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ON UNBOUNDED VOLTERRA OPERATORS

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Similarity invariants of certain families of Volterra operators acting in $L^p[0, 1]$ were first determined by Kalisch in [**4**] and [**5**]. Subsequently, Kantorovitz made an extensive study of perturbations of the form $T_\alpha = M + \alpha N$, where M is the operation of multiplication by x, and N is the Volterra operator

$$Nf(x) = \int_0^x f(t)dt,$$

both acting in $L^p[0, 1]$, and determined that Re α is a similarity invariant for the class of T_α's (cf. [**7**], and earlier work cited there). This result was eventually generalized to unbounded M (cf. [**6**]), and one direction was proved in [**2**] for both M and N unbounded: if Re α = Re β, then T_α is similar to T_β. Properties of the Riemann-Liouville semigroup of fractional integration play a key role, for in all cases the ensuing similarity turns out to be implemented by $N(i(\text{Im}(\alpha - \beta)))$, where $\{N(i\gamma)\}$ is the boundary group of the semigroup of fractional integration.

The similarity results in [**6**], where M is assumed unbounded, hinge on several basic facts: (i) M and N satisfy a suitable version of the "Volterra commutation relation" $[M, N] = N^2$; (ii) iT_α generates a strongly continuous group $\{T_\alpha(t)\}$; and (iii) this group satisfies certain growth conditions as a function of t. In the event that N is bounded, the group generation in (ii) is automatic. The unboundedness of N, however, raises the question of whether iT_α is in fact the generator of a strongly continuous group. Standard perturbation theorems (for example, cf. [**8**, IX.2]) indicate that group generation will no longer be the case for arbitrary α. Moreover, the crucial growth properties mentioned in (iii) no longer seem feasible, for $\|N\|$ enters into the estimates in a fundamental way. Consequently, the problem becomes considerably more difficult to formulate.

The purpose of this work, ultimately aimed at a viable formulation of the possible similarity invariance of Re α, is to establish a useful analogue of the group approach in [**6**]. A careful examination of that method reveals that "local" inequalities for semigroups could provide an adequate replacement, since semigroup generators are similar if and only if the semigroups are.

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Our first result (Theorem 1) is that for $|\alpha|$ sufficiently small, $-T_\alpha$ generates a holomorphic semigroup in the right half-plane. In Theorem 2, we exploit the approximation of N by the family of bounded operators $\{R(\epsilon; -D)\}_{\epsilon>0}$, where $D = N^{-1}$, to show that the semigroup $\{T_\alpha(\lambda)\}_{\lambda>0}$ generated by $-T_\alpha$ is the limit, in the strong operator topology, of $\{T_\alpha^\epsilon(\lambda)\}_{\lambda>0}$, generated by

$$-T_\alpha^\epsilon = -(M + \alpha R(\epsilon; -D)).$$

Theorem 3 then uses this approximation to obtain local estimates of $\|T_\alpha(\lambda)x\|$ for suitable $x \in X$.

We begin by recalling some of the terminology in [**6**].

Let $M(\cdot):R \to B(X)$ be a strongly continuous group of linear operators acting on the Banach space X, with infinitesimal generator iM, and let A be a bounded operator that commutes with M. Then a bounded operator N is said to be *M-Volterra with respect to A* if it commutes with A,

$$ND(M) \subset D(M),$$

and

$$[N, M] \subset AN^2$$

(cf. [**6**, Definition 1.1]). From [**6**], we have the following "standing hypotheses":

(i) $N(\zeta)$ is a given regular semigroup on C^+ (cf. [**6**, Definition 2.1]).

(ii) $M(\cdot)$ is a strongly continuous group of operators with generator iM.

(iii) A is a non-zero operator commuting with M and N.

(iv) $N(s + it)D(M) \subset D(M)$ for $s + it$ in some rectangle $0 \leqq s \leqq a$, $|t| \leqq a$.

(v) N is M-Volterra with respect to A.

For our purposes, the salient properties of $N(\zeta)$ are that it is a holomorphic semigroup of class (C_0) on C^+, with a strongly continuous boundary group $N(i\eta)$ on the imaginary axis. Moreover, $N(\zeta)$ is of exponential type $\nu < \infty$ in $\bar{C}^+$; i.e.,

$$\|N(\zeta)\| \leqq Ke^{\nu|\zeta|}, \quad \text{for } \operatorname{Re} \zeta \geqq 0.$$

We shall use these ideas in Proposition 3.

