is summed over \mathbb{Z}^d . Due to translation invariance, we can change the order of terms in (3.17) for a given $x \neq o$. Then (3.18) is also equal to

$$\sum_{\substack{y \in \mathbb{Z}^d \\ (y \neq x)}} S_{\geq 0}^{*2}(y) S_{\geq 0}^{*2}(x-y) S_{\geq 1}(x-y) = o \text{---} \triangle \text{---} x. \tag{3.19}$$

These diagrammatic representations will be used in the proof of Lemma 3.4 below. □

First we estimate G_2 . By the first inequality in (3.17) and the heat-kernel bound (see, e.g., [3, (1.6)]):

$$\|D^{*n}\|_{\infty} = O(\beta)n^{-d/2} \quad [n \in \mathbb{N}], \tag{3.20}$$

we can show

$$\frac{(\tau_{p_1} * D)(o)}{g_{p_1}} \leq \sup_{x \in \Lambda} \frac{\tau_{p_1}(x)}{g_{p_1}} \leq \sup_{x \in \Lambda} S_{\geq 1}(x) \leq \sum_{n=1}^{\infty} \|D^{*n}\|_{\infty} \stackrel{d \geq 2}{=} O(\beta). \tag{3.21}$$

Therefore,

$$|G_2| \leq G_0 \sum_{n=2}^{|\Lambda|} \sum_{\substack{Z \subset \Lambda \\ (|Z|=n)}} \prod_{y \in Z} \frac{\tau_{p_1}(y) D(y)}{g_{p_1}} \leq G_0 \sum_{n=2}^{\infty} \left(\frac{(\tau_{p_1} * D)(o)}{g_{p_1}} \right)^n = O(\beta^2). \tag{3.22}$$

Next we estimate G_1 in (3.15). By using (3.17), we have

$$\begin{aligned} 0 \leq S_{\geq 2}(o) - \frac{(\tau_{p_1} * D)(o)}{g_{p_1}} &= \sum_{x \in \mathbb{Z}^d} D(x) \left(S_{\geq 1}(x) - \frac{\tau_{p_1}(x)}{g_{p_1}} \right) \\ &\stackrel{(3.17)}{\leq} \sum_{x \in \mathbb{Z}^d} D(x) \sum_{\substack{y,z \in \mathbb{Z}^d \\ (y \neq z)}} S_{\geq 0}^{*2}(z-y) S_{\geq 0}(y) S_{\geq 1}(z-y) S_{\geq 0}(x-z) \\ &\leq \sum_{\substack{y,z \in \mathbb{Z}^d \\ (y \neq z)}} S_{\geq 0}^{*2}(z-y) S_{\geq 0}(y) S_{\geq 1}(z-y) S_{\geq 1}(z) \\ &\leq \sup_{w \neq o} S_{\geq 0}^{*2}(w) (S_{\geq 0} * S_{\geq 1}^{*2})(o). \end{aligned} \tag{3.23}$$

By the heat-kernel bound (3.20), we can estimate each term as

$$S_{\geq 0}^{*2}(w) = \sum_{s,t=0}^{\infty} D^{*(s+t)}(w) \stackrel{w \neq o}{\leq} \sum_{n=1}^{\infty} (n+1)D^{*n}(w) \stackrel{d \geq 4}{\equiv} O(\beta), \tag{3.24}$$

$$(S_{\geq 0} * S_{\geq 1}^{*2})(o) = \sum_{s=0}^{\infty} \sum_{t,u=1}^{\infty} D^{*(s+t+u)}(o) = \sum_{n=2}^{\infty} \binom{n}{2} D^{*n}(o) \stackrel{d \geq 6}{\equiv} O(\beta), \tag{3.25}$$

so that

$$0 \leq S_{\geq 2}(o) - \frac{(\tau_{p_1} * D)(o)}{g_{p_1}} = O(\beta^2). \tag{3.26}$$

Therefore, by (3.12),

$$\begin{aligned} G_1 &= \underbrace{(-S_{\geq 2}(o))}_{\stackrel{(3.21)}{\equiv} O(\beta)} + O(\beta^2) \left(e - \underbrace{\frac{e}{2} D^{*2}(o)}_{\stackrel{(3.20)}{\equiv} O(\beta)} + O(\beta^2) \right) \\ &= -eS_{\geq 2}(o) + O(\beta^2). \end{aligned} \tag{3.27}$$

Summarizing (3.12), (3.14), (3.22) and (3.27), we complete the proof of Lemma 3.2.

Proof of Lemma 3.3. First we recall

$$\frac{\tau_{p_1}(x)}{g_{p_1}} = \frac{1}{g_{p_1}} \sum_{T \in \mathcal{T}_{o,x}} \left(\frac{p_1}{|\Lambda|} \right)^{|E_T|}. \tag{3.28}$$

Since a tree $T \in \mathcal{T}_{o,x}$ can be divided into a unique path $\omega = (\omega_0, \omega_1, \dots, \omega_{|\omega|})$ from $\omega_0 = o$ to $\omega_{|\omega|} = x$, called a backbone, and disjoint subtrees $R_j \in \mathcal{T}_{\omega_j}$, called ribs (see Fig. 2), we can rewrite the above expression as

$$\frac{\tau_{p_1}(x)}{g_{p_1}} = \frac{1}{g_{p_1}} \sum_{\omega: o \rightarrow x} \left(\frac{p_1}{|\Lambda|} \right)^{|\omega|} \prod_{j=0}^{|\omega|} \sum_{R_j \in \mathcal{T}_{\omega_j}} \left(\frac{p_1}{|\Lambda|} \right)^{|E_{R_j}|} \prod_{s < t} \mathbb{1}_{\{V_{R_s} \cap V_{R_t} = \emptyset\}}. \tag{3.29}$$

If we replace the indicator $\prod_{s < t} \mathbb{1}_{\{V_{R_s} \cap V_{R_t} = \emptyset\}}$ by 1, then we obtain

$$\frac{1}{g_{p_1}} \sum_{\omega: o \rightarrow x} \left(\frac{p_1}{|\Lambda|} \right)^{|\omega|} \underbrace{\prod_{j=0}^{|\omega|} \sum_{R_j \in \mathcal{T}_{\omega_j}} \left(\frac{p_1}{|\Lambda|} \right)^{|E_{R_j}|}}_{g_{p_1}} \stackrel{(2.1)}{\equiv} \sum_{\omega: o \rightarrow x} \left(\frac{1}{|\Lambda|} \right)^{|\omega|} \stackrel{x \neq o}{\equiv} S_{\geq 1}(x). \tag{3.30}$$

Next we consider the remainder. Since $1 - \prod_{j=1}^n a_j \leq \sum_{j=1}^n (1 - a_j)$ as long as $0 \leq a_j \leq 1$ for all j , we can bound the remainder as

$$\begin{aligned} S_{\geq 1}(x) - \frac{\tau_{p_1}(x)}{g_{p_1}} &= \frac{1}{g_{p_1}} \sum_{\omega: o \rightarrow x} \left(\frac{p_1}{|\Lambda|} \right)^{|\omega|} \prod_{j=0}^{|\omega|} \sum_{R_j \in \mathcal{T}_{\omega_j}} \left(\frac{p_1}{|\Lambda|} \right)^{|E_{R_j}|} \left(1 - \prod_{s < t} \mathbb{1}_{\{V_{R_s} \cap V_{R_t} = \emptyset\}} \right) \\ &\leq \frac{1}{g_{p_1}} \sum_{\omega: o \rightarrow x} \left(\frac{p_1}{|\Lambda|} \right)^{|\omega|} \prod_{j=0}^{|\omega|} \sum_{R_j \in \mathcal{T}_{\omega_j}} \left(\frac{p_1}{|\Lambda|} \right)^{|E_{R_j}|} \sum_{s < t} \mathbb{1}_{\{V_{R_s} \cap V_{R_t} \neq \emptyset\}}. \end{aligned} \tag{3.31}$$

If $V_{R_s} \cap V_{R_t} \neq \emptyset$, then there must be a $w \in \mathbb{Z}^d$ that is shared by those two ribs. Therefore, the remainder is further bounded above as

$$\begin{aligned}
 S_{\geq 1}(x) - \frac{\tau_{p_1}(x)}{g_{p_1}} &\leq \frac{1}{g_{p_1}} \sum_{\omega:0 \rightarrow x} \left(\frac{p_1}{|\Lambda|}\right)^{|\omega|} \underbrace{\sum_{s < t} \prod_{j \neq s, t} \sum_{R_j \in \mathcal{T}_{\omega_j}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_j}|}}_{g_{p_1}^{|\omega|-1}} \\
 &\times \underbrace{\sum_{w \in \mathbb{Z}^d} \sum_{R_s \in \mathcal{T}_{\omega_s, w}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_s}|}}_{\tau_{p_1}(w - \omega_s)} \underbrace{\sum_{R_t \in \mathcal{T}_{\omega_t, w}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_t}|}}_{\tau_{p_1}(\omega_t - w)} \\
 &= \frac{1}{g_{p_1}^2} \sum_{\omega:0 \rightarrow x} \left(\frac{1}{|\Lambda|}\right)^{|\omega|} \sum_{s < t} \sum_{w \in \mathbb{Z}^d} \tau_{p_1}(w - \omega_s) \tau_{p_1}(\omega_t - w) \\
 &= \sum_{\substack{y, z \in \mathbb{Z}^d \\ (y \neq z)}} \frac{\tau_{p_1}^{*2}(z - y)}{g_{p_1}^2} \sum_{\omega:0 \rightarrow y \rightarrow z \rightarrow x} \left(\frac{1}{|\Lambda|}\right)^{|\omega|} \\
 &= \sum_{\substack{y, z \in \mathbb{Z}^d \\ (y \neq z)}} \frac{\tau_{p_1}^{*2}(z - y)}{g_{p_1}^2} S_{\geq 0}(y) S_{\geq 1}(z - y) S_{\geq 0}(x - z). \tag{3.32}
 \end{aligned}$$