Now, let M denote the operation of multiplication by x with maximal domain $D(M)$ acting in $X = L^p(0, \infty)$, for $1 < p < \infty$, and let

$$Nf(x) = \int_x^\infty f(t)dt,$$

also with maximal domain in $L^p(0, \infty)$. (Note. Here N is unbounded, and so does not satisfy the standing hypotheses above.) Then $-M$ generates a holomorphic semigroup of contractions $M(\zeta)$ in the right half-plane. Our main results (Theorems 1 and 3, respectively) are that for $|\alpha|$ sufficiently small, $-T_\alpha = -(M + \alpha N)$ generates a holomorphic semigroup in the right half-plane, and that a local estimate in the spirit of [**6**, Theorem 3.1] holds for suitable vectors in X. An interesting aspect of the proofs is the use of an approximating family of semigroups $\{T^\epsilon_\alpha(\zeta)\}$ generated by $-(M + \alpha R_\epsilon)$, where R_ϵ is a certain family of bounded operators (Proposition 3).

We begin with some technical results.

PROPOSITION 1. *For all* $k \geqq 1$, $D(M^k) \subset D(N^k)$.

Proof. Let

$$K^-_{\eta,\alpha} f(x) = \frac{1}{\Gamma(\alpha)} x^\eta \int_x^\infty (t - x)^{\alpha-1} t^{-\eta-\alpha} f(t)dt ,$$

where $f \in L^p(0, \infty)$, $1 \leqq p \leqq \infty$, and $\alpha, \eta \in \mathbf{C}$ with $\operatorname{Re} \alpha > 0$. Then if $\operatorname{Re} \eta > -1/p$, $K^-_{\eta,\alpha}$ is a bounded operator on $L^p(0, \infty)$ (cf. [**9**]). Now if $f \in D(M^k)$, then

$$t^k f(t) \in L^p(0, \infty);$$

taking $\eta = 0$ and $\alpha = k$, we have

$$K^-_{0,k}(M^k f) = N^k f \in L^p(0, \infty),$$

so that $f \in D(N^k)$. More precisely, we have

$$\|N^k f\| = \|K^-_{0,k}(M^k f)\| \leqq C_k \|M^k f\|, \tag{1.1}$$

where C_k is the L^p-bound of $K^-_{0,k}$.

In particular, (1.1) shows

PROPOSITION 2. N^k *is relatively bounded with respect to* M^k, *for all* $k \geqq 1$ (*cf.* [**8**, p. 190]).

We are now in a position to prove

THEOREM 1. *For* $|\alpha|$ *sufficiently small,* $-T_\alpha = -(M + \alpha N)$, *with* $D(T_\alpha) = D(M)$, *generates a holomorphic semigroup* $T_\alpha(\zeta)$ *for* $\operatorname{Re} \zeta > 0$.

Proof. We invoke a standard result from semigroup theory; namely, that a closed operator $-A$ generates a (uniformly bounded) holomorphic semigroup in the right half-plane if and only if $\sigma(A) \subset R^+$, and for each $\theta < \dfrac{\pi}{2}$ and $|\arg \lambda| \leqq \dfrac{\pi}{2} + \theta$,

$$\|R(\lambda; -A)\| \leqq \frac{K}{|\lambda|}; \tag{1.2}$$

here the constant K may depend on θ (cf. [**8**, IX.1.6] and [**10**, Theorem X.52]).

We now assume that $|\alpha| < 1/2C_1$, where C_1 is the constant in (1.1). Hence

$$\|\alpha Nf\| \leqq |\alpha|C_1\|Mf\|, \quad \text{for all } f \in D(M), \text{ and } |\alpha|C_1 < 1,$$

so by [**8**, Theorem IV.1.1] T_α is closed.

In light of Proposition 1, for all $\lambda \in \rho(-M)$, $NR(\lambda; -M)$ is a bounded operator. From the formula

$$(\lambda I + T_\alpha)x = (I + \alpha NR(\lambda; -M))(\lambda I + M)x,$$

which is valid for all $x \in D(M)$, we see that if

$$\|\alpha NR(\lambda; -M)\| < 1,$$

then for $\lambda \in \rho(-M)$,

$$1 \in \rho(\alpha NR(\lambda; -M)),$$

hence $\lambda \in \rho(-T_\alpha)$, and

$$(1.3) \quad R(\lambda; -T_\alpha) = R(\lambda; -M)[I + \alpha NR(\lambda; -M)]^{-1}.$$

Since

$$\|NR(\lambda; -M)\| \leqq C_1\|MR(\lambda; -M)\| \leqq 2C_1, \text{ if } |\alpha| < \frac{1}{2C_1},$$

then $\rho(-M) \subset \rho(-T_\alpha)$, and for $\theta < \pi/2$, utilizing (1.3), we have

$$(1.4) \qquad \|R(\lambda; -T_\alpha)\| \leqq \frac{K'}{|\lambda|}, \quad |\arg \lambda| \leqq \frac{\pi}{2} + \theta,$$

and so $-T_\alpha$ generates a semigroup $T_\alpha(\zeta)$ that is holomorphic in the right half-plane; for each $\theta < \pi/2$, $\|T_\alpha(\zeta)\|$ is uniformly bounded in the sector $|\arg \zeta| \leqq \theta$.

Remark. Theorem 1 is obviously valid for any M that generates a semigroup holomorphic in the right half-plane, and N that satisfies (1.1) for $k = 1$.

It is convenient at this point to introduce the "approximating family of semigroups" discussed in [**3**]. Let $D = N^{-1}$, and for $\epsilon > 0$, set $R_\epsilon = R(\epsilon; -D)$ ($\rho(-D)$ is the open right half-plane); concretely,

$$R_\epsilon f(x) = \int_x^\infty e^{-\epsilon(t-x)}f(t)dt.$$

Then $f \in D(N)$ if and only if $\lim_{\epsilon\to 0^+} R_\epsilon f$ exists, in which case

$$Nf = \lim_{\epsilon\to 0^+} R_\epsilon f.$$

(In fact,

$$N^\alpha f = \lim_{\epsilon\to 0^+} R_\epsilon^\alpha f$$

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for fractional powers (Re $\alpha > 0$) of the operators; cf [**3**, Theorem A].) As the R_ϵ's are bounded, for $\alpha \in \mathbf{C}$, $-T_\alpha^\epsilon = -(M + \alpha R_\epsilon)$ generates a holomorphic semigroup $T_\alpha^\epsilon(\zeta)$ in the right half-plane ([**10**, Theorem X.54]). The relationship between these semigroups and those generated by the $-T_\alpha$'s is made precise in the following:

THEOREM 2. *For* $|\alpha| < 1/2C_1$, *and* $\lambda > 0$,

$$T_\alpha(\lambda) = s - \lim_{\epsilon \to 0^+} T_\alpha^\epsilon(\lambda).$$

Proof. For $f \in L^p(0, \infty)$, and $\lambda \in \rho(-M)$,

$$\begin{aligned} \|R_\epsilon R(\lambda; -M)f\| &= \left(\int_x^\infty \left| \int_x^\infty e^{-\epsilon(t-x)} R(\lambda; -M)f(t)dt \right|^p dx\right)^{1/p} \\ &\leqq \left(\int_0^\infty \left(\int_x^\infty |R(\lambda; -M)f(t)|dt\right)^p dx\right)^{1/p} \\ &= \|N|R(\lambda; -M)f|\,\| \leqq C_1 \|M|R(\lambda; -M)f|\,\| \\ &\leqq 2C_1\|f\|. \end{aligned}$$

Hence if $|\alpha| < 1/2C_1$, $\|\alpha R_\epsilon R(\lambda; -M)\| < 1$, $\lambda \in \rho(-T_\alpha^\epsilon)$, and (1.3) holds, with T_α^ϵ replacing T_α, and R_ϵ replacing N. Moreover, for

$$\theta < \frac{\pi}{2}, \quad \text{and} \quad |\arg \lambda| \leqq \frac{\pi}{2} + \theta,$$

$$\|R(\lambda; -T_\alpha^\epsilon)\| \leqq \frac{K'}{|\lambda|}, \tag{1.5}$$

where the K' in (1.5) is the same as that in (1.4). From the integral representations

$$T_\alpha^\epsilon(\zeta) = \frac{1}{2\pi i} \int_\Gamma e^{\zeta\lambda} R(\lambda; -T_\alpha^\epsilon) d\lambda,$$

where Γ is a suitable curve that we may choose in

$$\rho(-M) \subset \rho(-T_\alpha^\epsilon) \cap \rho(-T_\alpha),$$

and the uniform bounds on the resolvents in (1.4) and (1.5), we see that for $\lambda > 0$,