The proof of (3.17) is completed by applying (3.29)–(3.30) to $(\tau_{p_1}/g_{p_1})^{*2}$ in the above bound. \square

3.2 Detailed analysis of H

To complete the proof of Lemma 3.1, it suffices to show the following:

Lemma 3.4. *For lattice trees with $d > 8$ and $L \uparrow \infty$,*

$$H = e \sum_{n=3}^{\infty} \frac{n-1}{2} D^{*n}(o) + O(\beta^2). \tag{3.33}$$

Proof. Recall the definition (3.7) of H :

$$H = \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 2)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y|} \prod_{y \in Y} \sum_{R_y \in \mathcal{T}_y \setminus \mathcal{T}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|} \left(1 - \prod_{\substack{u, v \in Y \\ (u \neq v)}} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} = \emptyset\}}\right). \tag{3.34}$$

First we split the indicator $1 - \prod_{u \neq v} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} = \emptyset\}}$ in (3.7) by introducing order among pairs of distinct vertices in Λ , called bonds. If a bond b is earlier than another bond b' in that order, we denote it by $b < b'$. Then we have

$$\begin{aligned}
 & 1 - \prod_{\{u,v\} \subset Y} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} = \emptyset\}} \\
 &= \sum_{\{u,v\} \subset Y} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}} \prod_{\substack{\{u',v'\} \subset Y \\ (\{u',v'\} < \{u,v\})}} \mathbb{1}_{\{V_{R_{u'}} \cap V_{R_{v'}} = \emptyset\}} \\
 &= \sum_{\{u,v\} \subset Y} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}} - \sum_{\{u,v\} \subset Y} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}} \left(1 - \prod_{\substack{\{u',v'\} \subset Y \\ (\{u',v'\} < \{u,v\})}} \mathbb{1}_{\{V_{R_{u'}} \cap V_{R_{v'}} = \emptyset\}} \right), \tag{3.35}
 \end{aligned}$$

where the second sum on the right is zero when $|Y| = 2$. Let H_1 be the contribution from the first sum on the right:

$$H_1 = \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 2)}} \left(\frac{p_1}{|\Lambda|} \right)^{|Y|} \prod_{y \in Y} \sum_{R_y \in \mathcal{T}_y \setminus \mathcal{T}_o} \left(\frac{p_1}{|\Lambda|} \right)^{|E_{R_y}|} \sum_{\{u,v\} \subset Y} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}}. \tag{3.36}$$

We will later show [after the derivation of (3.33); see (3.54)] that

$$H_2 := H_1 - H = O(\beta^2). \tag{3.37}$$

Next we investigate H_1 . Let H'_1 be the contribution from the case of $|Y| = 2$:

$$H'_1 = \sum_{\{u,v\} \subset \Lambda} \left(\frac{p_1}{|\Lambda|} \right)^2 \sum_{\substack{R_u \in \mathcal{T}_u \setminus \mathcal{T}_o \\ R_v \in \mathcal{T}_v \setminus \mathcal{T}_o}} \left(\frac{p_1}{|\Lambda|} \right)^{|E_{R_u}| + |E_{R_v}|} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}}. \tag{3.38}$$

By subadditivity, we already know that $H'_1 = O(\beta)$ for $d > 4$. By changing the order of sums, we can rewrite $H_1 - H'_1$ as

$$\begin{aligned}
 H_1 - H'_1 &= \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 3)}} \left(\frac{p_1}{|\Lambda|} \right)^{|Y|} \prod_{y \in Y} \sum_{R_y \in \mathcal{T}_y \setminus \mathcal{T}_o} \left(\frac{p_1}{|\Lambda|} \right)^{|E_{R_y}|} \sum_{\{u,v\} \subset Y} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}} \\
 &= \underbrace{\sum_{\{u,v\} \subset \Lambda} \left(\frac{p_1}{|\Lambda|} \right)^2 \sum_{\substack{R_u \in \mathcal{T}_u \setminus \mathcal{T}_o \\ R_v \in \mathcal{T}_v \setminus \mathcal{T}_o}} \left(\frac{p_1}{|\Lambda|} \right)^{|E_{R_u}| + |E_{R_v}|} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}}}_{H'_1} \\
 &\quad \times \sum_{\substack{Y' \subset \Lambda \setminus \{u,v\} \\ (|Y'| \geq 1)}} \left(\frac{p_1}{|\Lambda|} \right)^{|Y'|} \prod_{y' \in Y'} \underbrace{\sum_{R_{y'} \in \mathcal{T}_{y'} \setminus \mathcal{T}_o} \left(\frac{p_1}{|\Lambda|} \right)^{|E_{R_{y'}}|}}_{g_{p_1 - \tau_{p_1}}(y')}. \tag{3.39}
 \end{aligned}$$

Similarly to the proof of Lemma 3.2, we can show that the last line is estimated as

$$\begin{aligned}
 & \sum_{\substack{Y' \subset \Lambda \setminus \{u,v\} \\ (|Y'| \geq 1)}} \left(\frac{1}{|\Lambda|}\right)^{|Y'|} \prod_{y' \in Y'} \left(1 - \frac{\tau_{p_1}(y')}{g_{p_1}}\right) \\
 &= \left(1 + \frac{1}{|\Lambda|}\right)^{|\Lambda|-2} - 1 + \sum_{\substack{Y' \subset \Lambda \setminus \{u,v\} \\ (|Y'| \geq 1)}} \left(\frac{1}{|\Lambda|}\right)^{|Y'|} \left(\prod_{y' \in Y'} \left(1 - \frac{\tau_{p_1}(y')}{g_{p_1}}\right) - 1\right) \\
 &= e - 1 + O(\beta).
 \end{aligned} \tag{3.40}$$

Therefore,

$$H_1 - H'_1 = H'_1(e - 1 + O(\beta)), \tag{3.41}$$

or equivalently

$$H_1 = eH'_1 + O(\beta^2). \tag{3.42}$$

Next we investigate H'_1 . To do so, we first rewrite $\mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}}$ in (3.38) by introducing order among vertices in \mathbb{Z}^d . For a vertex set V and an element $x \in V$, we denote by $V^{<x}$ the set of vertices in V that are earlier than x in that order. By identifying the earliest element x among V_{R_u} that is also in V_{R_v} (so that $V_{R_u}^{<x} \cap V_{R_v} = \emptyset$), we can rewrite $\mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}}$ as

$$\begin{aligned}
 \mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}} &= \sum_{x \in V_{R_u}} \mathbb{1}_{\{x \in V_{R_v}\}} \mathbb{1}_{\{V_{R_u}^{<x} \cap V_{R_v} = \emptyset\}} \\
 &= \sum_{x \in \mathbb{Z}^d} \mathbb{1}_{\{x \in V_{R_u} \cap V_{R_v}\}} - \sum_{x \in V_{R_u} \cap V_{R_v}} \left(1 - \mathbb{1}_{\{V_{R_u}^{<x} \cap V_{R_v} = \emptyset\}}\right).
 \end{aligned} \tag{3.43}$$

Let H''_1 be the contribution from the first sum in the last line:

$$\begin{aligned}
 H''_1 &= \sum_{\{u,v\} \subset \Lambda} \left(\frac{p_1}{|\Lambda|}\right)^2 \sum_{x \in \mathbb{Z}^d} \sum_{\substack{R_u \in \mathcal{T}_{u,x} \setminus \mathcal{T}_o \\ R_v \in \mathcal{T}_{v,x} \setminus \mathcal{T}_o}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_u}| + |E_{R_v}|} \\
 &= \sum_{\{u,v\} \subset \Lambda} \left(\frac{p_1}{|\Lambda|}\right)^2 \sum_{x \in \mathbb{Z}^d} \left(\tau_{p_1}(u-x) - \tau_{p_1}^{(3)}(o, u, x)\right) \left(\tau_{p_1}(v-x) - \tau_{p_1}^{(3)}(o, v, x)\right) \\
 &\stackrel{(2.1)}{=} \sum_{\{u,v\} \subset \Lambda} \left(\frac{1}{|\Lambda|}\right)^2 \sum_{x \in \mathbb{Z}^d} \left(\frac{\tau_{p_1}(u-x)}{g_{p_1}} - \frac{\tau_{p_1}^{(3)}(o, u, x)}{g_{p_1}}\right) \left(\frac{\tau_{p_1}(v-x)}{g_{p_1}} - \frac{\tau_{p_1}^{(3)}(o, v, x)}{g_{p_1}}\right),
 \end{aligned} \tag{3.44}$$

where $\tau_p^{(3)}(o, u, x)$ is a 3-point function, defined as

$$\tau_p^{(3)}(o, u, x) = \sum_{T \in \mathcal{T}_{o,u,x}} W_p(T). \tag{3.45}$$

We will later show that

$$H''_2 := H''_1 - H'_1 = O(\beta^2). \tag{3.46}$$

Finally we investigate H_1' . The dominant contribution to H_1' comes from the product of 2-point functions:

$$\begin{aligned}
 H_1'' &:= \sum_{\{u,v\} \subset \Lambda} \left(\frac{1}{|\Lambda|}\right)^2 \sum_{x \in \mathbb{Z}^d} \frac{\tau_{p_1}(u-x)}{g_{p_1}} \frac{\tau_{p_1}(v-x)}{g_{p_1}} \\
 &= 2 \sum_{\{u,v\} \subset \Lambda} \left(\frac{1}{|\Lambda|}\right)^2 \frac{\tau_{p_1}(u-v)}{g_{p_1}} + \sum_{\{u,v\} \subset \Lambda} \left(\frac{1}{|\Lambda|}\right)^2 \sum_{x \neq u,v} \frac{\tau_{p_1}(u-x)}{g_{p_1}} \frac{\tau_{p_1}(v-x)}{g_{p_1}}, \tag{3.47}
 \end{aligned}$$