$$\|T_\alpha^\epsilon(\lambda)\| \leqq K_1 \text{ and } \|T_\alpha(\lambda)\| \leqq K_1,$$

where K_1 is independent of $\epsilon > 0$ (cf. [**8**, p. 491]). Moreover for $\lambda > 0$, and $x \in X$,

$$\begin{aligned}
&\|R(\lambda; -T_\alpha)x - R(\lambda; -T_\alpha^\epsilon)x\| \\
&\quad = \|R(\lambda; -M)\{[I + \alpha NR(\lambda; -M)]^{-1} \\
&\quad - [I + \alpha R_\epsilon R(\lambda; -M)]^{-1}\}x\| \\
&\quad \leqq \frac{1}{\lambda} \|(I + \alpha R_\epsilon R(\lambda; -M))\{\alpha(R_\epsilon - N)R(\lambda; -M)\} \\
&\quad \times (I + \alpha NR(\lambda; -M))x\| \\
&\quad \leqq \frac{2|\alpha|}{\lambda} \|(R_\epsilon - N)R(\lambda; -M)(I + \alpha NR(\lambda; -M))x\| \\
&\quad \to 0 \text{ as } \epsilon \to 0^+.
\end{aligned}$$

Convergence of the corresponding semigroups now follows from [**8**, Theorem 2.16].

Remark. The estimates $\|T_\alpha^\epsilon(\lambda)\| \leqq K_1$, $\|T_\alpha(\lambda)\| \leqq K_1$ are actually valid for all λ with $|\arg \lambda| \leqq \theta$, $\theta < \pi/2$ (here K_1 depends on θ), so that Vitali's convergence theorem yields the result of Theorem 2 for all λ in the right half-plane; moreover, the convergence is uniform in any compact subset.

Notation. We let $M(\lambda)$ ($\lambda > 0$) denote the semigroup generated by $-M$; note that this is not quite consistent with the notation $T_\alpha(t)$ for the group generated by iT_α that was used in [**6**].

The next result is valid in a more general context.

PROPOSITION 3. *Let N be M-Volterra with respect to A, and $-M$ be the generator of a semigroup holomorphic in the right half-plane. Then for all nonnegative integers k, and real $\lambda \geqq 0$,*

$$T_{-k}(\lambda) = M(\lambda)(I + \lambda AN)^k.$$

Proof. First, we note that if N is M-Volterra with respect to A, then N is $-M$-Volterra with respect to $-A$. Therefore by Theorem 1.2 (b) in [**6**], we have that

$$[N, R(\lambda; -M)] = -AR(\lambda; -M)N^2R(\lambda; -M), \quad \text{for } \lambda \in \rho(-M).$$

Proceeding as in the proof of part (c) of that theorem, we have for $x \in D(M)$, $\lambda > 0$ and γ sufficiently large

$$M(\lambda)x = \frac{1}{2\pi i}\int_{\mathbf{R}} e^{(\gamma + i\omega)\lambda}R(\gamma + i\omega; -M)x d\omega.$$

Thus (since $ND(M) \subset D(M)$)

$$\begin{aligned}
[N, M(\lambda)]x &= \frac{1}{2\pi i} \int_{\mathbf{R}} e^{(\gamma+i\omega)\lambda}[N, R(\gamma + i\omega; -M)]x d\omega \\
&= \frac{-A}{2\pi i} \int_{\mathbf{R}} e^{(\gamma+i\omega)\lambda} R(\gamma + i\omega; -M)N^2 \\
&\qquad \times R(\gamma + i\omega; -M)x d\omega \\
&= \frac{-AN}{2\pi i} \int_{\mathbf{R}} e^{(\gamma+i\omega)\lambda} R(\gamma + i\omega; -M)^2 N x d\omega \\
&= \frac{-AN}{2\pi} \int_{\mathbf{R}} e^{(\gamma+i\omega)\lambda} \frac{d}{d\omega} R(\gamma + i\omega; -M) N x d\omega \\
&= \frac{-\lambda AN}{2\pi i} \int_{\mathbf{R}} e^{(\gamma+i\omega)\lambda} R(\gamma + i\omega; -M) N x d\omega \\
&= -\lambda ANM(\lambda)Nx;
\end{aligned}$$

for the third equality, we used the fact that for $|\lambda|$ sufficiently large, $NR(\lambda; -M)$ commutes with $R(\lambda; -M)N$, and for the last, we integrated by parts and used the fact that

$$\|R(\gamma + i\omega); -M)\| \leqq \frac{K}{|\gamma + i\omega|} \to 0 \text{ as } |\omega| \to \infty,$$

since $-M$ generates a holomorphic semigroup in the right half-plane.