where we have used the identity $\tau_p(o) = g_p$. We will later show that the other contribution to H_1' which involves 3-point functions is estimated as

$$H_2'' := H_1' - H_1'' = O(\beta^2). \tag{3.48}$$

By Lemma 3.3, the first term in (3.47) is estimated as

$$\begin{aligned}
 &2 \sum_{\{u,v\} \subset \Lambda} \left(\frac{1}{|\Lambda|}\right)^2 \frac{\tau_{p_1}(u-v)}{g_{p_1}} \\
 &= \sum_{\substack{u,v \in \Lambda \\ (u \neq v)}} \left(\frac{1}{|\Lambda|}\right)^2 S_{\geq 1}(u-v) + \sum_{\substack{u,v \in \Lambda \\ (u \neq v)}} \left(\frac{1}{|\Lambda|}\right)^2 \left(\frac{\tau_{p_1}(u-v)}{g_{p_1}} - S_{\geq 1}(u-v)\right) \\
 &= S_{\geq 3}(o) - \underbrace{\frac{1}{|\Lambda|} S_{\geq 1}(o)}_{O(\beta^2) \text{ for } d > 2} + \underbrace{\sum_{\substack{u,v \in \Lambda \\ (u \neq v)}} \left(\frac{1}{|\Lambda|}\right)^2 \left(\frac{\tau_{p_1}(u-v)}{g_{p_1}} - S_{\geq 1}(u-v)\right)}_{\stackrel{(3.17)-(3.18)}{\leq} o \cdot \text{diamond}} \tag{3.49}
 \end{aligned}$$

where a gap next to the origin in the last diagram represents $1/|\Lambda|$. By translation invariance and (3.24)–(3.25), the last term is bounded above by

$$o \cdot \text{diamond} = \sum_{\substack{y \in \mathbb{Z}^d \\ y \neq o}} \underbrace{\text{diamond}}_{\leq \|S_{\geq 1}^{*2}\|_\infty} \leq \|S_{\geq 1}^{*2}\|_\infty (S_{\geq 0}^{*2} * S_{\geq 1})(o) = O(\beta^2). \tag{3.50}$$

Similarly, the second term in (3.47) is estimated as

$$\begin{aligned}
 &\sum_{\{u,v\} \subset \Lambda} \left(\frac{1}{|\Lambda|}\right)^2 \sum_{x \neq u,v} \frac{\tau_{p_1}(u-x)}{g_{p_1}} \frac{\tau_{p_1}(v-x)}{g_{p_1}} \\
 &= \frac{1}{2} S_{\geq 2}^{*2}(o) - \underbrace{\frac{1}{2|\Lambda|} S_{\geq 1}^{*2}(o)}_{O(\beta^2) \text{ for } d > 4} \\
 &+ \frac{1}{2} \sum_{\substack{u,v \in \Lambda \\ u \neq v}} \left(\frac{1}{|\Lambda|}\right)^2 \sum_{x \neq u,v} \underbrace{\left(\frac{\tau_{p_1}(u-x)}{g_{p_1}} \frac{\tau_{p_1}(v-x)}{g_{p_1}} - S_{\geq 1}(u-x) S_{\geq 1}(v-x)\right)}_{\leq 2 \cdot \text{diagram}}. \tag{3.51}
 \end{aligned}$$

By Lemma 3.3, (3.19) and the translation invariance, the last term of (3.51) is bounded above by

$$\begin{aligned}
 & \text{Diagram 1} + \frac{1}{2} \text{Diagram 2} = \sum_{\substack{y \in \mathbb{Z}^d \\ y \neq o}} \underbrace{\text{Diagram 3}}_{\leq \|S_{\geq 1}^{*3}\|_\infty} + \frac{1}{2} \sum_{\substack{y, z \in \mathbb{Z}^d \\ y, z \neq o}} \underbrace{\text{Diagram 4}}_{\leq \|S_{\geq 0}^{*2} * S_{\geq 1}^{*2}\|_\infty} \\
 & \leq \underbrace{\|S_{\geq 1}^{*3}\|_\infty (S_{\geq 0}^{*2} * S_{\geq 1}^{*2})(o)}_{O(\beta^2) \text{ for } d > 6} + \frac{1}{2} \underbrace{\|S_{\geq 0}^{*2} * S_{\geq 1}^{*2}\|_\infty (S_{\geq 0}^{*2} * S_{\geq 1}^{*2})(o)^2}_{O(\beta^3) \text{ for } d > 8} \\
 & = O(\beta^2).
 \end{aligned} \tag{3.52}$$

Therefore,

$$\begin{aligned}
 H_1'' &= S_{\geq 3}(o) + \frac{1}{2} S_{\geq 2}^{*2}(o) + O(\beta^2) = \sum_{n=3}^\infty D^{*n}(o) + \frac{1}{2} \sum_{n,m=2}^\infty D^{*(n+m)}(o) + O(\beta^2) \\
 &= \sum_{n=3}^\infty \frac{n-1}{2} D^{*n}(o) + O(\beta^2).
 \end{aligned} \tag{3.53}$$

Summarizing all the above estimates, we arrive at

$$\begin{aligned}
 H &\stackrel{(3.37)}{=} H_1 + O(\beta^2) \stackrel{(3.42)}{=} eH_1' + O(\beta^2) \stackrel{(3.46)}{=} eH_1'' + O(\beta^2) \stackrel{(3.48)}{=} eH_1''' + O(\beta^2) \\
 &\stackrel{(3.53)}{=} e \sum_{n=3}^\infty \frac{n-1}{2} D^{*n}(o) + O(\beta^2),
 \end{aligned} \tag{3.54}$$

as required. It remains to show (3.37), (3.46) and (3.48). □

Proof of (3.37): bounding H_2 . First we recall that H_2 is the contribution from the second sum on the right of (3.35):

$$\begin{aligned}
 H_2 &= \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 3)}} \left(\frac{p_1}{|\Lambda|} \right)^{|Y|} \prod_{y \in Y} \sum_{R_y \in \mathcal{T}_y \setminus \mathcal{T}_o} \left(\frac{p_1}{|\Lambda|} \right)^{|E_{R_y}|} \sum_{\{u, v\} \subset Y} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}} \\
 &\quad \times \left(1 - \prod_{\substack{\{u', v'\} \subset Y \\ (\{u', v'\} < \{u, v\})}} \mathbb{1}_{\{V_{R_{u'}} \cap V_{R_{v'}} = \emptyset\}} \right),
 \end{aligned} \tag{3.55}$$

which is nonnegative. Since we get an upper bound

$$1 - \prod_{\substack{\{u', v'\} \subset Y \\ (\{u', v'\} < \{u, v\})}} \mathbb{1}_{\{V_{R_{u'}} \cap V_{R_{v'}} = \emptyset\}} \leq \sum_{\substack{\{u', v'\} \subset Y \\ (\{u', v'\} < \{u, v\})}} \mathbb{1}_{\{V_{R_{u'}} \cap V_{R_{v'}} \neq \emptyset\}} \tag{3.56}$$

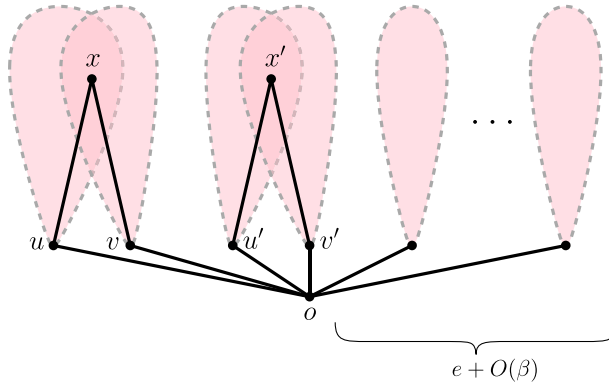


Figure 6. Schematic representation of $H_{2,4}$.

in a same manner as (3.35), we can bound H_2 as

$$\begin{aligned}
 H_2 &\leq \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 3)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y|} \prod_{y \in Y} \sum_{R_y \in \mathcal{T}_y \setminus \mathcal{T}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|} \sum_{\substack{\{u,v\} \subset Y \\ \{u',v'\} \subset Y \\ (\{u',v'\} \prec \{u,v\})}} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}} \mathbb{1}_{\{V_{R_{u'}} \cap V_{R_{v'}} \neq \emptyset\}} \\
 &= \frac{1}{2} \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 3)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y|} \prod_{y \in Y} \sum_{R_y \in \mathcal{T}_y \setminus \mathcal{T}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|} \sum_{\substack{\{u,v\} \subset Y \\ \{u',v'\} \subset Y \\ (\{u',v'\} \neq \{u,v\})}} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}} \mathbb{1}_{\{V_{R_{u'}} \cap V_{R_{v'}} \neq \emptyset\}}.
 \end{aligned} \tag{3.57}$$

Since $\{u, v\} \neq \{u', v'\}$, the union $\{u, v\} \cup \{u', v'\}$ consists of either three or four distinct vertices. We denote the contribution from the former by $H_{2,3}$, and that from the latter by $H_{2,4}$ and then we obtain

$$H_2 \leq \frac{1}{2}(H_{2,3} + H_{2,4}). \tag{3.58}$$

First we investigate $H_{2,4}$, which is bounded as (see Fig. 6)

$$\begin{aligned}
 H_{2,4} &= \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 4)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y|} \prod_{y \in Y} \sum_{R_y \in \mathcal{T}_y \setminus \mathcal{T}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|} \sum_{\substack{u,v,u',v' \in Y \\ (\text{distinct})}} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}} \mathbb{1}_{\{V_{R_{u'}} \cap V_{R_{v'}} \neq \emptyset\}} \\
 &= \sum_{\substack{u,v,u',v' \in \Lambda \\ (\text{distinct})}} \left(\frac{p_1}{|\Lambda|}\right)^4 \sum_{\substack{R_u \in \mathcal{T}_u \setminus \mathcal{T}_o \\ R_v \in \mathcal{T}_v \setminus \mathcal{T}_o \\ R_{u'} \in \mathcal{T}_{u'} \setminus \mathcal{T}_o \\ R_{v'} \in \mathcal{T}_{v'} \setminus \mathcal{T}_o}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_u}| + |E_{R_v}| + |E_{R_{u'}}| + |E_{R_{v'}}|} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}} \mathbb{1}_{\{V_{R_{u'}} \cap V_{R_{v'}} \neq \emptyset\}} \\
 &\quad \times \left(1 + \sum_{\substack{Y' \subset \Lambda \setminus \{u,v,u',v'\} \\ (|Y'| \geq 1)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y'|} \prod_{y \in Y'} \sum_{R_y \in \mathcal{T}_y \setminus \mathcal{T}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|}\right).
 \end{aligned} \tag{3.59}$$