Now to prove the proposition, we note that

$$\lambda \mapsto M(\lambda)(I + \lambda AN)^k$$

is a strongly continuous semigroup of class (C_0), whose generator U_k satisfies $-T_{-k} \subset U_k$; indeed, for λ, $\mu > 0$, we have

$$\begin{aligned}
&(I + \lambda AN)M(\mu)(I + \mu AN) \\
&= M(\mu)(I + \mu AN) + \lambda A(NM(\mu) + \mu ANM(\mu)N) \\
&= M(\mu)(I + \mu AN) + \lambda AM(\mu)N \\
&= M(\mu)(I + (\lambda + \mu)AN),
\end{aligned}$$

so that $\lambda \mapsto M(\lambda)(I + \lambda AN)$ is a semigroup (of class (C_0)). It follows by induction that $M(\lambda)(I + \lambda AN)^k$ is, as well, provided k is a positive integer. The statement about its generator is shown exactly as in [**6**, Theorem 1.3]. However, $U_k(\lambda)$ (the semigroup generated by U_k) is clearly of finite type, so there exists $\lambda \in \rho(-T_{-k}) \cap \rho(U_k)$. But this forces the two operators to be equal, and we are done.

Returning to our more concrete setting, we have

COROLLARY. *For each* $\epsilon > 0$, *all nonnegative integers* k, *and* $\lambda \geqq 0$,

$$T^{\epsilon}_{-k}(\lambda) = M(\lambda)(I - \lambda R_{\epsilon})^{k}.$$

Proof. It suffices to note that for each $\epsilon > 0$, R_{ϵ} is M-Volterra with respect to $-I$. In fact, R_{ϵ} satisfies the standing hypotheses for each $\epsilon > 0$ (cf. [**3**]).

Set

$$D^{\infty}(M) = \bigcap_{k \geqq 1} D(M^{k}).$$

We now prove

THEOREM 3. *Let* $x \in D^{\infty}(M)$, $\xi \geqq 0$, $\lambda \geqq 0$, $\eta \in \mathbf{R}$. *Then there exist constants* K *and* C, C *depending on* ξ *and* x, *such that if* $|-\xi + i\eta| < 1/2C_1$,

$$\|T_{-\xi+i\eta}(\lambda)x\| \leqq K^{2}C \exp(\nu\eta^{2})(1 + \lambda)^{\xi}.$$

Proof. Fix $x \in D^{\infty}(M)$. Using Proposition 3, we see that for $k \in \mathbf{Z}^{+}$, $\epsilon > 0$, and $\lambda \geqq 0$,

$$\begin{aligned}\|T^{\epsilon}_{-k}(\lambda)x\| &= \|M(\lambda)(I - \lambda R_{\epsilon})^{k}x\| \\ &\leqq \sum_{j=0}^{k} \binom{k}{j} \lambda^{j} \|R^{j}_{\epsilon}x\| \\ &\leqq (1 + \lambda)^{k}\bar{C}_{k},\end{aligned}$$

where

$$\bar{C}_{k} = \sup_{\substack{0 \leqq j \leqq k \\ \epsilon > 0}} \|R^{j}_{\epsilon}x\|;$$

$\bar{C}_{k} < \infty$, for $x \in D^{\infty}(M) \subset D^{\infty}(N)$ implies

$$\lim_{\epsilon \to 0^{+}} R^{j}_{\epsilon}x = N^{j}, \text{ for all } j \geqq 1.$$

Hence

$$(1.6) \qquad \|T^{\epsilon}_{-k}(\lambda)x\| \leqq \bar{C}_{k}(1 + \lambda)^{k}.$$