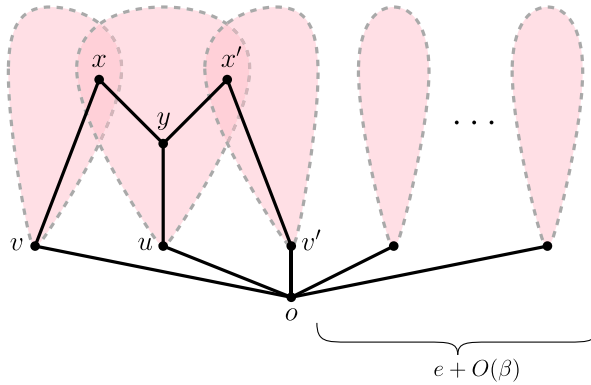


Figure 7. Schematic representation of $H_{2,3}$.

By (2.1) (i.e., $p_1 g_{p_1} = 1$), the last line is equal to

$$1 + \sum_{\substack{Y' \subset \Lambda \setminus \{u, v, u', v'\} \\ (|Y'| \geq 1)}} \left(\frac{1}{|\Lambda|}\right)^{|Y'|} = \left(1 + \frac{1}{|\Lambda|}\right)^{|\Lambda|-4} = e + O(\beta). \tag{3.60}$$

Then, by ignoring the constraint that $\{u, v\}$ and $\{u', v'\}$ are disjoint pairs and using the trivial inequality $\mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}} \leq \sum_x \mathbb{1}_{\{x \in V_{R_u} \cap V_{R_v}\}}$ as well as the relation $\mathcal{T}_{u,x} \setminus \mathcal{T}_o \subset \mathcal{T}_{u,x}$, $H_{2,4}$ is further bounded above as

$$\begin{aligned} H_{2,4} &\leq (e + O(\beta)) \left(\sum_{\{u,v\} \subset \Lambda} \left(\frac{p_1}{|\Lambda|}\right)^2 \sum_{\substack{R_u \in \mathcal{T}_u \setminus \mathcal{T}_o \\ R_v \in \mathcal{T}_v \setminus \mathcal{T}_o}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_u}| + |E_{R_v}|} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}} \right)^2 \\ &\leq (e + O(\beta)) \left(\sum_{\{u,v\} \subset \Lambda} \left(\frac{p_1}{|\Lambda|}\right)^2 \underbrace{\sum_{x \in \mathbb{Z}^d} \sum_{\substack{R_u \in \mathcal{T}_{u,x} \\ R_v \in \mathcal{T}_{v,x}}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_u}| + |E_{R_v}|}}_{\tau_{p_1}(u-x) \tau_{p_1}(v-x)} \right)^2 \\ &\stackrel{(2.1)}{=} (e + O(\beta)) \left(\sum_{\{u,v\} \subset \Lambda} \left(\frac{1}{|\Lambda|}\right)^2 \sum_{x \in \mathbb{Z}^d} \frac{\tau_{p_1}(u-x)}{g_{p_1}} \frac{\tau_{p_1}(v-x)}{g_{p_1}} \right)^2. \tag{3.61} \end{aligned}$$

Finally, by using $\tau_{p_1}(u-x)/g_{p_1} \leq S_{\geq 1}(u-x)$ for $x \neq u$ [cf., the first inequality in (3.17)] and $\tau_p(o) = g_p$ for $x = u$, we arrive at

$$H_{2,4} \leq (e + O(\beta)) \left(\sum_{u,v \in \Lambda} \left(\frac{1}{|\Lambda|}\right)^2 S_{\geq 0}^{*2}(u-v) \right)^2 \leq (e + O(\beta)) S_{\geq 1}^{*2}(o)^2 \stackrel{d \geq 4}{=} O(\beta^2). \tag{3.62}$$

Next we investigate $H_{2,3}$, which is bounded in a similar way to (3.61) as (see Fig. 7)

$$\begin{aligned}
 H_{2,3} &= \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 3)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y|} \prod_{y \in Y} \sum_{R_y \in \mathcal{T}_y \setminus \mathcal{T}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|} \sum_{\substack{u, v, v' \in Y \\ (\text{distinct})}} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \emptyset\}} \mathbb{1}_{\{V_{R_u} \cap V_{R_{v'}} \neq \emptyset\}} \\
 &\leq (e + O(\beta)) \sum_{\substack{u, v, v' \in \Lambda \\ (\text{distinct})}} \left(\frac{p_1}{|\Lambda|}\right)^3 \sum_{x, x' \in \mathbb{Z}^d} \sum_{\substack{R_u \in \mathcal{T}_{u, x'} \\ R_v \in \mathcal{T}_{v, x} \\ R_{v'} \in \mathcal{T}_{v', x'}}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_u}| + |E_{R_v}| + |E_{R_{v'}}|} \\
 &= (e + O(\beta)) g_{p_1}^2 \sum_{\substack{u, v, v' \in \Lambda \\ (\text{distinct})}} \left(\frac{1}{|\Lambda|}\right)^3 \sum_{x, x' \in \mathbb{Z}^d} \frac{\tau_{p_1}^{(3)}(u, x, x')}{g_{p_1}^3} \frac{\tau_{p_1}(v - x)}{g_{p_1}} \frac{\tau_{p_1}(v' - x')}{g_{p_1}}. \tag{3.63}
 \end{aligned}$$

Due to submultiplicativity, we can bound $\tau_p^{(3)}(u, x, x')$ as

$$\tau_p^{(3)}(u, x, x') \leq \sum_{y \in \mathbb{Z}^d} \tau_p(u - y) \tau_p(x - y) \tau_p(x' - y). \tag{3.64}$$

Then, by using $\tau_{p_1}(u - x)/g_{p_1} \leq S_{\geq 1}(u - x)$ for $x \neq u$ and $\tau_p(o) = g_p$ for $x = u$, we can bound the sum in (3.63) as

$$\begin{aligned}
 &\sum_{\substack{u, v, v' \in \Lambda \\ (\text{distinct})}} \left(\frac{1}{|\Lambda|}\right)^3 \sum_{y, x, x' \in \mathbb{Z}^d} \frac{\tau_{p_1}(u - y)}{g_{p_1}} \frac{\tau_{p_1}(x - y)}{g_{p_1}} \frac{\tau_{p_1}(x' - y)}{g_{p_1}} \frac{\tau_{p_1}(v - x)}{g_{p_1}} \frac{\tau_{p_1}(v' - x')}{g_{p_1}} \\
 &\leq \sum_{y, x, x' \in \mathbb{Z}^d} S_{\geq 0}(x - y) S_{\geq 0}(x' - y) \underbrace{\sum_{u, v, v' \in \Lambda} \left(\frac{1}{|\Lambda|}\right)^3 S_{\geq 0}(u - y) S_{\geq 0}(v - x) S_{\geq 0}(v' - x')}_{S_{\geq 1}(y) S_{\geq 1}(x) S_{\geq 1}(x')} \\
 &= \sum_{y \in \mathbb{Z}^d} (S_{\geq 0} * S_{\geq 1})(y)^2 S_{\geq 1}(y) \\
 &\leq \|S_{\geq 0} * S_{\geq 1}\|_{\infty} (S_{\geq 0} * S_{\geq 1}^{*2})(o) \stackrel{d > 6}{=} O(\beta^2). \tag{3.65}
 \end{aligned}$$

This together with (3.58) and (3.62) implies

$$H_2 \leq \frac{1}{2}(H_{2,3} + H_{2,4}) = O(\beta^2), \tag{3.66}$$

as required. □

Proof of (3.46): bounding H_2' . First we recall that H_2' is the contribution to H_1' from the second sum on the right of (3.43):

$$H_2' = \sum_{\{u, v\} \subset \Lambda} \left(\frac{p_1}{|\Lambda|}\right)^2 \sum_{x \in \mathbb{Z}^d} \sum_{\substack{R_u \in \mathcal{T}_{u, x} \setminus \mathcal{T}_o \\ R_v \in \mathcal{T}_{v, x} \setminus \mathcal{T}_o}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_u}| + |E_{R_v}|} \left(1 - \mathbb{1}_{\{V_{R_u}^{<x} \cap V_{R_v} = \emptyset\}}\right). \tag{3.67}$$

Notice that

$$1 - \mathbb{1}_{\{V_{R_u}^{<x} \cap V_{R_v} = \emptyset\}} \leq \sum_{x' \in \mathbb{Z}^d \setminus \{x\}} \mathbb{1}_{\{x' \in V_{R_u} \cap V_{R_v}\}}. \tag{3.68}$$

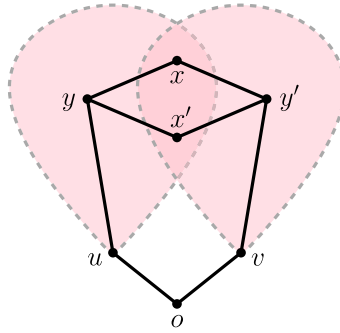


Figure 8. Schematic representation of the bound on H_2' due to (3.68).