Now fix $\lambda \geqq 0$, and set

$$\Phi^{\epsilon}_{\lambda}(\xi + i\eta) = \exp(\nu\zeta^{2})T^{\epsilon}_{\zeta}(\lambda)x,$$

where $\zeta = \xi + i\eta$. Then $\Phi^{\epsilon}_{\lambda}(\cdot)$ is entire, and

$$\|\Phi^{\epsilon}_{\lambda}(\xi + i\eta)\| \leqq K^{2} \exp(\nu(\xi^{2} - \eta^{2} + 2|\eta|))\,\|T^{\epsilon}_{\xi}(\lambda)\|\,\|x\|,$$

for

$$T^{\epsilon}_{\zeta}(\lambda) = R^{i\gamma}_{\epsilon}T^{\epsilon}_{\xi}(\lambda)R^{-i\gamma}_{\epsilon},$$

where $\{R_\epsilon^{i\gamma}\}_{\gamma\in\mathbf{R}}$ is the boundary group of $\{R_\epsilon^\alpha\}_{\mathrm{Re}\,\alpha>0}$, and $\|R_\epsilon^{i\eta}\| \leqq Ke^{\nu|\eta|}$, with K and ν independent of $\epsilon > 0$. Thus $\Phi_\lambda^\epsilon(\xi + i\eta)$ is bounded in the vertical strips $k - 1 \leqq \xi \leqq k$, since $T_\alpha^\epsilon(\lambda)$ is a holomorphic (entire) function of α (cf. [**8**, Theorem IX.2.1]). From (1.6),

$$\begin{aligned}\|\Phi_\lambda^\epsilon(-k + i\eta)\| &\leqq K \exp(\nu(k^2 - \eta^2 + |\eta|))\|T_{-k}^\epsilon(\lambda)R_\epsilon^{-i\eta}x\| \\ &\leqq \bar{C}_k K^2 \exp(\nu(k^2 + 1))(1 + \lambda)^k;\end{aligned}$$

we have used the fact that $R_\epsilon^{-i\eta}$ commutes with R_ϵ^j, so that

$$\|M(\lambda)(I - \lambda R_\epsilon)^k R_\epsilon^{-i\eta}x\| = \|M(\lambda)R_\epsilon^{-i\eta}(I - \lambda R_\epsilon)^k x\|.$$

By the Three Lines Theorem,

$$\|\Phi_\lambda^\epsilon(-\xi + i\eta)\| \leqq \bar{C}_k K^2 \exp(\nu(\xi^2 + 5/4))(1 + \lambda)^\xi,$$
$$-k \leqq -\xi \leqq -(k - 1).$$

Therefore

$$(1.7) \qquad \|T_\zeta^\epsilon(\lambda)x\| \leqq \bar{C}_k K^2 \exp(\nu(\eta^2 + 5/4))(1 + \lambda)^\xi, \quad \zeta = -\xi + i\eta,\ \xi \geqq 0;$$

here $\bar{C}_k$ depends on $[\xi]$ and x.

Now assume $|\zeta| < 1/2C_1$, so that $-T_\zeta$ generates a holomorphic semigroup $T_\zeta(\cdot)$ in the right half-plane that is, by Theorem 2, the strong limit of the $T_\alpha^\epsilon(\lambda)$'s, for $\lambda \geqq 0$. Let

$$k = [\,|\mathrm{Re}\,\zeta|\,] + 1.$$

Then for all $\epsilon > 0$, (1.7) holds, where $\bar{C}_k$ is independent of $\epsilon > 0$, but depends on $\mathrm{Re}\,\zeta$ and x. Taking the limit, as $\epsilon \to 0^+$, we obtain

$$\|T_{-\xi+i\eta}(\lambda)x\| \leqq C_k K^2 \exp(\nu(\eta^2 + 5/4))(1 + \lambda)^\xi,$$
$$\xi \geqq 0,\ |-\xi + i\eta| < \frac{1}{2C_1}.$$

Remark. For fixed $\zeta = -\xi + i\eta$, the estimate is actually valid for all

$$x \in \bigcap_{j=1}^{k} D(M^j), \quad \text{where } k = [\,|\mathrm{Re}\,\zeta|\,] + 1.$$

In conclusion, we make some comments regarding the difficulty in adapting additional techniques in [**6**] to this situation. In the proof of the inequality

$$\inf_t\,(1 + c|t|)^{-|\xi|}\|T_\xi(t)\| > 0$$

in [**6**, Theorem 3.2] it was assumed that the result fails for some $|\xi_0| > 1$.

Of course, there is no longer any guarantee that $-T_{\xi_0}$ will generate a semigroup for $|\xi_0| > 1$, and so any result on similarity invariance of Re α will only be valid in a suitable strip of the complex plane.

Although the full similarity result remains open, some of the difficulties inherent in its formulation and proof are brought into sharper focus by these results.

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