By the inclusion relation $\mathcal{T}_{u,x,x'} \setminus \mathcal{T}_o \subset \mathcal{T}_{u,x,x'}$ and using (3.64), (2.1) and (3.17), we can bound H_2' as (see Fig. 8)

$$\begin{aligned}
 H_2' &\leq \sum_{\{u,v\} \subset \Lambda} \left(\frac{p_1}{|\Lambda|} \right)^2 \sum_{\substack{x,x' \in \mathbb{Z}^d \\ (x \neq x')}} \sum_{\substack{R_u \in \mathcal{T}_{u,x,x'} \\ R_v \in \mathcal{T}_{v,x,x'}}} \left(\frac{p_1}{|\Lambda|} \right)^{|E_{R_u}|+|E_{R_v}|} \\
 &\quad \underbrace{\tau_{p_1}^{(3)}(u,x,x') \tau_{p_1}^{(3)}(v,x,x')} \\
 &\leq g_{p_1}^4 \sum_{\substack{x,x',y,y' \in \mathbb{Z}^d \\ (x \neq x')}} \frac{\tau_{p_1}(y-x)}{g_{p_1}} \frac{\tau_{p_1}(y-x')}{g_{p_1}} \frac{\tau_{p_1}(y'-x)}{g_{p_1}} \frac{\tau_{p_1}(y'-x')}{g_{p_1}} \\
 &\quad \times \frac{1}{2} \sum_{u,v \in \Lambda} \left(\frac{1}{|\Lambda|} \right)^2 \frac{\tau_{p_1}(u-y)}{g_{p_1}} \frac{\tau_{p_1}(v-y')}{g_{p_1}} \\
 &\leq \frac{g_{p_1}^4}{2} \sum_{\substack{x,x',y,y' \in \mathbb{Z}^d \\ (x \neq x')}} S_{\geq 0}(y-x) S_{\geq 0}(y-x') S_{\geq 0}(y'-x) S_{\geq 0}(y'-x') S_{\geq 1}(y) S_{\geq 1}(y'). \tag{3.69}
 \end{aligned}$$

Shifting the variables by $-x'$ and changing the variables $x-x', y-x', y'-x'$ to the new ones w, z, z' , respectively, we can rewrite the above sum as

$$\sum_{\substack{w,z,z' \in \mathbb{Z}^d \\ (w \neq o)}} S_{\geq 0}(z-w) S_{\geq 0}(z) S_{\geq 0}(z'-w) S_{\geq 0}(z') \underbrace{\sum_{x' \in \mathbb{Z}^d} S_{\geq 1}(z+x') S_{\geq 1}(z'+x')}_{S_{\geq 1}^{*2}(z-z')} \tag{3.70}$$

which is bounded above by

$$\begin{aligned}
 \|S_{\geq 1}^{*2}\|_{\infty} \sum_{w \neq o} S_{\geq 0}^{*2}(w)^2 &= \|S_{\geq 1}^{*2}\|_{\infty} \sum_{w \neq o} \left(\sum_{n=1}^{\infty} (n+1) D^{*n}(w) \right)^2 \\
 &\leq \|S_{\geq 1}^{*2}\|_{\infty} \sum_{t=2}^{\infty} D^{*t}(o) \underbrace{\sum_{n=1}^{t-1} (n+1)(t-n+1)}_{O(t^3)} \stackrel{d \geq 8}{\cong} O(\beta^2), \tag{3.71}
 \end{aligned}$$

as required. □

Proof of (3.48): bounding H_2'' . First we recall that H_2'' is the contribution to H_1'' which involves 3-point functions [cf., (3.44)]:

$$H_2'' = \sum_{\{u,v\} \subset \Lambda} \left(\frac{1}{|\Lambda|} \right)^2 \sum_{x \in \mathbb{Z}^d} \frac{\tau_{p_1}^{(3)}(o, u, x)}{g_{p_1}} \left(\frac{\tau_{p_1}^{(3)}(o, v, x)}{g_{p_1}} - 2 \frac{\tau_{p_1}(v-x)}{g_{p_1}} \right). \tag{3.72}$$

By (2.1), (3.64) and (3.17), we can readily conclude that

$$\begin{aligned} |H_2''| &\leq \frac{g_{p_1}^2}{2} \sum_{u,v \in \Lambda} \left(\frac{1}{|\Lambda|} \right)^2 \sum_{x,y \in \mathbb{Z}^d} \frac{\tau_{p_1}(y)}{g_{p_1}} \frac{\tau_{p_1}(y-u)}{g_{p_1}} \frac{\tau_{p_1}(y-x)}{g_{p_1}} \\ &\quad \times \left(g_{p_1}^2 \sum_{z \in \mathbb{Z}^d} \frac{\tau_{p_1}(z)}{g_{p_1}} \frac{\tau_{p_1}(z-v)}{g_{p_1}} \frac{\tau_{p_1}(z-x)}{g_{p_1}} + 2 \frac{\tau_{p_1}(v-x)}{g_{p_1}} \right) \\ &\leq g_{p_1}^2 \sum_{x,y \in \mathbb{Z}^d} S_{\geq 0}(y) S_{\geq 1}(y) S_{\geq 0}(y-x) \\ &\quad \times \left(\frac{g_{p_1}^2}{2} \sum_{z \in \mathbb{Z}^d} S_{\geq 0}(z) S_{\geq 1}(z) S_{\geq 0}(z-x) + S_{\geq 1}(x) \right) \\ &= g_{p_1}^2 \sum_{y \in \mathbb{Z}^d} S_{\geq 0}(y) S_{\geq 1}(y) \left(\frac{g_{p_1}^2}{2} \sum_{z \in \mathbb{Z}^d} S_{\geq 0}(z) S_{\geq 1}(z) S_{\geq 0}^*(y-z) + (S_{\geq 0} * S_{\geq 1})(y) \right) \\ &\leq g_{p_1}^2 \underbrace{(S_{\geq 0} * S_{\geq 1})(o)}_{O(\beta) \text{ for } d > 4} \left(\frac{g_{p_1}^2}{2} \underbrace{(S_{\geq 0} * S_{\geq 1})(o)}_{O(1) \text{ for } d > 4} \underbrace{\|S_{\geq 0}^*\|_\infty}_{O(1) \text{ for } d > 4} + \underbrace{\|S_{\geq 0} * S_{\geq 1}\|_\infty}_{O(\beta) \text{ for } d > 4} \right) \\ &= O(\beta^2), \end{aligned} \tag{3.73}$$

as required. □

4. Difference between lattice trees and lattice animals

Finally we prove Theorem 1.1 for lattice animals. Recall that, by Lemma 2.1, it suffices to investigate $p_1 = 1/g_{p_1}$ [cf., (2.1)]. The following is the key lemma:

Lemma 4.1. *For lattice animals with $d > 8$ and $L \uparrow \infty$,*

$$g_{p_1} = e \left(1 - \sum_{n=2}^{\infty} \frac{n+1}{2} D^{*n}(o) \right) + \frac{1}{2} S_{\geq 3}(o) + O(\beta^2). \tag{4.1}$$

Consequently,

$$p_1 = \frac{1}{e} + \sum_{n=2}^{\infty} \frac{n+1}{2e} D^{*n}(o) - \frac{1}{2e^2} S_{\geq 3}(o) + O(\beta^2). \tag{4.2}$$

Proof. As a first step, we want a similar decomposition to (3.3) for lattice animals. To do so, we identify the connected neighbours Y of the origin, just as done in (3.3). Then, we introduce $\Gamma(Y)$, which is the set of all partitions of Y . For example, if $Y = \{1, 2, 3\}$, then

$$\Gamma(Y) = \left\{ \{Y\}, \{\{1, 2\}, \{3\}\}, \{\{1, 3\}, \{2\}\}, \{\{1\}, \{2, 3\}\}, \{\{1\}, \{2\}, \{3\}\} \right\}. \tag{4.3}$$

For a partition $\gamma \in \Gamma(Y)$, we denote by $|\gamma|$ the number of sets in γ , so that $\gamma = \{\gamma_j\}_{j=1}^{|\gamma|}$. We can rewrite g_{p_1} as

$$g_{p_1} = 1 + \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 1)}} \left(\frac{p_1}{|\Lambda|} \right)^{|Y|} \sum_{\gamma \in \Gamma(Y)} \prod_{j=1}^{|\gamma|} \sum_{R_j \in \mathcal{A}_{\gamma_j} \setminus \mathcal{A}_o} \left(\frac{p_1}{|\Lambda|} \right)^{|E_{R_j}|} \prod_{i < j} \mathbb{1}_{\{V_{R_i} \cap V_{R_j} = \emptyset\}}. \tag{4.4}$$

The contribution from the maximum partition $\bar{\gamma} = \{\{y\}\}_{y \in Y}$ (i.e., $|\bar{\gamma}| = |Y|$) is equal to (3.3) (with \mathcal{T} replaced by \mathcal{A}) and can be decomposed into G and H as in (3.5) (with R_y regarded as animals instead of trees). Let I be the contribution from the remaining partitions $\gamma \in \Gamma(Y)$ with $|\gamma| < |Y|$, which is zero for lattice trees:

$$\begin{aligned} I &= g_{p_1} - (G - H) \\ &= \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 2)}} \left(\frac{p_1}{|\Lambda|} \right)^{|Y|} \sum_{\substack{\gamma \in \Gamma(Y) \\ (|\gamma| < |Y|)}} \prod_{j=1}^{|\gamma|} \sum_{R_j \in \mathcal{A}_{\gamma_j} \setminus \mathcal{A}_o} \left(\frac{p_1}{|\Lambda|} \right)^{|E_{R_j}|} \prod_{i < j} \mathbb{1}_{\{V_{R_i} \cap V_{R_j} = \emptyset\}}. \end{aligned} \tag{4.5}$$

To evaluate G, H and I for lattice animals, we cannot apply Lemma 3.3, which is a powerful tool for lattice trees to identify the coefficients of β as well as to estimate the error terms of $O(\beta^2)$. For the latter purpose for lattice animals, we will use the infrared bound (2.9) (and monotonicity in p , i.e., $\tau_{p_1} \leq \tau_{p_c}$); for the former purpose, we will use the following bounds that correspond to (3.26), (3.49) and (3.51), respectively:

Lemma 4.2. *For lattice animals with $d > 8$ and $L \uparrow \infty$,*

$$\left| \sum_{u \in \Lambda} \frac{1}{|\Lambda|} \frac{\tau_{p_1}(u)}{g_{p_1}} - S_{\geq 2}(o) \right| = O(\beta^2), \tag{4.6}$$

$$\left| \sum_{\{u,v\} \subset \Lambda} \left(\frac{1}{|\Lambda|} \right)^2 \frac{\tau_{p_1}(u-v)}{g_{p_1}} - \frac{1}{2} S_{\geq 3}(o) \right| = O(\beta^2), \tag{4.7}$$

$$\left| \sum_{\{u,v\} \subset \Lambda} \left(\frac{1}{|\Lambda|} \right)^2 \sum_{x \neq u,v} \frac{\tau_{p_1}(u-x)}{g_{p_1}} \frac{\tau_{p_1}(x-v)}{g_{p_1}} - \frac{1}{2} S_{\geq 2}^{*2}(o) \right| = O(\beta^2). \tag{4.8}$$

We will prove Lemma 4.2 after the proof of Lemma 4.1 is completed.

Now we resume the proof of Lemma 4.1 assuming the bounds in Lemma 4.2. First we recall $G = G_0 + G_1 + G_2$ [cf., (3.14)], where G_0 is independent of the models and estimated as (3.12); G_1 is defined as (3.15) and here we use (4.6) to show (3.27); G_2 is defined as (3.16) and obeys the same bound as (3.22). As a result, Lemma 3.2 also holds for lattice animals. Similarly, we can show $H = e(H_1'' + H_2'' - H_2' - H_2) + O(\beta^2)$ [cf., (3.37), (3.42), (3.46) and (3.48)], where H_1'' is defined in (3.47) and here we use (4.7)–(4.8) to show (3.53); H_2 is bounded by $H_{2,3} + H_{2,4}$, and $H_{2,3}$ and $H_{2,4}$ are further bounded as (3.59)–(3.61) and (3.63)–(3.65) (with \mathcal{T} replaced by \mathcal{A}), and

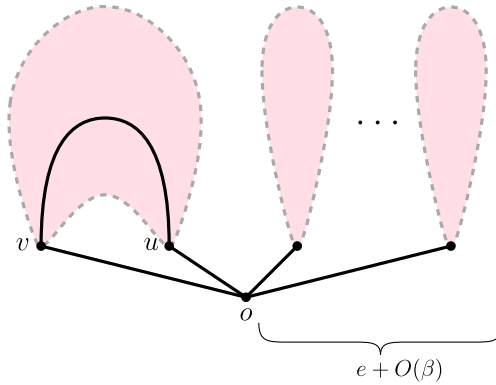


Figure 9. Schematic representation of the dominant contribution to I'_1 .

here we use the infrared bound (2.9) and the convolution bound on power functions [4, Lemma 3.2(i)] to show $H_2 = O(\beta^2)$, such as

$$\begin{aligned} & \sum_{\{u,v\} \subset \Lambda} \left(\frac{1}{|\Lambda|}\right)^2 \sum_{x \in \mathbb{Z}^d} \tau_{p_1}(u-x) \tau_{p_1}(v-x) \\ & \leq \sum_{\{u,v\} \subset \Lambda} \left(\frac{1}{|\Lambda|}\right)^2 \sum_{x \in \mathbb{Z}^d} \frac{O(L^{-2})}{(\|u-x\| \vee L)^{d-2}} \frac{O(L^{-2})}{(\|v-x\| \vee L)^{d-2}} \\ & \leq \sum_{\{u,v\} \subset \Lambda} \left(\frac{1}{|\Lambda|}\right)^2 \frac{O(L^{-4})}{(\|u-v\| \vee L)^{d-4}} = O(\beta^2). \end{aligned} \tag{4.9}$$

Similarly we can show that H'_2 and H''_2 are both $O(\beta^2)$ by using the infrared bound and the convolution bound, instead of bounding τ_{p_1}/g_{p_1} by $S_{\geq 0}$ or $S_{\geq 1}$, just as done for lattice trees. As a result, Lemma 3.4 also holds for lattice animals.

Next we investigate I , which is unique for lattice animals. Let I_1 be the contribution from $\gamma \in \Gamma(Y)$ with $|\gamma| = |Y| - 1$, i.e., consisting of a pair $\{u, v\}$ and $|Y| - 2$ singletons $\{y\}_{y \in Y \setminus \{u,v\}}$:

$$\begin{aligned} I_1 &= \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 2)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y|} \sum_{\{u,v\} \subset Y} \sum_{R \in \mathcal{A}_{u,v} \setminus \mathcal{A}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_R|} \prod_{y \in Y \setminus \{u,v\}} \sum_{R_y \in \mathcal{A}_y \setminus \mathcal{A}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|} \\ & \times \prod_{y \in Y \setminus \{u,v\}} \mathbb{1}_{\{V_R \cap V_{R_y} = \emptyset\}} \prod_{\substack{y,z \in Y \setminus \{u,v\} \\ (y \neq z)}} \mathbb{1}_{\{V_{R_y} \cap V_{R_z} = \emptyset\}}, \end{aligned} \tag{4.10}$$

where an empty product is regarded as 1. The dominant contribution to I_1 , denoted I'_1 , comes from when the last line is replaced by 1. By the tree-graph inequality (3.64), which is also true for lattice animals due to subadditivity, and then using the infrared bound (2.9), it is estimated as (see Fig. 9)

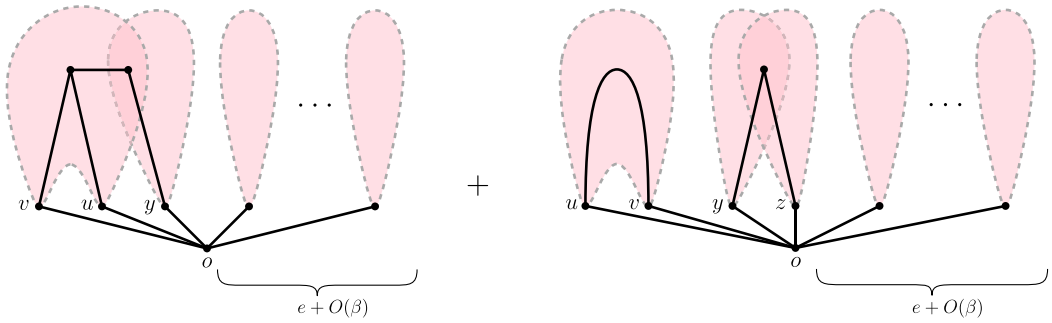


Figure 10. Schematic representation of the bound on (4.12).

$$\begin{aligned}
 I_1 &= \sum_{\{u,v\} \subset \Lambda} \left(\frac{p_1}{|\Lambda|} \right)^2 (\tau_{p_1}(u-v) - \tau_{p_1}^{(3)}(o, u, v)) \underbrace{\sum_{Y' \subset \Lambda \setminus \{u,v\}} \left(\frac{1}{|\Lambda|} \right)^{|Y'|} \prod_{y \in Y'} \left(1 - \frac{\tau_{p_1}(y)}{g_{p_1}} \right)}_{e + O(\beta)} \\
 &= \underbrace{p_1(e + O(\beta))}_{1 + O(\beta)} \sum_{\{u,v\} \subset \Lambda} \left(\frac{1}{|\Lambda|} \right)^2 \left(\frac{\tau_{p_1}(u-v)}{g_{p_1}} - \frac{\tau_{p_1}^{(3)}(o, u, v)}{g_{p_1}} \right) \\
 &= \sum_{\{u,v\} \subset \Lambda} \left(\frac{1}{|\Lambda|} \right)^2 \frac{\tau_{p_1}(u-v)}{g_{p_1}} + O(\beta^2) \\
 &\stackrel{(4.7)}{=} \frac{1}{2} S_{\geq 3}(o) + O(\beta^2). \tag{4.11}
 \end{aligned}$$

On the other hand, by using $1 - ab \leq (1 - a) + (1 - b)$ for any $a, b \in \{0, 1\}$, we can bound the difference $I'_1 - I_1 (\geq 0)$ as

$$\begin{aligned}
 I'_1 - I_1 &\leq \sum_{\{u,v\} \subset \Lambda} \left(\frac{p_1}{|\Lambda|} \right)^2 \sum_{R \in \mathcal{A}_{u,v} \setminus \mathcal{A}_o} \left(\frac{p_1}{|\Lambda|} \right)^{|E_R|} \sum_{Y' \subset \Lambda \setminus \{u,v\}} \left(\frac{p_1}{|\Lambda|} \right)^{|Y'|} \prod_{y \in Y'} \sum_{R_y \in \mathcal{A}_y \setminus \mathcal{A}_o} \left(\frac{p_1}{|\Lambda|} \right)^{|E_{R_y}|} \\
 &\quad \times \left(1 - \prod_{y \in Y'} \mathbb{1}_{\{V_R \cap V_{R_y} = \emptyset\}} + 1 - \prod_{\substack{y,z \in Y' \\ (y \neq z)}} \mathbb{1}_{\{V_{R_y} \cap V_{R_z} = \emptyset\}} \right) \\
 &\leq \sum_{\{u,v\} \subset \Lambda} \left(\frac{p_1}{|\Lambda|} \right)^2 \sum_{R \in \mathcal{A}_{u,v} \setminus \mathcal{A}_o} \left(\frac{p_1}{|\Lambda|} \right)^{|E_R|} \sum_{Y' \subset \Lambda \setminus \{u,v\}} \left(\frac{p_1}{|\Lambda|} \right)^{|Y'|} \prod_{y \in Y'} \sum_{R_y \in \mathcal{A}_y \setminus \mathcal{A}_o} \left(\frac{p_1}{|\Lambda|} \right)^{|E_{R_y}|} \\
 &\quad \times \left(\sum_{y \in Y'} \mathbb{1}_{\{V_R \cap V_{R_y} \neq \emptyset\}} + \sum_{\substack{y,z \in Y' \\ (y \neq z)}} \mathbb{1}_{\{V_{R_y} \cap V_{R_z} \neq \emptyset\}} \right). \tag{4.12}
 \end{aligned}$$

This is $O(\beta^2)$, as the contribution from the former (resp., latter) sum in the last line can be estimated in a similar way to showing $H_{2,3} = O(\beta^2)$ (resp., $H_{2,4} = O(\beta^2)$); see Fig. 10.

As a result,

$$I_1 = \frac{1}{2} S_{\geq 3}(o) + O(\beta^2). \tag{4.13}$$

Finally we estimate the difference $I - I_1$:

$$I - I_1 = \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 3)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y|} \sum_{\substack{\gamma \in \Gamma(Y) \\ (|\gamma| \leq |Y| - 2)}} \prod_{j=1}^{|\gamma|} \sum_{R_j \in \mathcal{A}_{\gamma_j} \setminus \mathcal{A}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_j}|} \prod_{i < j} \mathbb{1}_{\{V_{R_i} \cap V_{R_j} = \emptyset\}}. \tag{4.14}$$

Since $|\gamma| \leq |Y| - 2$, there are two possibilities: (i) there is a set in γ which includes at least three distinct neighbours of the origin, or (ii) there are at least two disjoint sets in γ both of which include exactly two distinct neighbours of the origin. Therefore,

$$\begin{aligned} I - I_1 &\leq \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 3)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y|} \sum_{\gamma \in \Gamma(Y)} \prod_{j=1}^{|\gamma|} \sum_{R_j \in \mathcal{A}_{\gamma_j} \setminus \mathcal{A}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_j}|} \prod_{i < j} \mathbb{1}_{\{V_{R_i} \cap V_{R_j} = \emptyset\}} \\ &\quad \times \left(\mathbb{1}_{\{\exists j, |\gamma_j| \geq 3\}} + \mathbb{1}_{\{\exists i \neq j, |\gamma_i| = |\gamma_j| = 2\}} \right) \\ &= I_3 + I_2, \end{aligned} \tag{4.15}$$

where I_3 and I_2 are the contributions from $\mathbb{1}_{\{\exists j, |\gamma_j| \geq 3\}}$ and $\mathbb{1}_{\{\exists i \neq j, |\gamma_i| = |\gamma_j| = 2\}}$, respectively.

For I_2 , we split the set Y of neighbours of the origin into U, V and $Y' = Y \setminus (U \cup V)$, where $U \cap V = \emptyset$ and $|U| = |V| = 2$. Partially ignoring the avoidance constraint among animals, we can bound I_2 as

$$\begin{aligned} I_2 &\leq \sum_{\substack{U \subset \Lambda \\ (|U|=2)}} \left(\frac{p_1}{|\Lambda|}\right)^2 \sum_{R \in \mathcal{A}_U \setminus \mathcal{A}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_R|} \sum_{\substack{V \subset \Lambda \setminus U \\ (|V|=2)}} \left(\frac{p_1}{|\Lambda|}\right)^2 \sum_{R' \in \mathcal{A}_V \setminus \mathcal{A}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R'}|} \\ &\quad \times \sum_{Y' \subset \Lambda \setminus (U \cup V)} \left(\frac{p_1}{|\Lambda|}\right)^{|Y'|} \sum_{\gamma \in \Gamma(Y')} \prod_{j=1}^{|\gamma|} \sum_{R_j \in \mathcal{A}_{\gamma_j} \setminus \mathcal{A}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_j}|} \prod_{i < j} \mathbb{1}_{\{V_{R_i} \cap V_{R_j} = \emptyset\}}. \end{aligned} \tag{4.16}$$

Notice that the second line is almost identical to g_{p_1} ; the only difference is the domain of summation over Y' , and therefore it is bounded above by g_{p_1} . Since $p_1 g_{p_1} = 1$, we obtain

$$I_2 \leq g_{p_1} \left(\sum_{\{u,v\} \subset \Lambda} \left(\frac{p_1}{|\Lambda|}\right)^2 \underbrace{\sum_{R \in \mathcal{A}_{u,v}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_R|}}_{\tau_{p_1}(u-v)} \right)^2 \stackrel{(4.7)}{\leq} p_1 \left(\frac{1}{2} S_{\geq 3}(o) + O(\beta^2) \right)^2 = O(\beta^2). \tag{4.17}$$

For I_3 , we split the set Y into X and $Y' = Y \setminus X$, where X includes at least 3 distinct vertices $x, y, z \in \Lambda$. Again, by partially ignoring the avoidance constraint among animals, we can bound I_3 as

$$\begin{aligned} I_3 &\leq \sum_{\{x,y,z\} \subset \Lambda} \sum_{\substack{X \subset \Lambda \\ (X \ni x,y,z)}} \left(\frac{p_1}{|\Lambda|}\right)^{|X|} \sum_{R \in \mathcal{A}_X \setminus \mathcal{A}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_R|} \\ &\quad \times \sum_{Y' \subset \Lambda \setminus X} \left(\frac{p_1}{|\Lambda|}\right)^{|Y'|} \sum_{\gamma \in \Gamma(Y')} \prod_{j=1}^{|\gamma|} \sum_{R_j \in \mathcal{A}_{\gamma_j} \setminus \mathcal{A}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_j}|} \prod_{i < j} \mathbb{1}_{\{V_{R_i} \cap V_{R_j} = \emptyset\}}. \end{aligned} \tag{4.18}$$

Notice again that the second line is bounded above by g_{p_1} . Using the relation $\mathcal{A}_X \setminus \mathcal{A}_o \subset \mathcal{A}_{x,y,z}$ and splitting X into $\{x, y, z\}$ and $X' = X \setminus \{x, y, z\}$, we obtain

$$I_3 \leq g_{p_1} \sum_{\{x,y,z\} \subset \Lambda} \left(\frac{p_1}{|\Lambda|}\right)^3 \underbrace{\sum_{R \in \mathcal{A}_{x,y,z}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_R|}}_{\tau_{p_1}^{(3)}(x,y,z)} \underbrace{\sum_{X' \subset \Lambda \setminus \{x,y,z\}} \left(\frac{p_1}{|\Lambda|}\right)^{|X'|}}_{\leq (1+p_1/|\Lambda|)^{|\Lambda|}}. \tag{4.19}$$

By the tree-graph inequality (3.64), we can show that

$$\begin{aligned} & \sum_{\{x,y,z\} \subset \Lambda} \left(\frac{p_1}{|\Lambda|}\right)^3 \tau_{p_1}^{(3)}(x, y, z) \\ & \leq \sum_{\{x,y,z\} \subset \Lambda} \left(\frac{1}{|\Lambda|}\right)^3 \sum_{w \in \mathbb{Z}^d} \frac{\tau_{p_1}(x-w)}{g_{p_1}} \frac{\tau_{p_1}(y-w)}{g_{p_1}} \frac{\tau_{p_1}(z-w)}{g_{p_1}} \\ & = \sum_{\{x,y,z\} \subset \Lambda} \left(\frac{1}{|\Lambda|}\right)^3 \left(\sum_{w \neq x,y,z} \frac{\tau_{p_1}(x-w)}{g_{p_1}} \frac{\tau_{p_1}(y-w)}{g_{p_1}} \frac{\tau_{p_1}(z-w)}{g_{p_1}} + \frac{3}{g_{p_1}} \frac{\tau_{p_1}(x-z)}{g_{p_1}} \frac{\tau_{p_1}(y-z)}{g_{p_1}} \right) \\ & \leq \frac{3}{g_{p_1}} \underbrace{\left(\|D * \tau_{p_1}\|_\infty + \|D\|_\infty \right)}_{O(\beta) \text{ (}\cdot\text{:}(2.9))} \underbrace{\sum_{\{x,y\} \subset \Lambda} \left(\frac{1}{|\Lambda|}\right)^2 \sum_{w \neq x,y} \frac{\tau_{p_1}(x-w)}{g_{p_1}} \frac{\tau_{p_1}(y-w)}{g_{p_1}}}_{O(\beta) \text{ (}\cdot\text{:}(4.8))}, \end{aligned} \tag{4.20}$$

hence $I_3 = O(\beta^2)$. This completes the proof of $I = \frac{1}{2}S_{\geq 3}(o) + O(\beta^2)$, hence the proof of Lemma 4.1.

Proof of Lemma 4.2. First we prove (4.6). By the inverse Fourier transform, we have the rewrite

$$\sum_{u \in \Lambda} \frac{1}{|\Lambda|} \frac{\tau_{p_1}(u)}{g_{p_1}} = \left(D * \frac{\tau_{p_1}(u)}{g_{p_1}} \right)(o) = \int_{[-\pi,\pi]^d} \hat{D}(k) \frac{\hat{\tau}_{p_1}(k)}{g_{p_1}} \frac{d^d k}{(2\pi)^d}. \tag{4.21}$$

Notice that the Fourier transform of the recursion equation (2.8) yields

$$\hat{\tau}_p(k) = \frac{g_p + \hat{h}_p(k) + \hat{\pi}_p(k)}{1 - (g_p + \hat{h}_p(k) + \hat{\pi}_p(k))p\hat{D}(k)}. \tag{4.22}$$

We use this identity at $p_1 = 1/g_{p_1}$. Let

$$H(x) = \frac{h_{p_1}(x) + \pi_{p_1}(x)}{g_{p_1}}. \tag{4.23}$$

Thanks to the symmetry, the Fourier transform $\hat{H}(k)$ is real. Moreover, by (2.11)–(2.12), we can show that, for $d > 8$ and $L \gg 1$, $|\hat{H}(k)| = O(\beta)$ uniformly in k . Then, we can rewrite $\hat{\tau}_{p_1}(k)/g_{p_1}$ as

$$\begin{aligned} \frac{\hat{\tau}_{p_1}(k)}{g_{p_1}} &= \frac{1 + \hat{H}(k)}{1 - (1 + \hat{H}(k))\hat{D}(k)} \\ &= \frac{1}{1 - \hat{D}(k)} + \frac{\hat{H}(k)}{1 - \hat{D}(k)} \underbrace{\frac{1}{1 - (1 + \hat{H}(k))\hat{D}(k)}}_{=:\hat{F}(k)}. \end{aligned} \tag{4.24}$$

Applying this to (4.21) yields the main term $S_{\geq 2}(o)$ as

$$\sum_{u \in \Lambda} \frac{1}{|\Lambda|} \frac{\tau_{p_1}(u)}{g_{p_1}} = \underbrace{\int_{[-\pi, \pi]^d} \frac{\hat{D}(k)}{1 - \hat{D}(k)} \frac{d^d k}{(2\pi)^d}}_{S_{\geq 1}(o) (= S_{\geq 2}(o))} + \int_{[-\pi, \pi]^d} \frac{\hat{D}(k)\hat{H}(k)}{1 - \hat{D}(k)} \hat{F}(k) \frac{d^d k}{(2\pi)^d}. \tag{4.25}$$

It remains to show that the second term on the right is $O(\beta^2)$. To do so, we want an effective bound on $\hat{F}(k)$. We will show at the end of the proof that, for $d > 8$ and $L \gg 1$, there is an L -independent constant $C < \infty$ such that

$$0 < \hat{F}(k) \leq \frac{C}{1 - \hat{D}(k)}, \tag{4.26}$$

uniformly in k . However, this results in a weaker bound than (4.6), because $\int \frac{|\hat{D}(k)|}{(1 - \hat{D}(k))^2} \frac{d^d k}{(2\pi)^d} = O(\beta^{1/2})$. Instead, we first rewrite $\hat{F}(k)$ as

$$\hat{F}(k) = \frac{1}{1 - \hat{D}(k)} + \frac{\hat{D}(k)\hat{H}(k)}{1 - \hat{D}(k)} \hat{F}(k). \tag{4.27}$$

Then, the second term on the right of (4.25) equals

$$\underbrace{\int_{[-\pi, \pi]^d} \frac{\hat{D}(k)\hat{H}(k)}{(1 - \hat{D}(k))^2} \frac{d^d k}{(2\pi)^d}}_{(D * S_{\geq 0}^{*2} * H)(o)} + \int_{[-\pi, \pi]^d} \left(\frac{\hat{D}(k)\hat{H}(k)}{1 - \hat{D}(k)} \right)^2 \hat{F}(k) \frac{d^d k}{(2\pi)^d}. \tag{4.28}$$

Notice that, due to the identity (4.27), we have $\hat{D}(k)^2$ in the numerator of the second integrand in (4.28). The first term is readily bounded by $\|D * S_{\geq 0}^{*2}\|_{\infty} \hat{H}(0) = O(\beta^2)$. For the second term, we use $|\hat{H}(k)| = O(\beta)$ and (4.26) to obtain that

$$\int_{[-\pi, \pi]^d} \left(\frac{\hat{D}(k)\hat{H}(k)}{1 - \hat{D}(k)} \right)^2 \hat{F}(k) \frac{d^d k}{(2\pi)^d} \leq O(\beta^2) \underbrace{\int_{[-\pi, \pi]^d} \frac{\hat{D}(k)^2}{(1 - \hat{D}(k))^3} \frac{d^d k}{(2\pi)^d}}_{(D^{*2} * S_{\geq 0}^{*3})(o)} = O(\beta^3). \tag{4.29}$$

This completes the proof of (4.6).

We can also prove (4.7)–(4.8) in a similar manner by assuming (4.26). Hence we here prove only (4.7). By the inverse Fourier transform, we can rewrite the sum in (4.7) as

$$\begin{aligned} \sum_{\{u, v\} \subset \Lambda} \left(\frac{1}{|\Lambda|} \right)^2 \frac{\tau_{p_1}(u - v)}{g_{p_1}} &= \frac{1}{2} \left(D^{*2} * \frac{\tau_{p_1}}{g_{p_1}} \right) (o) - \frac{1}{2} D^{*2}(o) \\ &= \frac{1}{2} \int_{[-\pi, \pi]^d} \hat{D}(k)^2 \frac{\hat{\tau}_{p_1}(k)}{g_{p_1}} \frac{d^d k}{(2\pi)^d} - \frac{1}{2} D^{*2}(o). \end{aligned} \tag{4.30}$$

Then, by the identity (4.24), we can extract the main term $\frac{1}{2} S_{\geq 3}(o)$ as

$$\frac{1}{2} \underbrace{\int_{[-\pi, \pi]^d} \frac{\hat{D}(k)^2}{1 - \hat{D}(k)} \frac{d^d k}{(2\pi)^d}}_{\frac{1}{2} S_{\geq 3}(o)} - \frac{1}{2} D^{*2}(o) + \frac{1}{2} \int_{[-\pi, \pi]^d} \frac{\hat{D}(k)^2 \hat{H}(k)}{1 - \hat{D}(k)} \hat{F}(k) \frac{d^d k}{(2\pi)^d}. \tag{4.31}$$

Similarly to (4.29), the second term is bounded as

$$\left| \frac{1}{2} \int_{[-\pi, \pi]^d} \frac{\hat{D}(k)^2 \hat{H}(k)}{1 - \hat{D}(k)} \hat{F}(k) \frac{d^d k}{(2\pi)^d} \right| \stackrel{(4.26)}{\leq} O(\beta) \underbrace{\int_{[-\pi, \pi]^d} \frac{\hat{D}(k)^2}{(1 - \hat{D}(k))^2} \frac{d^d k}{(2\pi)^d}}_{(D^{*2} * S_{\geq 0}^{*2})(\alpha)} = O(\beta^2), \tag{4.32}$$

hence the completion of the proof of (4.7).

Finally we prove the inequality (4.26), for $\|k\| \geq \frac{1}{L}$ and $\|k\| \leq \frac{1}{L}$ separately. We begin with the former case. It is known (cf., e.g., [12]) that our D satisfies [14, Assumption D]; in particular, there is an L -independent constant $\eta \in (0, 1)$ such that

$$-1 + \eta \leq \forall_k \hat{D}(k) \stackrel{\|k\| \geq \frac{1}{L}}{\leq} 1 - \eta. \tag{4.33}$$

Since $|\hat{H}(k)| = O(\beta)$, we obtain that, for $L \gg 1$,

$$-1 + \frac{\eta}{2} \leq \forall_k (1 + \hat{H}(k)) \hat{D}(k) \stackrel{\|k\| \geq \frac{1}{L}}{\leq} 1 - \frac{\eta}{2}, \tag{4.34}$$

hence

$$0 < \frac{1}{2 - \eta/2} \leq \forall_k \hat{F}(k) \stackrel{\|k\| \geq \frac{1}{L}}{\leq} \frac{2}{\eta} \leq \frac{2}{\eta} \frac{2 - \eta}{1 - \hat{D}(k)}. \tag{4.35}$$

It remains to show that $\hat{F}(k)$ is bounded above by a multiple of $(1 - \hat{D}(k))^{-1}$ uniformly in $\|k\| \leq \frac{1}{L}$. We note that

$$\begin{aligned} \hat{F}(k)^{-1} &= 1 - (1 + \hat{H}(0)) + (1 + \hat{H}(0))(1 - \hat{D}(k)) + (\hat{H}(0) - \hat{H}(k))\hat{D}(k) \\ &= -\hat{H}(0) + \left(1 + \hat{H}(0) + \frac{\hat{H}(0) - \hat{H}(k)}{1 - \hat{D}(k)} \hat{D}(k) \right) (1 - \hat{D}(k)). \end{aligned} \tag{4.36}$$

Since $-\hat{H}(0)$ is bounded below by a positive multiple of β (as explained in the beginning of the proof of Lemma 2.1), ignoring this term yields a lower bound on $\hat{F}(k)^{-1}$. Moreover, since $|k \cdot x| \leq \|k\| \|x\| \leq 1$ for $x \in \Lambda$ and $\|k\| \leq \frac{1}{L}$, and since $1 - \cos t \geq \frac{2}{\pi^2} t^2$ for $|t| \leq 1$, there is a $c > 0$ such that

$$1 - \hat{D}(k) = \sum_{x \in \Lambda} \frac{1 - \cos(k \cdot x)}{|\Lambda|} \geq \frac{2}{\pi^2} \sum_{x \in \Lambda} \frac{(k \cdot x)^2}{|\Lambda|} = \frac{2\|k\|^2}{d\pi^2} \underbrace{\sum_{x \in \Lambda} \frac{\|x\|^2}{|\Lambda|}}_{\geq cL^2}. \tag{4.37}$$

On the other hand, by $1 - \cos t \leq \frac{1}{2} t^2$ for any t and using the x -space bounds (2.7) and (2.12), we have

$$|\hat{H}(0) - \hat{H}(k)| \leq \sum_x \frac{(k \cdot x)^2}{2} |H(x)| \leq \frac{\|k\|^2}{2d} \underbrace{\sum_x \|x\|^2 |H(x)|}_{O(L^2 \beta)}. \tag{4.38}$$

Therefore, by taking L sufficiently large, $\hat{F}(k)^{-1}$ is bounded below by a positive multiple of $1 - \hat{D}(k)$, uniformly in $\|k\| \leq \frac{1}{L}$. Combined with (4.35), this completes the proof of the inequality (4.26), hence the completion of the proof of Lemma 4.2. \square

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